PRACTICAL MATHEMATICS

A TEXTBOOK COVERING THE SYLLABUS OF THE B.Sc. EXAMINATIONS IN THIS SUBJECT AND SUITABLE FOR ADVANCED CLASSES IN TECHNICAL COLLEGES

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PRACTICAL
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CONTENTS

CHAPTER I

LIMITS—CONVERGENCY AND DIVERGENCY OF SERIES—BINOMIAL AND EXPONENTIAL SERIES—HYPERBOLIC FUNCTIONS—COMPLEX NUMBERS

1. **Limits or Limiting Values.** The result of substituting $2 + h$ for x in $\dfrac{3x-2}{x+4}$ is $\dfrac{4+3h}{6+h}$, and as h is indefinitely diminished (i.e. as x approaches the value 2) this fraction approaches the value $\dfrac{2}{3}$, which is called the *limit* or *limiting value* of $\dfrac{3x-2}{x+4}$ as x tends to the value 2. Briefly we write $\underset{x \to 2}{Lt.} \dfrac{3x-2}{x+4} = \dfrac{2}{3}$. It will be noted that in this case the value of the limit is equal to the value of the fraction when $x = 2$.

By substituting $2 + h$ for x in $\dfrac{x^2-4}{x-2}$ we obtain $4 + h$, which approaches the value 4 as h is indefinitely diminished, i.e. $\underset{x \to 2}{Lt.} \dfrac{x^2-4}{x-2} = 4$. If, however, we simply substitute $x = 2$ in the expression $\dfrac{x^2-4}{x-2}$ we obtain, not the limit 4, but the meaningless or undetermined form $\dfrac{0}{0}$. The essential point to note in the above two cases is that, as x continually approaches the value 2, the function considered continually approaches a definite value from which it can be made to differ by as small an amount as we please. We are thus led to the following general definition of the *limit of a function.*

A function $f(x)$ approaches a limiting value L as x approaches a value x_1, if by choosing δ sufficiently small the value of $L - f(x_1 \pm \delta)$ can be made less than any assignable magnitude, however small. (The sign $\pm$ means "either plus or minus.")

It may happen that, as x approaches the value x_1, the absolute value of $f(x)$ ultimately becomes and thereafter remains greater than any assignable positive magnitude, however large. In such cases we write $\underset{x \to x_1}{Lt.} f(x) = +\infty$ or $\underset{x \to x_1}{Lt.} f(x) = -\infty$; but it must be

clearly understood that infinity (∞) is not a limit as given in the definition above, but a symbol to indicate that under the conditions $f(x)$ increases or decreases without restriction. Thus, if A is a constant other than zero, $\underset{x \to 0}{Lt.} \frac{A}{x} = +\infty$ or $-\infty$ according as the sign of x as it approaches its limit is the same as or opposite to the sign of A. Also $\underset{x \to \infty}{Lt.} \frac{A}{x} = 0$, whether A is positive or negative.

2. **Continuous Function.** If the value assumed by $f(x)$ when $x = x_1$ is the limit to which $f(x)$ tends as x tends to the value x_1 in any manner, then $f(x)$ is said to be continuous for $x = x_1$. More briefly, $f(x)$ is continuous for $x = x_1$ if $\underset{x \to x_1}{Lt.} f(x) = f(x_1)$; and when this condition does not hold $f(x)$ is discontinuous for $x = x_1$. When $f(x)$ is continuous for every value of x in an interval $x = x_1$ to $x = x_2$, it is said to be continuous throughout that interval.

EXAMPLE

Let $f(x) = \frac{3x - 2}{x + 4}$; then since $\underset{x \to 2}{Lt.} f(x) = f(2)$ (by Art. 1), $f(x)$ is continuous for $x = 2$.

The functions considered in this book are in general continuous. We shall, however, meet with functions, such as $\frac{1}{x}$ and $\tan x$, which are discontinuous for certain isolated values; $\frac{1}{x}$ is continuous for all values of x except $x = 0$, and $\tan x$ is continuous for all values of x except $x = \frac{k\pi}{2}$, where k is any odd integer, positive or negative.

3. **Theorems on Limits.** The following theorems relating to limiting values are fundamental. We assume that the limits are all finite.

(*a*) The limit of the sum of a finite number of functions equals the sum of the limits of the separate functions.

(*b*) The limit of the product of a finite number of functions equals the product of the limits of the separate functions.

(*c*) The limit of the quotient of two functions equals the quotient of the limits of the separate functions, provided that the limit of the divisor is not zero.

We shall prove the second of these theorems and leave the reader to prove the other two.

Let $Lt.\ f(x) = l$, $Lt.\ \phi(x) = m$. We can write $f(x) = l + \alpha$, $\phi(x) = m + \beta$, where α and β are small quantities, which vanish in the limit.

$$\therefore Lt.\ [f(x) \cdot \phi(x)] = Lt.\ [(l + \alpha)(m + \beta)]$$
$$= Lt.\ (lm + m\alpha + l\beta + \alpha\beta)$$
$$= lm \text{ (since } m\alpha,\ l\beta,\ \alpha\beta \text{ all vanish in the limit)}$$

$$\therefore \qquad Lt.\ [f(x) \cdot \phi(x)] = Lt.\ f(x) \times Lt.\ \phi(x) \quad . \qquad \text{(I.1)}$$

This result can obviously be extended to the product of three, four, and any finite number of functions.

4. **Two Important Limits.** The following limits follow from geometrical considerations.

$$\underset{\theta \to 0}{Lt.} \frac{\sin \theta}{\theta} = 1 : \underset{\theta \to 0}{Lt.} \frac{\tan \theta}{\theta} = 1 \quad . \quad . \qquad \text{(I.2)}$$

The arc AB (Fig. 1) subtends an angle θ radians $\left(\theta < \frac{\pi}{2}\right)$ at the centre O of a circle. AC is the tangent at A meeting OB produced at C.

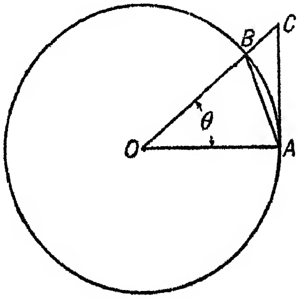


FIG. 1

Then area of $\Delta OAB <$ area of sector $OAB <$ area of ΔOAC, i.e. $\frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan \theta$.

$$\therefore \qquad \sin \theta < \theta < \tan \theta \quad . \quad . \quad . \qquad \text{(I.3)}$$

Dividing by $\sin \theta$, $\qquad 1 < \dfrac{\theta}{\sin \theta} < \sec \theta.$

If θ now tend to the limit zero, $\underset{\theta\to 0}{Lt.}\ \frac{\theta}{\sin\theta}$ must lie between the constant 1 and $\underset{\theta\to 0}{Lt.} \sec\theta$, which is also 1.

Hence, $\underset{\theta\to 0}{Lt.}\ \frac{\theta}{\sin\theta} = 1$, or $\underset{\theta\to 0}{Lt.}\ \frac{\sin\theta}{\theta} = 1$.

Dividing (1.3) by $\tan\theta$, $\cos\theta < \frac{\theta}{\tan\theta} < 1$, and it follows by similar reasoning to the above that $\underset{\theta\to 0}{Lt.}\ \frac{\tan\theta}{\theta} = 1$.

Other important limits will be established in the course of the present chapter.

5. **Convergency and Divergency of Series.** An infinite series is a series with an endless succession of terms. If S_n denote the sum of the first n terms of the series, then the series is said to be convergent if $\underset{n\to\infty}{Lt.} S_n$ is a finite fixed quantity, and it is said to be divergent if $\underset{n\to\infty}{Lt.} S_n$ is $+\infty$ or $-\infty$.

The reader will already be familiar with the Geometric Series $a + ar + ar^2 + ar^3 + \ldots$ Here $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$.

Now $\underset{n\to\infty}{Lt.} r^n = 0$, provided that $|r|^* < 1$; and therefore $\underset{n\to\infty}{Lt.} S_n = \frac{a}{1-r}$. If $|r| \nless 1$, $\underset{n\to\infty}{Lt.} S_n = \pm\infty$. The Geometric Series is, then, convergent if $|r| < 1$, and divergent if $|r| \nless 1$.

6. **Tests for Convergency or Divergency.** The removal of a finite number of terms from an infinite series will not make a divergent series convergent or a convergent series divergent. For let S_n be the sum of the original infinite series and S_T the sum of the terms removed. Then since S_T is a finite quantity we have $\underset{n\to\infty}{Lt.} (S_n - S_T) = \underset{n\to\infty}{Lt.} S_n - S_T$, and if $\underset{n\to\infty}{Lt.} S_n = \infty$, the left-hand side of this relation is also infinite. Thus if the original series is divergent the series with its terms removed is also divergent. Consequently when applying tests of convergency we may ignore any finite number of terms of the series. The following tests will be found sufficient when

* $|r|$ means the numerical value of r, or as we term it, "the absolute value of r."

dealing with those infinite series which usually occur in engineering mathematics.

Test 1. AN INFINITE SERIES IS CONVERGENT IF NONE OF ITS TERMS IS NUMERICALLY GREATER THAN THE CORRESPONDING TERM OF ANOTHER SERIES OF POSITIVE TERMS WHICH IS KNOWN TO BE CONVERGENT. If each term of the first series is equal to the corresponding term of the second series, the two series are identical and the rule is obviously true. If some of the terms of the first series are less than the corresponding terms of the second series the first series, having a smaller sum than the second series, is therefore convergent.

Test 2. THE RATIO TEST OF CONVERGENCE. Let $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ be an infinite series. Then the series is convergent if $\underset{n\to\infty}{Lt.} \frac{u_{n+1}}{u_n} < 1$ and divergent if $\underset{n\to\infty}{Lt.} \frac{u_{n+1}}{u_n} > 1$. If $\underset{n\to\infty}{Lt.} \frac{u_{n+1}}{u_n} = 1$, this test is inconclusive and a further test is required.

Suppose that after the nth term of the series the ratio of any term to the preceding term is less than k where $k < 1$, then since

$$\frac{u_{n+1}}{u} < k, \frac{u_{n+2}}{u_{n+1}} < k, \frac{u_{n+3}}{u_{n+2}} < k, \text{ etc. } \ldots$$

we have

$$u_n + u_{n+1} + u_{n+2} + \ldots < u_n(1 + k + k^2 + \ldots)$$
$$< \frac{u_n}{1-k}$$

Hence, the sum of the series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is finite and the series is convergent.

EXAMPLE

Test for convergence the series

$$(1)\ 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$$

$$(2)\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

In (1) $u_n = \frac{x^{n-1}}{\lfloor n-1}$, $u_{n+1} = \frac{x^n}{\lfloor n}$ and $\frac{u_{n+1}}{u_n} = \frac{x}{n}$.

Hence, $\underset{n\to\infty}{Lt.} \frac{u_{n+1}}{u_n} = 0$, and the series is convergent for all values of x.

In (2) $u_n = (-1)^{n-1}\frac{x^n}{n}$, $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$, and $\frac{u_{n+1}}{u_n} = -\frac{nx}{n+1}$.

Hence, $\underset{n\to\infty}{Lt.} \frac{u_{n+1}}{u_n} = -x$.

The series is therefore convergent if $|x| < 1$ and divergent if $|x| > 1$.

If $x = 1$, the series becomes

$$S_1 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \quad . \quad . \quad . \quad (I.4)$$

the nth term of which is $(-1)^{n-1} \cdot \dfrac{1}{n}$; and if $x = -1$ the series becomes

$$S_2 = -1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{5} - \quad . \quad . \quad . \quad (I.5)$$

The series (I.4) may be written in either of the forms

$$S_1 = (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \ . \ . \ .$$

or

$$S_1 = 1 - (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{4} - \tfrac{1}{5}) - \ . \ . \ .$$

The former of these shows that S_1 is positive, and the latter shows that S_1 is less than unity and is therefore finite.

The absolute value of the sum of all the terms after the nth is less than $\dfrac{1}{n+1}$ and greater than $\dfrac{1}{n+1} - \dfrac{1}{n+2} = \dfrac{1}{(n+1)(n+2)}$ and as both these approach the limit 0 as $n \to \infty$, the series (I.4) has a sum, and is therefore convergent.

The series (I.5) may be written

$$S_2 = -(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ . \ . \ ),$$

the absolute value of S_2 being given by the series in the bracket. We have, therefore,

$$\begin{aligned} |S_2| &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \ . \ . \ . \\ &> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \ . \ . \ . \ . \\ &> 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ . \ . \ . \end{aligned}$$

Hence, $|S_2| = \infty$, and the series is divergent.

7. **Binomial Series.** The expansion of $(a + x)^n$ in ascending powers of x is as follows—

$$\begin{aligned} (a + x)^n = a^n + na^{n-1}x &+ \frac{n(n-1)}{\underline{|2}} a^{n-2}x^2 \\ &+ \frac{n(n-1)(n-2)}{\underline{|3}} a^{n-3}x^3 + \ . \ . \ . \\ . \ . \ . + \frac{n(n-1)(n-2) \ . \ . \ . \ (n-r+1)}{\underline{|r}} & a^{n-r}x^r + \ . \ . \ . \quad (I.6) \end{aligned}$$

When n is a positive integer, the series terminates, the last term being x^n. For the proof of the expansion in this case the reader is

referred to an elementary textbook on algebra. When n is not a positive integer, the series still holds under certain conditions.

In the series as given

$$u_r = \frac{n(n-1)(n-2)\ldots(n-r+2)}{\underline{|r-1}}\,a^{n-r+1}x^{r-1}$$

and $$u_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}\,a^{n-r}x^{r}$$

$$\therefore \quad \frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}\cdot\frac{x}{a} = \left(\frac{n+1}{r} - 1\right)\frac{x}{a}$$

Hence, $\underset{r \to \infty}{Lt.}\ \dfrac{u_{r+1}}{u_r} = -\dfrac{x}{a}$ numerically; so that if $x < a$ numerically, the series is convergent.

Consider now the expansion of $(1 - x)^n$ when $n = -1$.

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

Put $x = 3$. Then $(-2)^{-1} = 1 + 3 + 3^2 + 3^3 + \ldots$, a result which is obviously absurd. The series $1 + x + x^2 + x^3 + \ldots$ is an infinite geometrical series, and if $|x| < 1$, its sum is $\dfrac{1}{1-x} = (1 - x)^{-1}$. It follows then that the series $1 + x + x^2 + x^3 + \ldots$ is the arithmetical equivalent of $(1 - x)^{-1}$ only so long as $|x| < 1$.

In general, when n is not a positive integer, the infinite series

$$1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots$$

is the arithmetical equivalent of $(1 + x)^n$, provided that the series is convergent, i.e. provided that $|x| < 1$.

That is

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots \qquad \text{(I.7)}$$

The expansion (I.6) is obtained from Maclaurin's Theorem in Art. 56.

EXAMPLE 1

Find the 4th term in the expansion of $(1 - x)^{\frac{3}{4}}$.

4th term $= \dfrac{\frac{3}{4}(\frac{3}{4} - 1)(\frac{3}{4} - 2)}{\underline{|3}}\cdot(-x)^3 = \dfrac{3(-1)(-5)}{4^3 \cdot 6}(-x^3) = -\dfrac{5}{128}x^3$

EXAMPLE 2

Expand $\sqrt{\frac{1-a}{1+a}}$ as far as the term in a^5, where $a < 1$.

$$\sqrt{\frac{1-a}{1+a}} = \frac{1-a}{\sqrt{1-a^2}} = (1-a)(1-a^2)^{-\frac{1}{2}}$$

$$= (1-a)\left[1 + (-\tfrac{1}{2})(-a^2) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{\underline{|2}}(-a^2)^2 + \dots\right]$$

$$= (1-a)(1 + \tfrac{1}{2}a^2 + \tfrac{3}{8}a^4 + \dots)$$

$$= 1 - a + \tfrac{1}{2}a^2 - \tfrac{1}{2}a^3 + \tfrac{3}{8}a^4 - \tfrac{3}{8}a^5 + \dots$$

8. **Approximations.** If x is small compared with unity, then using (I.7) we obtain the following as first approximations.

$$(1+x)^n = 1 + nx \quad . \quad . \quad . \quad . \qquad (I.8)$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x \quad . \quad . \qquad (I.9)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x \quad . \quad . \qquad (I.10)$$

Also $$\frac{(1+a)(1+b)}{(1+c)(1-d)} = (1+a+b)(1+c)^{-1}(1-d)^{-1}$$

$$= (1+a+b)(1-c)(1+d)$$

$$= 1 + a + b - c + d$$

where a, b, c, d are all small compared with unity.

If greater accuracy is required, we can take

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2$$

as a second approximation.

EXAMPLE 1

Show how to approximate to $\left(\frac{a}{b}\right)^n$, when a and b are nearly equal. Find the value of $\sqrt{\frac{97}{101}}$ correct to 5 figures.

Let $a = b + h$ where h will be a small positive or negative quantity. Then

$$\left(\frac{a}{b}\right)^n = \left(1 + \frac{h}{b}\right)^n = 1 + n\cdot\frac{h}{b} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{h^2}{b^2} + \dots$$

and we can take as many terms as will give us the required degree of accuracy,

$$\sqrt{\frac{97}{101}} = \left(1-\frac{4}{101}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\cdot\frac{4}{101} - \frac{1}{8}\cdot\left(\frac{4}{101}\right)^2 - \frac{1}{16}\left(\frac{4}{101}\right)^3 \ldots$$

$$= 1 - 0{\cdot}0198020 - 0{\cdot}0001961 - 0{\cdot}0000039 \ldots$$

$$= 1 - 0{\cdot}020002$$

$$= 0{\cdot}98000 \text{ (to 5 figures)}$$

EXAMPLE 2

Prove that for small changes of the angular velocity ω radians per second, the percentage change in the energy E ft-lb of a rotating flywheel is twice the percentage change in the speed.

If I engineers' units is the moment of inertia of the flywheel $E = \frac{1}{2}I\omega^2$, and if the speed changes to $\omega + \delta\omega$ and the energy becomes $E + \delta E$, we have

$$E + \delta E = \tfrac{1}{2}I(\omega + \delta\omega)^2$$

$$= \tfrac{1}{2}I\omega^2\left(1 + \frac{\delta\omega}{\omega}\right)^2$$

$$= \tfrac{1}{2}I\omega^2\left(1 + 2\frac{\delta\omega}{\omega}\right) \text{ approximately.}$$

By subtraction $\qquad \delta E = I\omega\delta\omega$

and $\qquad \dfrac{\delta E}{E} = \dfrac{I\omega\delta\omega}{\frac{1}{2}I\omega^2}$

$\therefore \qquad 100\dfrac{\delta E}{E} = 2\dfrac{\delta\omega}{\omega} \times 100$

that is, the percentage variation of the energy is twice that of the speed.

EXAMPLE 3

A given mass of gas is expanding according to the law $\dfrac{pv}{T} = C$ (constant). Find, correct to the first order of small quantities, the increase in pressure due to increases δv in v and δT in T.

$p = \dfrac{CT}{v}$, and if p_1 be the new pressure, $p_1 = \dfrac{C(T + \delta T)}{v + \delta v} = \dfrac{CT}{v}\cdot\dfrac{1 + \dfrac{\delta T}{T}}{1 + \dfrac{\delta v}{v}}$.

i.e. $\qquad p_1 = p\left(1 + \dfrac{\delta T}{T}\right)\left(1 + \dfrac{\delta v}{v}\right)^{-1}$

$$= p\left(1 + \frac{\delta T}{T}\right)\left(1 - \frac{\delta v}{v}\right) \text{ nearly}$$

$$= p\left(1 + \frac{\delta T}{T} - \frac{\delta v}{v}\right) \text{ nearly}$$

$\therefore$ increase of pressure $= p\left(\dfrac{\delta T}{T} - \dfrac{\delta v}{v}\right)$

EXAMPLE 4

To find exact and approximate expressions for the distance from crankshaft to cross head in the simple steam-engine

Let OP (Fig 2) be the crank, r ft long, and CP the connecting-rod, l ft long and let $x = OC$ be the distance in feet from the crankshaft to the crosshead Suppose that the crank starts in the position OA where $A\hat{O}A' = \alpha$, and moves for t seconds with uniform angular velocity ω radians per second After t seconds

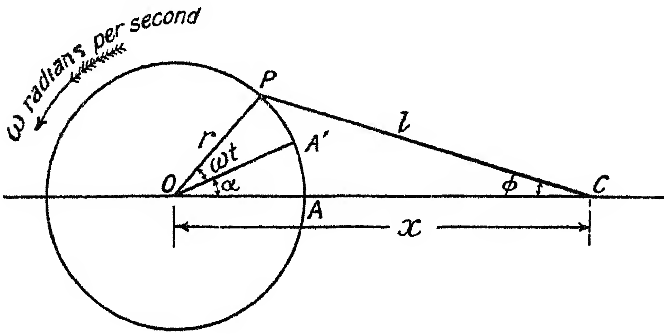


FIG 2

it has moved to OP and $A\hat{O}P = \omega t + \alpha$ radians Let $P\hat{C}O = \phi$. Then, by the sine rule

$$\frac{l}{\sin(\omega t + \alpha)} = \frac{r}{\sin\phi} \qquad (1)$$

Now x = sum of projection of OP and PC on OC

$$= r\cos(\omega t + \alpha) + l\cos\phi$$

$$= r\cos(\omega t + \alpha) + l\sqrt{1 - \sin^2\phi}$$

$$= r\cos(\omega t + \alpha) + l\left[1 - \frac{r^2}{l^2}\sin^2(\omega t + \alpha)\right]^{\frac{1}{2}} \text{ from (1),}$$

which gives x as a function of t

Approximate Value for x By (I 8) $(1 + x)^n = 1 + nx$ approx , and

$$\left[1 - \frac{r^2}{l^2}\sin^2(\omega t + \alpha)\right]^{\frac{1}{2}} = 1 - \frac{r^2}{2l^2}\sin^2(\omega t + \alpha) \text{ approx} \qquad (2)$$

$$= 1 - \frac{r^2}{2l^2}\,\frac{1 - \cos(2\omega t + 2\alpha)}{2}$$

$$= 1 - \frac{r^2}{4l^2} + \frac{r^2}{4l^2}\cos(2\omega t + 2\alpha)$$

The third term in the expansion (2) would be

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{\underline{|2}}\frac{r^4}{l^4}\sin^4(\omega t+\alpha) = -\frac{1}{8}\frac{r^4}{l^4}\sin^4(\omega t+\alpha)$$

which is numerically less than $\frac{1}{648}$ if $l > 3r$, as is usually the case. Hence, neglecting this in comparison with unity

$$x = r\cos(\omega t+\alpha) + l\left(1-\frac{r^2}{4l^2}\right) + \frac{r^2}{4l}\cos(2\omega t+2\alpha) \text{ approx}$$

EXAMPLE 5

Prove that $\underset{x\to a}{Lt}\dfrac{x^n-a^n}{x-a} = na^{n-1}$ for all values of n

Let $x = a + h$, so that $h\to 0$ as $x\to a$

Then $$\frac{x^n-a^n}{x-a} = \frac{(a+h)^n-a^n}{h} = a^n\frac{\left(1+\frac{h}{a}\right)^n-1}{h}$$

Now $\frac{h}{a} < 1$, so that we can expand $\left(1+\frac{h}{a}\right)^n$ in a binomial series

Hence,

$$\underset{x\to a}{Lt}\frac{x^n-a^n}{x-a} = \underset{h\to 0}{Lt}\, a^n\left[\frac{\frac{nh}{a}+\frac{n(n-1)}{\underline{|2}}\frac{h^2}{a^2}+\cdots}{h}\right]$$

$$= \underset{h\to 0}{Lt}\, a^n\left(\frac{n}{a}+\text{terms involving positive powers of } h\right)$$

$$\underset{x\to a}{Lt}\frac{x^n-a^n}{x-a} = a^n\frac{n}{a} = na^{n-1} \qquad (1.11)$$

9 **Exponentials and Exponential Series.** The quantity a^x, in which a is any constant positive number and x is a variable which may have positive or negative values, is known as an *exponential.* Fig. 3 shows the graph of $y = a^x$.

Consider the two chords AR and PQ whose projections on OX are $\overline{OL} = \overline{MN} = h$ and let $\overline{OM} = x$ Then we have

$$\text{Gradient of chord } PQ = \frac{\overline{QN}-\overline{PM}}{\overline{MN}} = \frac{a^{x+h}-a^x}{h} = \frac{a^x(a^h-1)}{h}$$

Also,

$$\text{Gradient of chord } AR = \frac{\overline{RL}-\overline{AO}}{h} = \frac{a^h-a^0}{h} = \frac{a^h-1}{h}$$

Hence,

$$\frac{\text{Gradient of chord } PQ}{\text{Gradient of chord } AR} = a^x \qquad \text{(I.12)}$$

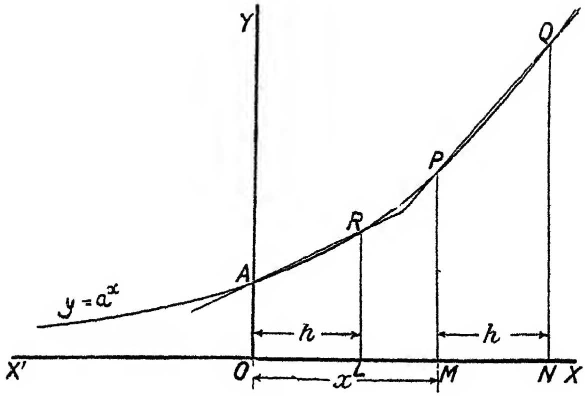


FIG. 3

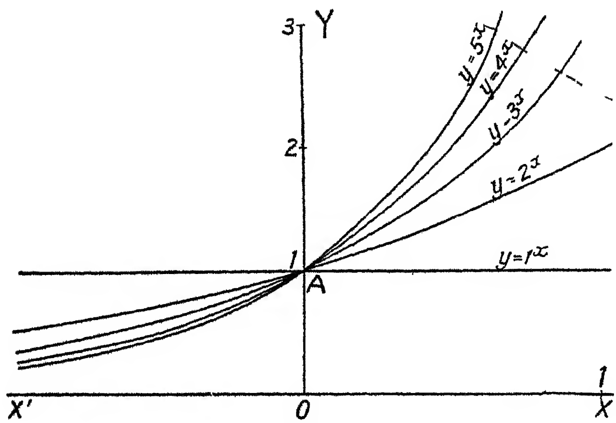


FIG. 4

As h approaches the limit zero the chord PQ approaches its limiting position, which is that of the tangent at P, and similarly the chord AR approaches as its limit the tangent at A. Hence, in the limit we have

$$\frac{\text{Gradient of tangent at } P}{\text{Gradient of tangent at } A} = a^x \qquad . \qquad . \qquad \text{(I.13)}$$

Fig. 4 shows the graphs of 1^x, 2^x, 3^x, 4^x, and 5^x, i.e. of a^x for several values of a. By inspection of the graphs it will be seen that they all pass through the point $x = 0$, $y = 1$. This follows from the fact that, if a is not zero, $a^0 = 1$ for all values of a. It also appears that the gradient of the tangent at A to the graph increases with a, being zero when $a = 1$ and large when a is large. The gradient of the tangent at A to $y = 2^x$ is less than 1, and that of the tangent at A to $y = 3^x$ is greater than 1. Consequently there must be some value of a between $a = 2$ and $a = 3$ for which the gradient of the tangent at A to the graph $y = a^x$ is unity. Let e be this value of a. We then have for the equation to the graph, $y = e^x$, and since the gradient of the tangent at A is unity, we have also from the relation proved above

$$\text{Gradient of tangent at } P = e^x \quad . \quad \text{(I.14)}$$

Thus the gradient of the tangent at any point of the graph of $y = e^x$ is numerically equal to the ordinate at that point. We shall show below that the value of e is 2·71828 . . ., a non-terminating, non-recurring decimal, and we shall also prove at a later stage the relation

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \frac{c^4x^4}{\underline{|4}} + \frac{c^5x^5}{\underline{|5}} + \ldots \quad . \quad \text{(I.15)}$$

Putting $c = 1$ in this we have

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \ldots \quad . \quad . \quad \text{(I.16)}$$

Now putting $x = 1$, we have the series

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \ldots \quad . \quad \text{(I.17)}$$

The relation (I.15) is obtained by means of Maclaurin's Theorem in Art. 56. The reader should notice that we have really defined e by the relation

$$\underset{h \to 0}{Lt.} \frac{e^h - 1}{h} = 1 \quad . \quad . \quad \text{(I.18)}$$

It can be shown that e is given also by

$$e = \underset{n \to \infty}{Lt.} \left(1 + \frac{1}{n}\right)^n \quad . \quad . \quad . \quad \text{(I.19)}$$

and that e^x is given by either of the limits

$$e^x = \underset{n \to \infty}{Lt.} \left(1 + \frac{x}{n}\right)^n \quad . \quad . \quad . \quad (I.20)$$

or

$$e^x = \underset{n \to \infty}{Lt.} \left(1 + \frac{1}{n}\right)^{nx} \quad . \quad . \quad . \quad (I.21)$$

We do not attempt the proofs of these three relations, but suggest a method of proof in Example 57, page 40. This proof is, however, not rigorous, as it depends on the assumption that the limit of the product of an infinite number of factors is always the product of their limits. For a rigorous proof the reader is referred to any modern textbook of algebra.

If we write $c = \log_e a$, then $a = e^c$ and (I.15) becomes

$$a^x = 1 + x(\log_e a) + \frac{x^2}{\lfloor 2}(\log_e a)^2 + \frac{x^3}{\lfloor 3}(\log_e a)^3 + \quad . \quad (I.22)$$

The series (I.15) and (I.16) are known as *exponential series*, the variable x being in the exponent or index. We have shown in Art. 6 that the series (I.16) is convergent for all values of x. Hence, since c and $\log_e a$ are finite quantities the series (I.15) and (I.22) are also convergent. The series (I.17) may be used to calculate the value of e to any required degree of accuracy. Since the $(n + 1)$th term of the series is obtained from the nth term by dividing by n, the calculation is conveniently arranged as follows—

Calculate e correct to six significant figures.

1st term	= 1			= 1·0
2nd „	= 1st term	÷	1	= 1·0
3rd „	= 2nd „	÷	2	= 0·5
4th „	= 3rd „	÷	3	= 0·166667
5th „	= 4th „	÷	4	= 0·041667
6th „	= 5th „	÷	5	= 0·008333
7th „	= 6th „	÷	6	= 0·001389
8th „	= 7th „	÷	7	= 0·000198
9th „	= 8th „	÷	8	= 0·000025
10th „	= 9th „	÷	9	= 0·000003
11th „	= 10th „	÷	10	= 0·000000
	∴ e = sum of terms			= 2·71828

correct to six significant figures.

10. Some Limiting Values.

(1) $$\underset{h\to 0}{Lt.}\frac{a^h-1}{h}=\log_e a\ (a>0) \qquad . \qquad (I.23)$$

In Art. 9 we defined e so that

$$\underset{h\to 0}{Lt.}\frac{e^h-1}{h}=1 \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Writing $a=e^c$ so that $c=\log_e a$, we have

$$\begin{aligned}\underset{h\to 0}{Lt.}\frac{a^h-1}{h}&=\underset{h\to 0}{Lt.}\frac{e^{ch}-1}{h}\\&=\underset{ch\to 0}{Lt.}\frac{e^{ch}-1}{ch}\times c\\&=c\underset{ch\to 0}{Lt.}\frac{e^{ch}-1}{ch}\\&=c\times 1 \text{ by (i) above}\end{aligned}$$

Hence, $$\underset{h\to 0}{Lt.}\frac{a^h-1}{h}=\log_e a$$

(2) $$\underset{x\to 0}{Lt.}\,x\log_a x=0\ (a>0) \qquad . \quad . \qquad (I.24)$$

Since x tends ultimately to 0, we can assume $x<1$ and let $\log_a x=-k$, where k is positive, so that $x=a^{-k}$

$$\therefore\ x\log_a x=-ka^{-k}=-\frac{k}{1+k\log_e a+\frac{k^2}{\lfloor 2}(\log_e a)^2+\ldots}$$

$$=-\frac{1}{\frac{1}{k}+\log_e a+\frac{k}{\lfloor 2}(\log_e a)^2+\ldots}$$

Now, as $x\to 0$, $k\to\infty$, so that the limit of

$$\frac{1}{k}+\log_e a+\frac{k}{\lfloor 2}(\log_e a)^2+\ldots \text{ is } \infty$$

and hence

$$\underset{x\to 0}{Lt.}\,x\log_a x=0$$

It follows that $\underset{x\to 0}{Lt.}\,x\log_e x=0 \quad . \quad . \quad . \quad . \qquad (I.25)$

(3) $$\underset{x\to\infty}{Lt.}\ \frac{x}{\log_a x} = \infty \quad (a > 0) \quad . \quad . \qquad (I.26)$$

Let $x = \frac{1}{y}$, where y will be positive and will tend to 0 as x tends to ∞. Hence,

$$\frac{x}{\log_a x} = -\frac{1}{y \log_a y}$$

and $$\underset{y\to 0}{Lt.}\ y \log_a y = 0 \text{ by (I.24)}$$

$\therefore$ $$\underset{x\to\infty}{Lt.}\ \frac{x}{\log_a x} = \infty$$

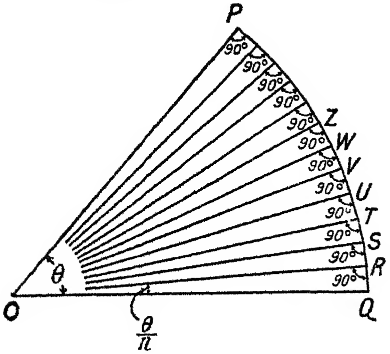


FIG. 5

(4) $$\underset{n\to\infty}{Lt.}\ na^n = 0 \ (a \text{ positive and } < 1) \quad . \quad . \qquad (I.27)$$

Let $a^n = z$; then, since $a < 1$, $z \to 0$ as $n \to \infty$.

$\therefore$ $$\underset{n\to\infty}{Lt.}\ na^n = \underset{z\to 0}{Lt.}\ z \log_a z = 0, \text{ by (I.24)}$$

(5) $$\underset{x\to 0}{Lt.}\ x^x = 1 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (I.28)$$

Let $x = e^z$, so that $z = \log_e x$.

$\therefore$ $$x^x = e^{zx} = e^{x \log_e x}$$

Now $\underset{x\to 0}{Lt.}\ x \log_e x = 0$, by (I.25), so that $\underset{x\to 0}{Lt.}\ x^x = e^0 = 1$.

(6) $$\underset{n\to\infty}{Lt.}\ \left(\cos\frac{\theta}{n}\right)^n = 1 \quad . \quad . \quad . \qquad (I.29)$$

Let $P\hat{O}Q = \theta$ (Fig. 5), and let the lines OR, OS, OT, OU, OV, OW, . . . divide POQ into a number n of equal parts each of which

is of magnitude θ/n. Let QR, RS, ST, ... be drawn perpendicular to OR, OS, OT, ... respectively. Then

$$OR = OQ \cos \frac{\theta}{n}$$

$$OS = OR \cos \frac{\theta}{n} = OQ \cos^2 \frac{\theta}{n}$$

$$OT = OS \cos \frac{\theta}{n} = OQ \cos^3 \frac{\theta}{n}$$

. . . . .

etc.

$$OP = OQ \cos^n \frac{\theta}{n}$$

Consider the triangle OVU. The angle $OUV = 90^\circ - \frac{\theta}{n}$, and as n tends to the limit ∞, $O\hat{U}V$ tends to the limit 90°. In the limit $QRSTUV \ldots P$ becomes a curve, the limiting position of UV is that of the tangent at U, and the angle between OU and this tangent is 90°. In the same way we see that the tangent at any other point of the curve is perpendicular to the line joining that point to O, and therefore the curve is an arc of a circle with centre O. Hence, in the limit $OQ = OP$, and since

$$\underset{n\to\infty}{Lt.}\left(\cos\frac{\theta}{n}\right)^n = \underset{n\to\infty}{Lt.}\frac{OP}{OQ}, \text{ we have } \underset{n\to\infty}{Lt.}\left(\cos\frac{\theta}{n}\right)^n = 1$$

11. **Hyperbolic Functions.** Corresponding to each trigonometrical function there is a hyperbolic function. Thus we have the functions defined by the following—

$$\left.\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \operatorname{cosech} x &= \frac{1}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x} \end{aligned}\right\} \tag{I.30}$$

These quantities are known as the hyperbolic sine of x, the hyperbolic cosine of x, etc.

Fig. 6 shows the graphs of $y = e^x$, $y = e^{-x}$, $y = \cosh x$, and $y = \sinh x$, and Fig. 7 shows the graph of $y = \tanh x$. From the

expressions for $\sinh x$ and $\cosh x$ we obtain by addition and subtraction respectively

$$\cosh x + \sinh x = e^{x}$$

and $$\cosh x - \sinh x = e^{-x}$$

and by multiplication of these we have

$$\cosh^2 x - \sinh^2 x = 1 \qquad (I\ 31)$$

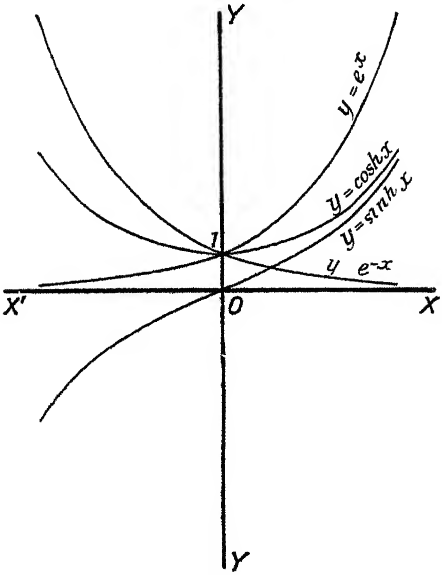


FIG. 6

Dividing through this in turn by $\cosh^2 x$ and $\sinh^2 x$, and noticing that $\dfrac{\sinh x}{\cosh x} = \tanh x$, we have

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad (I\ 32)$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x \qquad (I\ 33)$$

We leave the reader to establish the following formulae—

$$\left.\begin{aligned} \cosh (x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \\ \sinh (x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \\ \tanh 2x &= \frac{2 \tanh x}{1 + \tanh^2 x} \end{aligned}\right\} \qquad (I\ 34)$$

Infinite series for sinh x and cosh x are found from those for e^x and e^{-x}, thus—

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{x^5}{\underline{|5}} + \ldots$$

Hence, $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} + \dfrac{x^7}{\underline{|7}} + \ldots$ (I.35)

and $\cosh x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} + \dfrac{x^6}{\underline{|6}} + \ldots$ (I.36)

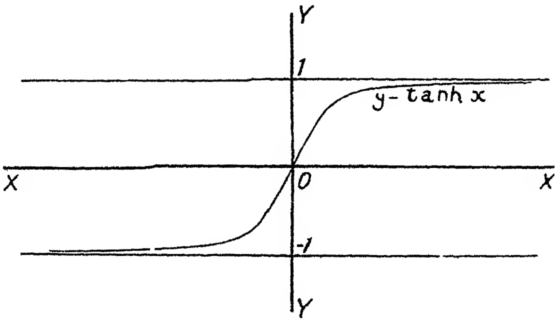


FIG 7

12. **Inverse Hyperbolic Functions.** For definitions of these functions the reader should refer to Chapter II.

If $y = \sinh^{-1} x$, then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$\therefore \quad e^{2y} - 2xe^y = 1$

$\therefore \quad e^{2y} - 2xe^y + x^2 = x^2 + 1,$

whence $\quad e^y - x = \pm \sqrt{x^2 + 1}$

and $\quad e^y = x + \sqrt{x^2 + 1}$ (since e^y is positive)

$\therefore \quad y = \sinh^{-1} x = \log_e [x + \sqrt{x^2 + 1}]$ (I.37)

Similarly, we obtain

$$\cosh^{-1} x = \log_e [x \pm \sqrt{x^2 - 1}] \qquad (I.38)$$

$$\tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1+x}{1-x} \text{ if } x^2 < 1 \qquad (I.39)$$

$$\coth^{-1} x = \tfrac{1}{2} \log_e \frac{x+1}{x-1} \text{ if } x^2 > 1 \qquad (I.40)$$

The reader will see by reference to Fig. 6 why there is a double sign on the right-hand side of (I.38); the graph shows that there are two values of x for a given value of $\cosh x$. If the axes of x and y in Fig. 6 are interchanged the graphs then become the graphs of $y = \log_e x$, $y = -\log_e x$, $y = \cosh^{-1} x$, and $y = \sinh^{-1} x$.

13. **Complex Numbers.** If a and b are real positive or negative numbers and $i = \sqrt{-1}$, ai or bi is called an *imaginary* number, and $a + bi$ is a *complex* number. We shall see that the number $i = \sqrt{-1}$ can be interpreted and that it is not really imaginary, though we shall retain the term as a description of quantities which are the product of i and a positive or negative real number.

Two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are also equal. For if a, b, c, d are all real, and

$$a + bi = c + di$$

then

$$a - c = (d - b)i$$

This relation can only be true if $a = c$ and $b = d$, for otherwise we should have a real quantity equal to an imaginary quantity, which is impossible. The sum or difference of two complex quantities is also a complex quantity for

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

The product of two complex quantities is a complex quantity for $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

On dividing this last relation through by $c + di$ we see that when one complex quantity is divided by another, the quotient is a complex quantity. This may be shown otherwise, for

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$c - di$ is known as the *conjugate* of $c + di$ and vice versa, and the method used here of multiplying numerator and denominator by the expression conjugate to that in the denominator, is generally used for simplifying the division of one complex quantity by another.

EXAMPLE

Simplify $\dfrac{3 + 2i}{(4 - 5i)(2 + i)}$.

Simplifying the denominator first, we have

$$\frac{3 + 2i}{(4 - 5i)(2 + i)} = \frac{3 + 2i}{13 - 6i} = \frac{(3 + 2i)(13 + 6i)}{(13 - 6i)(13 + 6i)} = \frac{27 + 44i}{13^2 + 6^2} = 0.1317 + 0.2146i$$

14. **Modulus and Argument.** We can express a complex number in terms of trigonometrical functions, thus—

Let $$a = r \cos \theta \text{ and } b = r \sin \theta \qquad (I.41)$$

Then $$a + bi = r(\cos \theta + i \sin \theta) \qquad (I.42)$$

From (I.41) we find on squaring and adding the two expressions that $r = \sqrt{a^2 + b^2}$, and on dividing the second of the expressions by the first we have $\tan \theta = \dfrac{b}{a}$ or $\theta = \tan^{-1} \dfrac{b}{a}$. r is called the *modulus* of the complex number, and θ its *argument* or *amplitude*. The modulus is sometimes denoted by $|a + bi|$.

EXAMPLE

Express $(2 + 3i)(3 - 2i)$ as a complex number, and find its modulus and argument.

$$(2 + 3i)(3 - 2i) = 12 + 5i = r(\cos \theta + i \sin \theta)$$

By the above, $r = \sqrt{12^2 + 5^2} = 13$, and $\theta = \tan^{-1} \dfrac{5}{12} = 22° \, 37'$.

The modulus is 13 and the argument $22° \, 37'$.

The modulus of the product of two complex numbers is the product of their moduli, and the argument of the product is the sum of their arguments. For

$$\begin{aligned} \{r_1(\cos \theta_1 + i \sin \theta_1)\}\{r_2(\cos \theta_2 + i \sin \theta_2)\} \\ = r_1 r_2\{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\} \\ = r_1 r_2\{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\} \end{aligned} \qquad (I.43)$$

a complex quantity whose modulus is $r_1 r_2$ and whose argument is $\theta_1 + \theta_2$.

Dividing each side of (I.43) by $r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$r_1(\cos + i\sin\theta_1) = \frac{r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

which shows that the result of dividing one complex quantity by another is a complex quantity whose modulus is the quotient of the moduli, and whose argument is the difference of the arguments.

It is often convenient to represent the complex number $a + bi$, or $r(\cos\theta + i\sin\theta)$, by the symbol $r[\theta]$, the relations between r, θ, and a, b being those given above.

15. **De Moivre's Theorem.** This states that *if n is any real number, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.*

We shall consider three cases.

(1) Let n be a positive integer.

By actual multiplication we saw above that

$$(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)$$
$$= \cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)$$

Similarly

$$(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)(\cos\alpha_3 + i\sin\alpha_3)$$
$$= \cos(\alpha_1+\alpha_2+\alpha_3) + i\sin(\alpha_1+\alpha_2+\alpha_3)$$

Continuing this process we find that

$$(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)(\cos\alpha_3 + i\sin\alpha_3)$$
$$\ldots(\cos\alpha_n + i\sin\alpha_n)$$
$$= \cos(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n)$$
$$+ i\sin(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n)$$

If now we write $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \ldots = \alpha_n = \theta$ we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \quad . \quad (I.44)$$

(2) Let n be a negative integer, and put $n = -k$ where k is positive. Then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-k} = \frac{1}{(\cos\theta + i\sin\theta)^k}$$
$$= \frac{1}{\cos k\theta + i\sin k\theta}$$

and multiplying numerator and denominator by the complex quantity $\cos k\theta - i \sin k\theta$ (see Art. 13)

$$(\cos \theta + i \sin \theta)^n = \frac{\cos k\theta - i \sin k\theta}{(\cos k\theta + i \sin k\theta)(\cos k\theta - i \sin k\theta)}$$

$$= \frac{\cos k\theta - i \sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} = \cos k\theta - i \sin k\theta$$

$$= \cos(-n\theta) - i \sin(-n\theta)$$

$$\therefore \ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

since $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$.

(3) Let $n = \frac{p}{q}$, where p is any integer positive or negative and q is a positive integer. Then by (1)

$$\left(\cos \frac{\theta}{q} + i \sin \frac{\theta}{q}\right)^q = \cos \theta + i \sin \theta$$

Taking the qth root of each side

$$\cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \text{ is one of the values of } (\cos \theta + i \sin \theta)^{\frac{p}{q}}$$

Hence, $$\left(\cos \frac{\theta}{q} + i \sin \frac{\theta}{q}\right)^p = (\cos \theta + i \sin \theta)^{\frac{1}{q}}$$

But by (1) the left-hand side is equal to $\left(\cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}\right)$, and therefore

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

The relation (I.44) is true for all real values of n, though we have only proved it for values of n which are the ratios of two positive or negative integers.

16. **The Geometrical Representation of Complex Numbers.** Complex numbers may be represented graphically by a diagram known as the *Argand diagram* (Fig. 8). $X'OX$ and YOY' are two perpendicular axes. A real number a is represented by a length a measured along or parallel to OX, just as in graphical algebra, and an imaginary number bi is represented by a length b along or parallel to OY. Thus, in Fig. 9, the lengths OA_1, OA_2, and OA_3 represent the real numbers -3, -2, and $+5$ respectively, while the lengths A_1P_1,

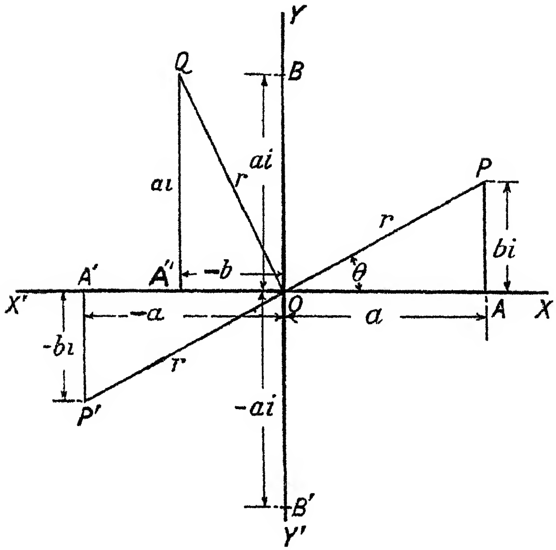

Y
Q
B
ai
ai
r
P
r
bi
A′
A″
−b
θ
X′
O
a
A
X
−a
−bi
r
P′
−ai
B′
Y′

FIG 8

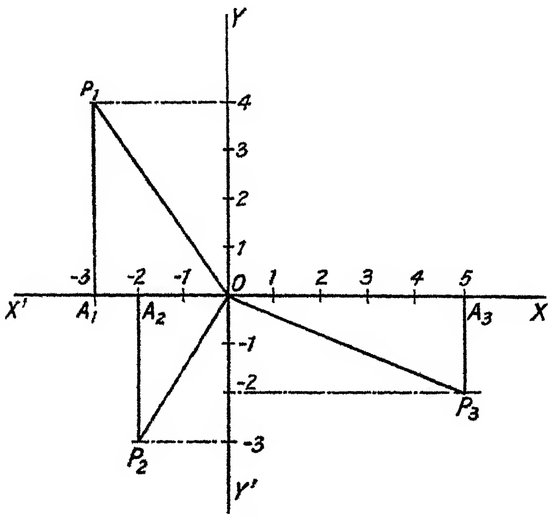

Y
P1
4
3
2
1
-3
-2
-1
0
1
2
3
4
5
X′
A1
A2
A3
X
-1
-2
P3
-3
P2
Y′

FIG. 9

A_2P_2, and A_3P_3 represent the imaginary numbers $4i, -3i$, and $-2i$ respectively. We may look upon the lines OP_1, OP_2, and OP_3 (or, if we choose, upon the points P_1, P_2, and P_3) as representing the complex numbers $-3+4i, -2-3i$, and $5-2i$.

In Fig. 8 the lines OA, OB, OA', OB', OP, and OP' represent respectively the quantities a, ai, $-a$ or ai^2 (since $i^2 = -1$), $-ai$ or ai^3, $a+bi$, and $-a-bi$. The line OQ is drawn perpendicular to OP and $\overline{OP} = \overline{OQ} = r$. The angle $XOP = \theta$. The product of a and i is ai, and the effect in the diagram of multiplying a by i is to turn the line OA, which represents a, into the position OB, which represents ai. Similarly the effect of multiplying ai by i is to turn the line OB into the position OA' which represents ai^2 or $-a$. The effect of multiplying $a+bi$ by i, $(a+bi)\times i = -b+ai$, is to turn the line OP into the position OQ, for OQ is seen to represent $-b+ai$.

The reader should test the effects of multiplying by i in turn the numbers represented by OA', OB', OQ and OP', and he will see that in each case the line representing the number will be found to have turned through a right angle in the anti-clockwise direction. Multiplying a complex number by i corresponds to the operation of rotating the line representing the complex number through a right angle in the anti-clockwise direction.

The relation (I.43), which may be written $r_1[\theta_1]\times r_2[\theta_2] = r_1r_2[\theta_1+\theta_2]$, shows that the effect of operating on $r_2[\theta_2]$ with the quantity $r_1[\theta_1]$ is to convert the line representing $r_2[\theta_2]$ into the line representing $r_1r_2[\theta_1+\theta_2]$ by rotating the former line through an angle θ_1, and multiplying its length by r_1. This result is shown in Fig. 10, as also is the result of the division

$$\frac{r_2\,[\theta_2]}{r_1\,[\theta_1]} = \frac{r_2}{r_1}\,[\theta_2-\theta_1]$$

the product being represented by OR and the quotient by OR'.

In Art. 13 we saw that the sum of two given complex numbers was a complex number whose real and imaginary parts were respectively the sum of the real and of the imaginary parts of the given complex numbers. This result is shown in Fig. 11. The figure $OACB$ is a parallelogram; OA and OB represent the quantities a_1+ia_2 and b_1+ib_2 respectively the sum of which is $a_1+b_1+i(a_2+b_2)$. On examination of the figure we see that the projections of OC on OX and OY respectively are a_1+b_1 and a_2+b_2, so that OC represents $a_1+b_1+i(a_2+b_2)$, the sum of the two complex quantities represented by OA and OB.

Similarly, the line OD equal and parallel to AB represents the

difference between the quantities represented by OB and OA, i.e. $(b_1 + ib_2) - (a_1 + ia_2)$, or $(b_1 - a_1) + i(b_2 - a_2)$. The engineering student will notice that the graphical method of adding complex

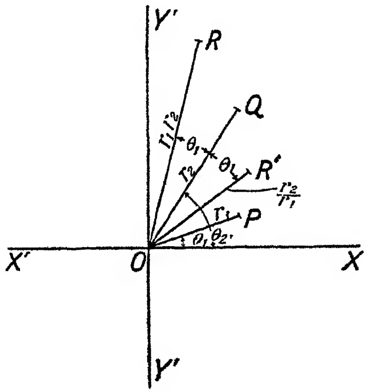


FIG. 10

numbers is the same as the method of finding the resultant $\overrightarrow{OC}$ of two coplanar vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, whilst the operation of subtracting

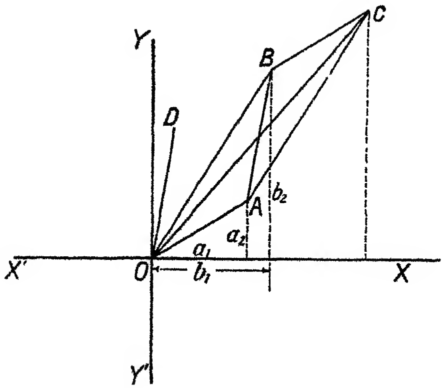


FIG. 11

complex numbers is the same as that of finding the difference $\overrightarrow{AB}$ of the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ for $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

EXAMPLE

Represent on an Argand diagram the number $3 + 4i$, its square, and its square roots. Apply De Moivre's theorem to express $\cos 5\theta$ as a function of $\cos \theta$. (U.L.)

OP_1 (Fig. 12) represents $3 + 4i$. $(3 + 4i)^2 = 9 + 24i - 16 = -7 + 24i$, which is represented by OP_2 in the diagram.

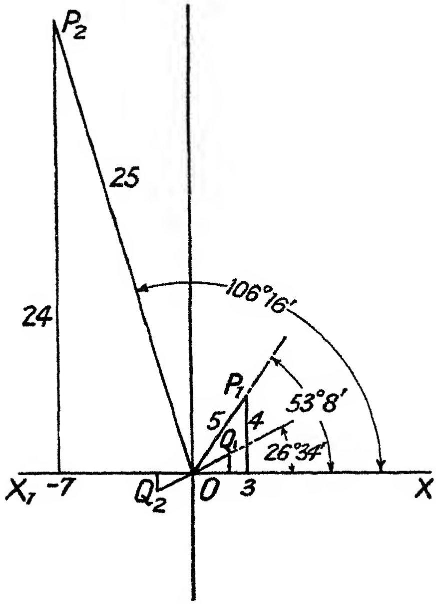


FIG. 12

Let $3 + 4i = r(\cos\theta + i\sin\theta)$, so that $r = \sqrt{3^2 + 4^2} = 5$, and $\theta = \tan^{-1}(\frac{4}{3}) = 53^\circ\ 8'$.

Then $(3 + 4i)^{\frac{1}{2}} = r^{\frac{1}{2}}\left(\cos\dfrac{\theta + k \times 360^\circ}{2} + i\sin\dfrac{\theta + k \times 360^\circ}{2}\right)$ where $k = 0, 1$. [See Art. 18]

$= \sqrt{5}\,(\cos 26^\circ\ 34' + i\sin 26^\circ\ 34')$, or $\sqrt{5}\,(\cos 206^\circ\ 34' + i\sin 206^\circ\ 34')$.

These two square roots are represented by OQ_1 and OQ_2 respectively in the diagram, where $OQ_1 = OQ_2 = \sqrt{5}$, $X\hat{O}Q_1 = 26^\circ\ 34'$, $X\hat{O}Q_2$ (reflex) $= 206^\circ\ 34'$.

Let $\qquad x = \cos\theta + i\sin\theta. \quad \therefore \dfrac{1}{x} = \cos\theta - i\sin\theta.$

Also $\qquad x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\qquad \dfrac{1}{x^n} = \cos n\theta - i\sin n\theta. \quad \therefore x^n + \dfrac{1}{x^n} = 2\cos n\theta.$

Put $n = 5$: $\quad 2\cos 5\theta = x^5 + \dfrac{1}{x^5}$

$$= (\cos\theta + i\sin\theta)^5 + (\cos\theta - i\sin\theta)^5$$

$$= 2[\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta]$$

$$= 2[\cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)]$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

17. **Exponential Values of Sin θ and Cos θ. Sine and Cosine Series.** For all real values of z,

$$e^z = \underset{n\to\infty}{Lt.}\left(1+\frac{z}{n}\right)^n = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \ldots \text{ (See Art. 9)}$$

Let us now extend the definition of e^z so that for *all* values of z, real or complex, e^z denotes the series

$$1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \ldots \text{ or } \underset{n\to\infty}{Lt.}\left(1+\frac{z}{n}\right)^n$$

Let $z = i\theta$; then $e^{i\theta} = \underset{n\to\infty}{Lt.}\left(1+\dfrac{i\theta}{n}\right)^n = \underset{n\to\infty}{Lt.}\left(1 + i\tan\dfrac{\theta}{n}\right)^n$

where we have replaced $\dfrac{\theta}{n}$ by $\tan\dfrac{\theta}{n}$ since $\underset{n\to\infty}{Lt.}\left(\dfrac{\tan\frac{\theta}{n}}{\frac{\theta}{n}}\right) = 1$

$$e^{i\theta} = \underset{n\to\infty}{Lt.}\left(\frac{\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}}{\cos\frac{\theta}{n}}\right)^n = \underset{n\to\infty}{Lt.}\frac{\cos\theta + i\sin\theta}{\left(\cos\frac{\theta}{n}\right)^n}$$

i.e.
$$e^{i\theta} = \cos\theta + i\sin\theta \quad . \quad . \quad (\text{I.45})$$

for
$$\underset{n\to\infty}{Lt.}\left(\cos\frac{\theta}{n}\right)^n = 1 \ldots \{\text{by (I.29)}\}$$

Since $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(-\theta)$, we have on substituting $(-\theta)$ for θ in the above relation

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$\therefore \quad e^{-i\theta} = \cos\theta - i\sin\theta \quad . \quad . \quad . \quad (\text{I.46})$$

A complex number can be expressed in any of the forms $a + bi$, $r(\cos\theta + i\sin\theta)$, or $re^{i\theta}$. This last form gives in a striking manner the results of Art. 14. For, if $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ be two complex numbers,

then $r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$ and $\dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}\cdot e^{i(\theta_1-\theta_2)}$

By addition from (I.45) and (I.46),

$$\left.\begin{aligned}
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh i\theta \\
&= 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \dots \\
\text{By subtraction,}\quad \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{i}\sinh i\theta \\
&= \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \dots
\end{aligned}\right\} \quad (I.47)$$

Let $\theta = ix$; then

$$\left.\begin{aligned}
\cos ix &= \cosh i^2x = \cosh(-x) = \cosh x \\
\text{and}\quad \sin ix &= \frac{1}{i}\sinh(-x) = \frac{(-1)}{i}\cdot\sinh x = i\sinh x \\
\text{By division,}\quad & \\
\tan ix &= \frac{i\sinh x}{\cosh x} = i\tanh x
\end{aligned}\right\} \quad (I.48)$$

EXAMPLE 1

If u, v, and θ are all real, and if $u + iv = \dfrac{1}{i}\log_e\dfrac{1+ie^{i\theta}}{1-ie^{i\theta}}$, show that $v = \frac{1}{2}\log_e(\sec\theta+\tan\theta)^2$ and $u = 2n\pi + \frac{1}{2}\pi$, n being an integer. (U.L.)

$$\frac{1+ie^{i\theta}}{1-ie^{i\theta}} = \frac{(1+ie^{i\theta})(1+ie^{-i\theta})}{(1-ie^{i\theta})(1+ie^{-i\theta})} = \frac{1+i(e^{i\theta}+e^{-i\theta})-1}{1-i(e^{i\theta}-e^{-i\theta})+1}$$ (Compare Art. 19, Ex. 1).

$$= \frac{2i\cos\theta}{2-i(2i\sin\theta)} = \frac{i\cos\theta}{1+\sin\theta} = \frac{i}{\sec\theta+\tan\theta}$$

$$\therefore\quad \frac{1}{i}\log_e\frac{1+ie^{i\theta}}{1-ie^{i\theta}} = \frac{1}{i}\left[\log_e i - \log_e(\sec\theta+\tan\theta)\right]$$

But $$\log_e i = \log_e e^{i\left(2n\pi+\frac{\pi}{2}\right)} = i\left(2n\pi+\frac{\pi}{2}\right);$$

and $$-\frac{1}{i}\log_e(\sec\theta+\tan\theta) = i\log_e(\sec\theta+\tan\theta)$$

$$u + iv = 2n\pi + \frac{\pi}{2} + i \log_e (\sec\theta + \tan\theta).$$

Equating real and imaginary parts

$$u = 2n\pi + \frac{\pi}{2}$$

and $$v = \log_e (\sec\theta + \tan\theta) = \tfrac{1}{2} \log_e (\sec\theta + \tan\theta)^2$$

EXAMPLE 2

If $x + iy = \cosh (u + iv)$, show that

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1; \quad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1. \qquad \text{(U.L.)}$$

$$x + iy = \cosh u . \cosh iv + \sinh u . \sinh iv = \cosh u . \cos v + \sinh u(i \sin v).$$

Equating real and imaginary parts, $x = \cosh u . \cos v$; $y = \sinh u . \sin v$.

Now $\quad \cos^2 v + \sin^2 v = 1. \quad \therefore \quad \dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1.$

Also $\quad \cosh^2 u - \sinh^2 u = 1. \quad \therefore \quad \dfrac{x^2}{\cos^2 v} - \dfrac{y^2}{\sin^2 v} = 1.$

18. **The nth Roots of any Real or Complex Number.** Let $a + bi = r(\cos \theta + i \sin \theta)$ be any complex number. Since $\cos \theta + i \sin \theta$ remains unaltered when we add $2k\pi$ to θ, where k is an integer, we may write

$$a + bi = r\{\cos(\theta + 2k\pi) + i \sin (\theta + 2k\pi)\}$$

Hence,

$$(a + bi)^{\frac{1}{n}} = r^{\frac{1}{n}} \left\{ \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right\} \quad . . . \qquad \text{(I.49)}$$

Since k may be any positive or negative integer, it would appear that we might obtain an infinite number of values of $(a + bi)^{\frac{1}{n}}$. This, however, is not so, as every time the angle $\dfrac{\theta + 2k\pi}{n}$ is increased by 2π, i.e. every time k is increased by n, the cosine and sine repeat their values and the function is unaltered. It is sufficient, therefore, to give to k the values $0, 1, 2, 3, . . ., n - 2, n - 1$. In this way we obtain n different values of the nth root of $a + bi = r(\cos \theta + i \sin \theta)$, and we see that in general any complex number has n distinct nth roots, each of which is complex. If b is zero one, or two at the most, of the roots will be real.

EXAMPLE 1

Find the three cube roots of 1.

Here $a + bi = 1 + 0i = \cos 0 + i \sin 0$

and $$\sqrt[3]{1} = (a + bi)^{\frac{1}{3}} = \cos \frac{0 + 2\pi k}{3} + i \sin \frac{0 + 2\pi k}{3}$$

where k has the values 0, 1, 2, in order.

Hence, the roots are $$\cos 0 + i \sin 0 = 1$$

$$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

and $$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

In Fig. 13, OA represents 1, OP represents $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, and OQ represents $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$, the angles AOP, POQ, QOA, being each 120°.

EXAMPLE 2

Find one cube root of $6 \cdot 4 - 4 \cdot 8i$.

Let $6 \cdot 4 - 4 \cdot 8i = r(\cos \theta + i \sin \theta) = r[\theta]$.

Then $r = \sqrt{6 \cdot 4^2 + 4 \cdot 8^2} = 8$ and $\theta = \tan^{-1}\left(-\frac{4 \cdot 8}{6 \cdot 4}\right) = 323° \, 8'$.

The cube roots are given by

$$r^{\frac{1}{3}} \left\{ \cos \frac{\theta + 2k\pi}{3} + i \sin \frac{\theta + 2k\pi}{3} \right\}$$

where $k = 0, 1, 2$, in order. Putting $k = 0$, we obtain the root

$$r^{\frac{1}{3}} \left\{ \cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right\} = 8^{\frac{1}{3}} (\cos 107° \, 43' + i \sin 107° \, 43')$$
$$= 2(-0 \cdot 3043 + 0 \cdot 9526i)$$
$$= -0 \cdot 609 + 1 \cdot 905i$$

Substituting $k = 1$ and $k = 2$ will give the other two roots.

In Fig. 14, OA represents the quantity $6 \cdot 4 - 4 \cdot 8i$, the length of OA is 8 units, and the reflex angle $XOA = 323° \, 8'$. The lines Oq, Or, and Os represent the three cube roots of $6 \cdot 4 - 4 \cdot 8i$; the length of each line is two units, and the angles XOQ, XOR, and XOS are respectively $107° \, 43'$ as above, $\frac{1}{3}(360° + 323° \, 8') = 227° \, 43'$ and $\frac{1}{3}(720° + 323° \, 8') = 347° \, 43'$ respectively.

The reader should notice that $\sqrt[n]{r[\theta]}$ has n values, represented by $r^{\frac{1}{n}} \left[\frac{\theta}{n}\right]$, $r^{\frac{1}{n}} \left[\frac{\theta}{n} + \frac{360°}{n}\right]$, $r^{\frac{1}{n}} \left[\frac{\theta}{n} + \frac{2 \times 360°}{n}\right]$, etc., and that the n lines in an Argand diagram representing the roots, form a system of equally-spaced radiating lines each of length $r^{\frac{1}{n}}$, the angle between each pair of which is $\frac{360}{n}$ degrees.

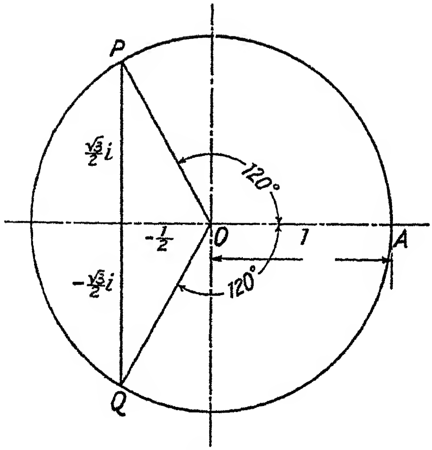

P
$\frac{\sqrt{3}}{2}i$
120°
$-\frac{1}{2}$
O
1
A
$-\frac{\sqrt{3}}{2}i$
120°
Q

FIG. 13

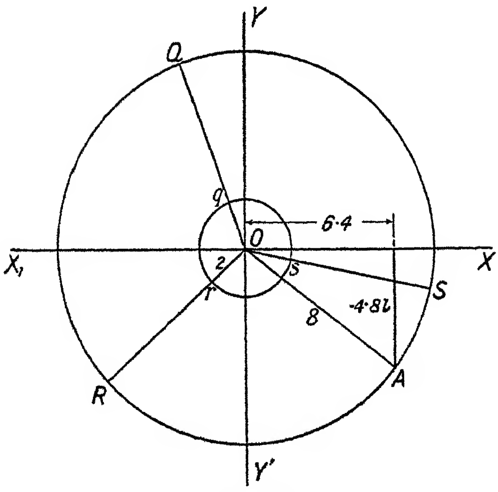

Y
Q
q
6·4
O
X_1
2
s
X
S
r
-4·8i
8
A
R
Y'

FIG. 14

19. Summation of Trigonometrical Series. If we know the sum of a series such as

$$\Sigma_n = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n \quad \text{(I.50)}$$

we can obtain from this the sum of the two series

$$\left.\begin{aligned} C_n = A_0 + A_1x\cos\theta + A_2x^2\cos 2\theta \\ + A_3x^3\cos 3\theta + \ldots + A_nx^n\cos n\theta \end{aligned}\right\} \quad \text{(I.51)}$$

$$\text{and}\quad \left.\begin{aligned} S_n = A_1x\sin\theta + A_2x^2\sin 2\theta + A_3x^3\sin 3\theta \\ + \ldots + A_nx^n\sin n\theta \end{aligned}\right\} \quad \text{(I.52)}$$

For, multiplying (I.52) through by i and adding to (I.51)

$$\begin{aligned} C_n + iS_n = A_0 &+ A_1x(\cos\theta + i\sin\theta) \\ &+ A_2x^2(\cos 2\theta + i\sin 2\theta) \\ &+ A_3x^3(\cos 3\theta + i\sin 3\theta) + \ldots \\ &+ A_nx^n(\cos n\theta + i\sin n\theta) \end{aligned}$$

$$\left.\begin{aligned} \therefore C_n + iS_n = A_0 + A_1xe^{i\theta} + A_2x^2e^{2i\theta} + A_3x^3e^{3i\theta} \\ + \ldots + A_nx^ne^{ni\theta} \end{aligned}\right\} \quad \text{(I.53)}$$

which series may be obtained from (I.50) by writing $xe^{i\theta}$ for x. Hence, if the sum of (I.50) is known so also is that of (I.53). By equating the real and imaginary parts of the two sides of (I.53) we can find C_n and S_n.

EXAMPLE 1

Find the sum of each of the series

$$C_n = 1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \ldots + x^{n-1}\cos(n-1)\theta$$

$$S_n = \quad x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \ldots + x^{n-1}\sin(n-1)\theta$$

By the above method

$$\begin{aligned} C_n + iS_n &= 1 + xe^{i\theta} + x^2e^{2i\theta} + x^3e^{3i\theta} + \ldots + x^{n-1}e^{(n-1)i\theta} \\ &= \frac{1 - x^ne^{ni\theta}}{1 - xe^{i\theta}} \text{ since the series is a geometrical progression.} \end{aligned}$$

In order to put the right-hand side in the form $a + ib$ it is necessary to eliminate the imaginary term from the denominator. This is effected by multiplying both numerator and denominator by the expression $1 - xe^{-i\theta}$. [In general, if the denominator is of the type $A + Be^{ki\theta}$ where A and B may be any real expressions and k any real number, we multiply by $A + Be^{-ki\theta}$.]

We have then

$$C \quad iS \quad \frac{(1 \quad x e^{i\theta})(1 \quad e \quad)}{(1 \quad xe \quad)(1 \quad xe \quad)}$$

$$\frac{1 \quad xe \quad x e \quad x \quad e \quad 1)}{1 \quad 2x\cos\theta \quad x}$$

$$\frac{1 \quad x(\cos\theta \quad i\sin\theta) \quad x\,(\cos n\theta \quad i\sin n\theta) \quad x \quad (\cos\overline{n \quad 1}\theta \quad i\sin\overline{n-1}\theta)}{1 \quad 2x\cos\theta \quad x}$$

and equating real and imaginary parts

$$C \quad \frac{1 \quad x\cos\theta \quad x \quad \cos n\theta \quad x \quad {}^{1}\cos\overline{n-1}\theta}{1 \quad 2x\cos\theta + x}$$

and

$$S \quad \frac{x\sin\theta \quad x \quad \sin n\theta \quad x \quad {}^{1}\sin\overline{n-1}\theta}{1 \quad 2x\cos\theta \quad x}$$

EXAMPLE 2

Find the sum of the infinite series

$$C \quad 1 \quad \frac{1}{3}\cos x \quad \frac{1}{9}\cos 2x \quad \frac{1}{27}\cos 3x$$

Suppose $\quad S \quad \frac{1}{3}\sin x \quad \frac{1}{9}\sin 2x \quad \frac{1}{27}\sin 3x$

Then $C \quad iS \quad 1 \quad \frac{1}{3}e^{ix} \quad \frac{1}{9}e \quad \frac{1}{27}e^{3}$

$$\frac{1}{1 \quad \frac{1}{3}e} \qquad \frac{1 \quad \frac{1}{3}e}{\left(1 \quad \frac{1}{3}e\right)\left(1 \quad \frac{1}{3}e\right)}$$

$$\frac{1 \quad \frac{1}{3}(\cos x \quad i\sin x)}{\frac{10}{9} \quad \frac{2}{3}\cos x} \qquad \frac{9}{10 \quad 6\cos x}\left(1 \quad \frac{1}{3}\cos x \quad \frac{1}{3}i\sin x\right)$$

$$C \quad \frac{9-3\cos x}{10 \quad 6\cos x}$$

EXAMPLES I

Evaluate the following

(1) $Lt_{x \to} \frac{3 \quad 4x}{2 \quad x}$ [*Hint* Divide through numerator and denominator by x.]

(2) $Lt_{x \to 0} \frac{4x \quad 5x}{5x \quad 4}$ (3) $Lt_{x \to} \frac{x \quad 4}{x \quad 2}$ [Put $x \quad 2 \quad h$] (4) $Lt_{x \to 2} \frac{x - 4}{x \quad 2}$

(5) $Lt_{x \to} \frac{x - 7x \quad 10}{x^{3} \quad 4x \quad 8x \quad 8}$ (6) $Lt_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a}}{x}$ (7) $Lt_{x \to \infty} \tan^{-1} x$

(8) Show that $\underset{x \to \frac{\pi}{2}}{Lt}(\tan x)$ is or according as x approaches the limit by increasing or decreasing

(9) A straight line AB slides with its ends A and B on the axes of x and y respectively $A\ B$ is another position of the line near to AB and P is the point of intersection Find the limiting position of P as $A\ B$ approaches the position AB Let α be the angle APA and let angle OAB θ

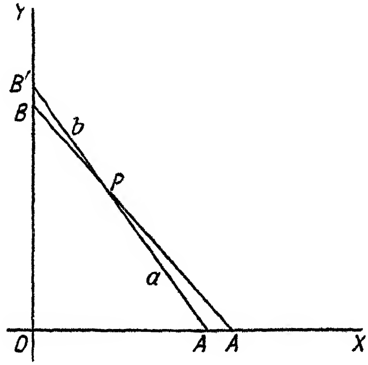


FIG 14A

Then by the sine rule $\dfrac{\sin\theta}{\overline{PA}} = \dfrac{\sin\alpha}{\overline{A\ A}}$ and $\dfrac{\cos\theta}{\overline{PB}} \quad \dfrac{\sin\alpha}{\overline{BB}}$

hence $\qquad \dfrac{\overline{A\ A}}{\overline{BB}} = \dfrac{a}{b}\cot\theta$ where $a \quad PA$ and $b - PB$

Since $\qquad \overline{AB} = \overline{A\ B},\ \overline{OA} + \overline{OB} - OA \quad OB$

i e $\qquad (\overline{OA} \quad \overline{A\ A})^2 \quad (\overline{OB} \quad \overline{BB}) \quad OA \quad OB$

$$2\ \overline{OA}\ \overline{AA} \quad 2\ \overline{OB}\ \overline{BB} \qquad AA \quad BB$$

or $\qquad 2\ \overline{OB}\ BB \quad 2\ \overline{OA}\ \overline{BB}\dfrac{a}{b}\cot\theta \qquad AA \quad BB$

i e $\qquad 2\left(\overline{OB} - \overline{OA}\ \dfrac{a}{b}\cot\theta\right) \quad - A\ A\ \dfrac{a}{b}\cot\theta - BB$

As $A\ B$ approaches AB, the right-hand side tends to the value zero and in the limit we have $\qquad OB = OA\ \dfrac{a}{b}\cot\theta,\ PA - a$ and $PB - b$

or $\qquad \dfrac{a}{b} - \tan^2\theta$

Thus in the limiting position $\dfrac{PA}{PB} - \tan^2\theta$

(10) Find the limiting position of P in the last example if for the condition $\overline{AB}$ constant we substitute the condition $\overline{OA} + \overline{OB} = \overline{OA'} + \overline{OB'} =$ constant.

Test for convergency or divergency the series—

(11) $x \quad \frac{x^2}{2} \quad \frac{x^3}{3} \quad \frac{x^4}{4} \quad \frac{x^5}{5} \quad \ldots$

(12) $\frac{1}{2^2} \quad \frac{2^2}{3^3} \quad \frac{3^3}{4^4} \quad \frac{4^4}{5^5} \quad \ldots$

(13) $\frac{x}{1.2} \quad \frac{x^2}{2.3} \quad \frac{x^3}{3.4} \quad \frac{x^4}{4.5} \quad \ldots$

(14) $\frac{1}{1.2} \quad \frac{1}{2.3} \quad \frac{1}{3.4} \quad \frac{1}{4.5} \quad \ldots$

(15) $1^3 \quad 2^3x \quad 3^3x^2 \quad 4^3x^3 \quad \ldots$

(16) Write down the first five terms in the expansion of $\left(1 - \frac{r^2}{l^2}\sin^2 \omega t\right)^{\frac{1}{2}}$.

(17) Write down the $(r+1)$th term in $(x+a)^n$ and the 6th term in $(1+2x)^{12}$.

(18) The equation $y = c \cosh \frac{x}{c}$ represents the shape of a hanging flexible wire or string. (Fig. 6 shows the graph for the case $c = 1$.) Show that for small values of x, i.e. near the lowest point of the graph, the equation is approximately $y = c\left(1 + \frac{x^2}{2c^2}\right)$ which is the equation to a parabola.

(19) The equation to a catenary, or chain curve, as in the last example, is $y = c \cosh \frac{x}{c}$. A chain is suspended from two points at the same horizontal level and L ft apart. Prove that the sag in the middle is $c\left(\cosh \frac{L}{2c} - 1\right)$, and show that, if the sag s is small compared with L, i.e. if c is large compared with L, then $s = \frac{L^2}{8c}$.

(20) Show that if n is large compared with unity, $\sqrt{n^2 - \sin^2 \theta}$ is approximately equal to $n - \frac{\sin^2 \theta}{2n}$, and express this as a function of 2θ by substituting $\frac{1}{2}(1 - \cos 2\theta)$ for $\sin^2 \theta$.

(21) If $\sqrt{1 - \frac{1}{n^2}\sin^2 \theta} = 1 + \frac{A}{n^2}\sin^2 \theta + \frac{B}{n^4}\sin^4 \theta + \frac{C}{n^6}\sin^6 \theta + \ldots$ find A, B, and C.

(22) By the methods of Art. 8, evaluate $\frac{1 \cdot 023}{0 \cdot 996}$, $\frac{0 \cdot 997}{0 \cdot 993}$, $\sqrt{\frac{1 \cdot 02}{0 \cdot 97}}$, $\sqrt{\frac{102}{97}}$.

(23) Show that if $t = 2\pi\sqrt{\frac{l}{g}}$, an error of 2 per cent in measuring l or g will cause an error of 1 per cent in the time, whilst errors of 2 per cent in both l and g may cause either no error or an error of 2 per cent in the value of t.

(24) Obtain the first four terms of the expansion of $(N^5 + a)^{\frac{1}{5}}$ by the binomial theorem, where $a < N^5$; and, hence, show that $N\dfrac{5N^5 + 3a}{5N^5 + 2a}$ is an approximate value of $(N^5 + a)^{\frac{1}{5}}$.

Deduce that the difference between (100100) and $\dfrac{50030}{5002}$ is of the order $\dfrac{1}{10^{10}}$.

(U.L.)

(25) From the relation (1.16) calculate the values of e, e^{-1}, and $e^{\frac{1}{2}}$, correct to five significant figures, and verify by multiplication that $(e^{\frac{1}{2}})^2 = e$.

(26) Find the sum of the binomial series $1 + \dfrac{2}{6} + \dfrac{2\,.\,5}{6\,.\,12} + \dfrac{2\,.\,5\,.\,8}{6\,.\,12\,.\,18} + \ldots$ to ∞. [*Hint.* Assume the sum to be $(1 + x)^n = 1 + nx + \dfrac{n\,.\,\overline{n-1}}{\underline{|2}}x^2 + \dfrac{n\,.\,\overline{n-1}\,\overline{n-2}}{\underline{|3}}x^3 + \ldots$ and by equating the coefficients of the second terms and those of the third terms, determine n and x. Show that the fourth terms agree.]

(27) Sum by the method indicated in the last example the binomial series $1 - 6x + 6x^2 + 4x^3 + \ldots$ to ∞

(28) Show that if θ_1, θ_2, θ_3, . . . are a series of quantities in arithmetical progression, and these values are substituted for θ in turn in the expression $y = ae^{k\theta}$, where a and k are any constants, the corresponding values of y, i.e. y_1, y_2, y_3, . . . are in geometric progression. Hence, or otherwise, show that if $y = ae^{-kt}$, where a and k are positive and t represents time, and if y falls to one-half of its original value in time t_0, then y will fall to one-fourth, one-eighth, one-sixteenth, one-thirty-second, etc., respectively of its original value in total times $2t_0$, $3t_0$, $4t_0$, $5t_0$, etc.

(29) Prove the relations (1.34).

(30) Prove that $\qquad \sinh^{-1} x = \log_e(\sqrt{1 + x^2} + x)$

and $\qquad \cosh^{-1} x = \log_e(\sqrt{x^2 - 1} + x)$

Hence, or otherwise, prove that $\sinh^{-1} x = \cosh^{-1} \sqrt{1 + x^2}$

Calculate $\sinh^{-1} \dfrac{9}{40}$ correct to four significant figures.

(31) Show that $e^{\tanh^{-1} x} = \sqrt{\dfrac{1 + x}{1 - x}}$.

(32) Find by means of the Argand diagram the five solutions of $x^5 - 1 = 0$, i.e. the five fifth roots of unity.

(33) Find all three values of $\sqrt[3]{27}$ and both values of $\sqrt{36 + 64i}$.

(34) Using the exponential values of $\sin\theta$ and $\cos\theta$ (1.47), prove the formulae $\sin^2\theta + \cos^2\theta = 1$ and $\cos 2\theta = 2\cos^2\theta - 1$. By expanding the right-hand side of $(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$ and equating real and imaginary parts, obtain expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(35) By the method indicated in the last question, obtain series for $\cos n\theta$ and $\sin n\theta$ where n is a positive integer.

(36) If X and Y are the rectangular components along OX and OY respectively of a force $F[\theta]$, F being in pounds and θ the angle measured in the anti-clockwise direction from OX to the line of action of F, the senses of all forces being away from the origin, we may write $F[\theta] = X + iY$. Find, by the method of Art. 16, the resultant of the system of coplanar forces: 10[22°], 15[103°], 17[220°] and 12[320°]. If $R[\alpha]$ is the resultant state the values of R and α, and give $R[\alpha]$ in the form $a + ib$.

(37) Define a complex number and show that any complex number may be expressed in the form $r(\cos\theta + i\sin\theta)$. Express $\dfrac{5}{3-4i}$ in the form $r(\cos\theta + i\sin\theta)$.

(38) Define the *modulus* and *amplitude* of a complex number. Show that the product of two complex numbers is a complex number whose modulus is the product of their moduli, and whose amplitude is the sum of their amplitudes. Express $(2 + 3i)(3 - 2i)$ as a complex number, and determine its modulus and amplitude.

If $\sin(x + iy) = r(\cos\theta + i\sin\theta)$, find the numerical values of r and θ when $x = 1$ and $y = 1$. (U.L.)

(39) Show how complex numbers may be represented graphically. O, A, and B are the points which represent zero, unity, and $a + ib$ in the diagram of the complex variable; OC is the internal bisector of the angle BOA, and the parallel through B to OC meets AO in D. Show that the point in which the circle through B, D, and A meets OC represents one of the values of $(a + ib)^{\frac{1}{2}}$. Indicate the point which represents the second value. (U.L.)

(40) Express each of the following fractions in the form $a + ib$, where a and b are real, and mark the representative points on an Argand diagram.

$$(1)\ \frac{2+i}{2-i}. \qquad (2)\ \left(\frac{2+i}{2-i}\right)^2. \qquad (3)\ \frac{1}{(2+i)^2} - \frac{1}{(2-i)^2}.$$

(41) Write down the general value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$, where p and q are integers, and prove that the expression has q different values and no more.

Find all the values of $(-1)^{1}$ and show that their representative points on an Argand diagram lie on a circle. (U.L.)

(42) Find the square roots of $5 + 12i$ and represent them on an Argand diagram.

Express i in the form $r(\cos\theta + i\sin\theta)$, and, hence, find the continued product of the three values of $i^{\frac{1}{3}}$.

(43) If $x = \cos\alpha + i\sin\alpha$, show that $\cos\alpha = \frac{1}{2}\left(x + \dfrac{1}{x}\right)$ and $i\sin\alpha = \frac{1}{2}\left(x - \dfrac{1}{x}\right)$. Deduce the expansions of $\cos^5\alpha$ and $\sin^4\alpha$ in series of cosines of multiples of α.

(44) Write down the values of $\sin x$ and $\cos x$ in series of ascending powers of x.

Prove that the length of an arc of a circle is given approximately by the formula $\frac{1}{3}(8b - a)$ where a is the chord of the arc and b the chord of half the arc. Show that the error made in the length of an arc which subtends an angle of 90° at the centre of the circle calculated by this formula is about $\frac{1}{800}$ of the radius. (U.L.)

(45) Give, with proofs, the exponential values of $\cos\theta$ and $\sin\theta$, and deduce

their connection with the hyperbolic functions. From the identity $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$ find the corresponding value of $\tanh 2x$.

Show that by a proper choice of A and B, $Ae^{2i\theta} + Be^{-2i\theta}$ (where $i = \sqrt{-1}$) can be made equal to $5\cos 2\theta - 7\sin 2\theta$. (U.L.)

(46) Suppose $x + iy = Ae^{int} + Be^{-int}$ ($i = \sqrt{-1}$) where $A = a_1 + ia_2$, $B = b_1 + ib_2$. Assuming that x, y, a_1, a_2, b_1, b_2, n, t are all real, find the values of x and y; and find an equation connecting x and y which does not contain t. (U.L.)

(47) Show that $\sin ix = i\sinh x$ and $\cos ix = \cosh x$, where i denotes $\sqrt{-1}$. Show also that $\dfrac{\sinh x}{1 + \cosh x}$ is an odd function, and expand it in ascending powers of x as far as the term in x^5. (U.L.)

(48) Write down expressions for $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ where i denotes $\sqrt{-1}$.

Find, in the form $a + ib$, *one* value of each of the following expressions—

(i) $\log_e \dfrac{3-i}{3+i}$. (ii) $\cos^{-1}\dfrac{3i}{4}$. (U.L.)

(49) Prove that if $\sin(x + iy)$ be expressed in the form $r(\cos\theta + i\sin\theta)$, then $r = \sqrt{\dfrac{\cosh 2y - \cos 2x}{2}}$ and $\theta = \tan^{-1}(\cot x \,.\, \tanh y)$.

(50) If $x + iy = \sinh(3 + 4i)$, find the numerical values of x and y to three places of decimals.

Prove that the values of z satisfying the equation $\sin z = 3$ are $n\pi + (-1)^n\{\frac{1}{2}\pi + i\log(3 + 2\sqrt{2})\}$. (U.L.)

(51) Show that when a and b are both positive, $a + ib$ may be expressed in the form $\sqrt{(a^2 + b^2)}e^{i\theta}$ where $\tan\theta = \dfrac{b}{a}$.

Examine the cases when a and b are not both positive, illustrating by a diagram.

Given that $\dfrac{1}{\rho} = \dfrac{1}{Lpi} + Cpi + \dfrac{1}{R}$, express ρ in the form $Ae^{i\theta}$, giving the values of A and θ. (U.L.)

(52) If $\tan\dfrac{x}{2} = \tanh\dfrac{u}{2}$, prove that $\sinh u = \tan x$, $\cosh u = \sec x$, and $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$. Calculate by the tables $\sinh(0{\cdot}5)$ and $\cosh(0{\cdot}5)$. (U.L.)

(53) Prove that (1) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(2) $\sinh 2x + \sinh 2y = 2\sinh(x + y) \,.\, \cosh(x - y)$

(3) $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

(54) Prove that

$$\cos a + \cos (a + \delta) + \cos (a + 2\delta) + \ldots + \cos (a + \overline{n-1}\,\delta)$$

$$= \cos (a + \tfrac{1}{2}\overline{n-1}\,\delta) \times \frac{\sin \frac{n}{2}\delta}{\sin \frac{\delta}{2}}$$

and find the corresponding sum for a series of sines.

(55) Sum the series –

$$x \sin \theta + x^2 \sin 2\theta + x^3 \sin 3\theta + \ldots \text{ to } n \text{ terms}$$

$$1 + x \cos \theta + \frac{x^2}{\underline{|2}} \cos 2\theta + \frac{x^3}{\underline{|3}} \cos 3\theta + \ldots \text{ to infinity}$$

(56) Sum the series

$$1 + n \sin \theta + \frac{n(n-1)}{\underline{|2}} \sin 2\theta + \frac{n(n-1)(n-2)}{\underline{|3}} \sin 3\theta + \ldots$$
$$\ldots + \sin n\theta$$

(57) Expand $\left(1 + \frac{1}{n}\right)^{nx}$ by the binomial theorem, and show that

$$\underset{n \to \infty}{Lt} \left(1 + \frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

Hence, show that $e^x = \underset{n \to \infty}{Lt.} \left(1 + \frac{1}{n}\right)^{nx}$. Criticize this proof.

(58) Find the sum of n terms of the series

$$\sin a + \sin (a + \beta) + \sin (a + 2\beta) + \sin (a + 3\beta) + \ldots$$

Find the sum of the lengths of the lines joining a vertex of a regular polygon of n sides to each of the other vertices, in terms of the length of a side of the polygon. (U.L.)

CHAPTER II

DIFFERENTIATION

20. **Changes in a Function of One Independent Variable.** When applying mathematics to problems in engineering or applied science, we require to know not only the values of a function $f(x)$, say, for different values of x, but we need also a knowledge of how $f(x)$ increases or decreases when x changes in a certain manner. Thus the engineer knowing the positions of the several pieces, or elements, of a machine at different times, must obtain from this a knowledge of their velocities and accelerations. From the latter he finds the necessary accelerating forces. He is thus enabled with the extended information at his disposal to examine whether the machine will function properly, i.e. whether it will produce the required motion or motions and will transmit the necessary force or forces.

When designing a portion of a machine or structure it is usually necessary to find where the stress in the material is greatest, so that this greatest stress may be kept below the safe working stress. By considering the rate at which the stress varies from point to point we are able to see where it is greatest, and thus to produce a correct design. When considering variations such as the above we may assume one variable quantity, the distance in the above cases, to vary in any arbitrary manner. This variable is then known as the "independent variable," and the other variable, the velocity, acceleration, or stress in the above cases, then varies in a certain defined manner depending upon (1) the variations in the independent variable, and (2) the relation connecting the two variables. On this account the second variable is known as the "dependent variable."

In $y = f(x)$, x is the independent and y the dependent variable. Without changing the relation between two variables we may sometimes change its form so as to interchange the dependent and independent variables. Thus, over the range inside of which it is defined below, $y = \sin^{-1}x$ means the same as $x = \sin y$, $y = \cos^{-1} x$ as $x = \cos y$, $y = \log_e x$ as $x = e^y$, etc. In each of these and in similar cases the first expression is known as the inverse of the second.

We shall proceed to investigate rates of change of functions of one independent variable of the type $y = f(x)$. Before doing so we shall define more explicitly the meaning of certain inverse functions.

21. **Inverse Functions.** $y = \sin^{-1} x$ is the angle in radians between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose sine is x. Since there are an infinite number of solutions to the equation $\sin y = x$, the reader will see that in order that $\sin^{-1} x$ may be a single valued function of x, it is necessary to limit the definition so as to include only angles between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

$y = \cos^{-1} x$ is the angle in radians between 0 and π, whose cosine is x.

$y = \tan^{-1} x$ is the angle in radians between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose tangent is x.

There are similar definitions for $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$.

The inverse functions given in Art. 12 are defined as follows:

$y = \cosh^{-1} x$ is the quantity defined by $\cosh y = x$

$y = \sinh^{-1} x$ is the quantity defined by $\sinh y = x$

$y = \tanh^{-1} x$ is the quantity defined by $\tanh y = x$

As there is only one solution to $\sinh y = x$ or $\tanh y = x$ (see Figs. 6 and 7, the axes of x and y being interchanged), the second and third of these are single-valued functions of x. There are, however, two solutions to $\cosh y = x$, and, consequently, $\cosh^{-1} x$ is a double-valued function of x (Fig. 6). Generally, the inverse of $y = f(x)$ is $x = f^{-1}(y)$, and the definition of an inverse function of x may be written $f(f^{-1}(x)) = x$.

22. **Differential Coefficient of $f(x)$.** Let $y = f(x)$. Assuming a fixed initial value for x, let Δy be the increment of y corresponding to an increment Δx of x; so that we have

$$y + \Delta y = f(x + \Delta x)$$

and $\therefore$ $$\Delta y = f(x + \Delta x) - y = f(x + \Delta x) - f(x)$$

Dividing by Δx, we obtain the ratio of the increment of y to the increment of x, i.e.

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit of this ratio when Δx tends to zero is called the first derivative or differential coefficient of y with respect to x, and is denoted by the symbol $\frac{dy}{dx}$.

i.e.
$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt.} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad . \quad . \quad (\text{II}.1)$$

The differential coefficient of $y = f(x)$ with respect to x is often denoted by $\frac{d}{dx}f(x)$, $f'(x)$, or simply $D_x y$.

EXAMPLE

If $y = \sqrt{x}$, find $\frac{dy}{dx}$ from first principles.

By definition $\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt.} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$

$$= \underset{\Delta x \to 0}{Lt.} \frac{(x + \Delta x) - x}{(\sqrt{x + \Delta x} + \sqrt{x}) \cdot \Delta x} = \underset{\Delta x \to 0}{Lt.} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

23. **Dynamical Illustration.** The equation $s = f(t)$ gives the distance s feet moved by a particle from a fixed point in a straight line to any other point in it in time t seconds. To find a meaning for $\frac{ds}{dt}$.

By the definition above $\frac{ds}{dt} = \underset{\Delta t \to 0}{Lt.} \frac{f(t + \Delta t) - f(t)}{\Delta t}$

Now $f(t + \Delta t) - f(t)$ = distance gone in the time Δt;

and $\therefore$ $\frac{f(t + \Delta t) - f(t)}{\Delta t}$ = average speed during the time Δt.

Hence, $\frac{ds}{dt}$ = limit of average speed as the interval Δt becomes small and smaller = actual speed at time t.

If we denote the speed $\frac{ds}{dt}$ by v and assume, what is true in general, that $v = \phi(t)$, then by a similar argument to the above we find that $\frac{dv}{dt}$ denotes the acceleration of the body at time t.

24. **Relations Between the Bending Moment, Shearing Force, and Load per Foot Run on an Horizontal Beam Carrying a Load of *w* lb per Foot Run.** Let PQ (Fig. 15) be a small portion of the beam, the vertical sections at P and Q being at distances x and $x + \Delta x$ respectively from some fixed point O in the beam, as shown in the figure. Let F and M be the shearing force and bending moment respectively at the section P, and let the only load on the portion PQ be one of w lb per foot run, where w is either a constant or a function of x. The reader will know that F is the algebraic sum of

all the forces on one side only, left or right, of the section at P, and that M is the algebraic sum of the turning moments about P of all the forces on one side only of the section. We shall assume that F is positive when the portion of the beam to the left of the section at P exerts an upward force on the portion to the right of P, and vice versa. We shall also assume M to be positive when the portion of the beam to the left of P exerts an anticlockwise turning moment on the portion to the right of P. The arrows in the figure show the correct senses for positive shear and positive bending moment,

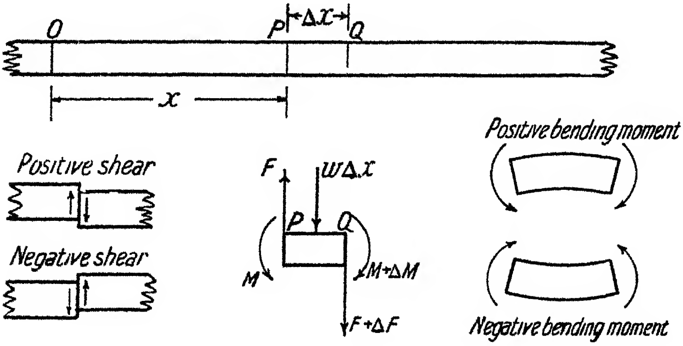


FIG. 15

the forces and moments being those exerted on the portion PQ of the beam by the adjacent portions. Since F and M are both functions of x their values will vary, and we assume that the corresponding values at Q will be $F + \Delta F$ and $M + \Delta M$ respectively.

Now consider the equilibrium of the piece of beam PQ. In addition to the forces F and $F + \Delta F$ and the couples M and $M + \Delta M$ shown, there is a load of amount $w\Delta x$ due to the distributed load of w lb per foot run. Equating to zero the sum of the vertical forces, we have

$$F - w\Delta x - F - \Delta F = 0 \quad . \quad . \quad . \quad \text{(II.2)}$$

and taking moments about Q,

$$M - F\,.\,\Delta x + \frac{w(\Delta x)^2}{2} - M - \Delta M = 0 \quad . \quad \text{(II.3)}$$

From (II.2) $-\Delta F = w\Delta x$, i.e. $\dfrac{\Delta F}{\Delta x} = -w$,

or in the limit
$$\frac{dF}{dx} = - w \quad . \quad . \quad . \quad . \quad . \quad \text{(II.4)}$$

From (II.3)

$$-F\,.\,\Delta x + \frac{w(\Delta x)^2}{2} = \Delta M, \text{ or } \quad F + \frac{w\,.\,\Delta x}{2} = \frac{\Delta M}{\Delta x},$$

and in the limit
$$\frac{dM}{dx} = -F \quad . \quad . \quad . \quad . \quad \text{(II.5)}$$

These relations (II.4) and (II.5) are of great importance to students of strength of materials.

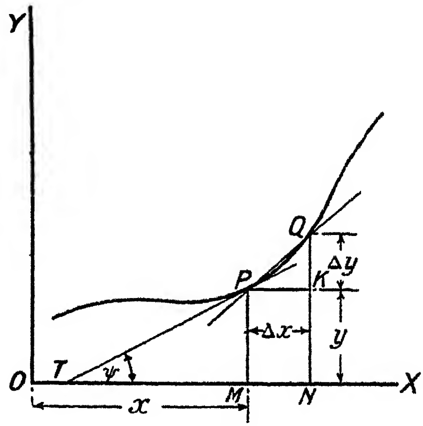


Fig. 16

25. **Geometrical Meaning of the Derivative.** Let P be any point (x, y) on the curve $y = f(x)$. Referring to Fig. 16, we have

$$y = f(x) = MP$$
$$\Delta x = MN = PK$$
$$y + \Delta y = f(x + \Delta x) = NQ$$
$$\therefore \quad \Delta y = f(x + \Delta x) - f(x) = KQ$$
$$\therefore \quad \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{KQ}{PK} = \tan KPQ$$
$$= \text{gradient of chord } PQ$$

Now, as Δx becomes smaller and smaller, the point Q will approach nearer and nearer to P, and the chord PQ will rotate about P and approach the position of the tangent PT to the curve at P as its limiting position.

Hence,

$$\underset{\Delta x \to 0}{Lt.} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \text{limit of the gradient of chord } PQ$$
$$= \text{gradient of the tangent } PT$$
$$= \tan \psi$$

(where ψ is the angle between the tangent at P and the positive direction of the x axis = angle PTX)

i.e. $$\frac{dy}{dx} = \tan \psi \quad . \quad . \quad . \quad . \quad \text{(II.6)}$$

[Prove also that $\cos \psi = \frac{dx}{ds}$ and $\sin \psi = \frac{dy}{ds}$, where Δs = length of arc PQ]

The value of $\frac{dy}{dx}$ at any point on a curve is, then, the gradient of the tangent to the curve at that point.

It follows that if $y = c$ where c is constant, $\frac{dy}{dx} = 0$, since the graph is a straight line parallel to the x-axis; and if $y = mx + c$, $\frac{dy}{dx} = m$, since m is the gradient of the straight line.

26. **Differentiation from First Principles. Standard Forms.** Following the method of Art. 22 we can, theoretically at any rate, differentiate any ordinary function, but this process, if applied generally, would involve too great labour and difficulty. To avoid this we establish, once and for all, the differential coefficients of certain standard functions, and use the results, together with a few general rules, in the differentiation of other functions.

To prove that $\frac{d}{dx} x^n = nx^{n-1}$ *for all values of n.* By definition

$$\frac{d}{dx} x^n = \underset{\Delta x \to 0}{Lt.} \frac{(x + \Delta x)^n - x^n}{\Delta x} = \underset{x + \Delta x \to x}{Lt.} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x}$$
$$= n \, . \, x^{n-1} \text{ (by Art. 8, Ex. 5)} \quad . \quad . \quad . \quad \text{(II.7)}$$

Note that on substitution of the numbers, 4, 3, 2, 1, 0, — 1 — 2, etc., for n in (II.7) the differential coefficients are respectively $4x^3$, $3x^2$, $2x$, 1, 0, $-\frac{1}{x^2}$, $-\frac{2}{x^3}$, etc., in which series of powers of x, $x^{-1} = \frac{1}{x}$ does not appear.

EXAMPLE

$$\frac{d}{dx}(x^8) = 8x^7;\ \frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{d}{dx}(x^{-\frac{1}{3}}) = -\tfrac{1}{3}\,.\,x^{-\frac{4}{3}} = -\frac{1}{3\sqrt[3]{x^4}}$$

To prove that $\frac{d}{dx}[c\,.\,f(x) + d] = c\,.\,\frac{d}{dx}[f(x)]$, *where* c *is a constant.*

$$\frac{d}{dx}[c\,.\,f(x) + d] = \underset{\Delta x \to 0}{Lt.}\frac{c\,.\,f(x+\Delta x) + d - c\,.\,f(x) - d}{\Delta x}$$

$$= c\,.\,\underset{\Delta x \to 0}{Lt.}\frac{f(x+\Delta x) - f(x)}{\Delta x} = c\,.\,\frac{d}{dx}[f(x)]$$

$$\therefore \frac{d}{dx}[c\,.\,f(x) + d] = c\,.\,\frac{d}{dx}f(x)\,. \qquad \text{(II.8)}$$

EXAMPLE

If $pv^{1.2} = 550$, find $\frac{dp}{dv}$ when $v = 30$.

Here $p = 550v^{-1.2}$. $\therefore \frac{dp}{dv} = 550\,.\,\frac{d}{dv}(v^{-1.2}) = 550(-1{\cdot}2v^{-2.2}) = -\frac{660}{v^{2.2}}$

and when $v = 30$, $\frac{dp}{dv} = -\frac{660}{30^{2.2}} = -0{\cdot}372$.

27. **Differentiation of a Sum.** Let $y = u - v + w + \ldots$ where $u, v, w, \ldots$ are all functions of x; and let $\Delta u, \Delta v, \Delta w, \ldots \Delta y$, be increments of $u, v, w, \ldots y$, corresponding to an increment Δx of x.

Then $y + \Delta y = (u + \Delta u) - (v + \Delta v) + (w + \Delta w) + \ldots$

$$\therefore \qquad \Delta y = \Delta u - \Delta v + \Delta w + \ldots$$

and
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} - \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x} + \ldots$$

In the limit when $\Delta x \to 0$, we have by Art. 3,

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} + \frac{dw}{dx} + \ldots \qquad \text{(II.9)}$$

Thus, the differential coefficient of the algebraic sum of a finite number of functions is equal to the algebraic sum of the differential coefficients of the several functions.

EXAMPLE

Find $\dfrac{d}{dx}\left[\dfrac{4}{x} - 7x^{0.3} + 6\sqrt[4]{x} - 9\right]$

$$\frac{d}{dx}[4x^{-1} - 7x^{0.3} + 6x^{\frac{1}{4}} - 9] = 4(-x^{-2}) - 7(0.3x^{-0.7}) + 6(\tfrac{1}{4}x^{-\frac{3}{4}}) - 0$$

$$= -\frac{4}{x^2} - \frac{2.1}{x^{0.7}} + \frac{3}{2\sqrt[4]{x^3}}$$

28. To prove that $\dfrac{d}{dx}(\sin x) = \cos x$.

$$\frac{d}{dx}(\sin x) = \underset{\Delta x \to 0}{Lt.} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt.} \frac{2\cos\left(x + \dfrac{\Delta x}{2}\right) . \sin\dfrac{\Delta x}{2}}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt.} \cos\left(x + \frac{\Delta x}{2}\right) . \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}$$

$$\therefore \qquad \frac{d}{dx}(\sin x) = \cos x \quad . \quad . \quad . \quad . \quad . \quad (II.10)$$

$$\left(\text{since } \underset{\Delta x \to 0}{Lt.} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x, \text{ and } \underset{\Delta x \to 0}{Lt.} \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}} = 1\right)$$

Similarly writing $x + \dfrac{\pi}{2}$ for x in the above we have

$$\frac{d}{dx}(\cos x) = -\sin x$$

for
$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$$

$$\therefore \qquad \frac{d}{dx}(\cos x) = -\sin x \quad . \quad . \quad . \quad . \quad . \quad (II.11)$$

The following geometrical method of differentiating $\sin x$ is illuminating. Let the arc AP of a circle of unit radius (Fig. 17) subtend an angle x radians at the centre O and let angle $POQ = \Delta x$.

Draw PM and QN perpendicular to OA and PK perpendicular to QN. Then

$$\sin(x + \Delta x) - \sin x = NQ - MP = KQ$$

$$\therefore \quad \underset{\Delta x \to 0}{Lt.} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \underset{\Delta x \to 0}{Lt.} \frac{KQ}{PQ} = \cos x$$

(since $\widehat{KQP}$ approaches the limit x).

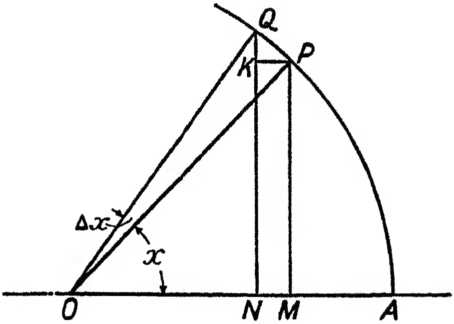


FIG. 17

Hence, $\frac{d}{dx} \sin x = \cos x$. The reader should try this method for the differentiation of $\cos x$.

29. To Find Expressions for e^x and for e^{cx} using the Fact Proved in Art. 9 that the Gradient at any Point on the Graph of $y = e^x$ is Equal to the Ordinate at that Point. Since the gradient is represented by $\frac{dy}{dx}$, this relation may be written $\frac{dy}{dx} = y$.

We shall assume that y (or e^x) may be expanded in the form of an infinite series, each term of which contains a power of x. Thus assume

$$y = e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Then $$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

(For the conditions under which the relation (II.9) is applicable to an infinite number of terms the reader is referred to any standard treatise on the differential calculus. We shall content ourselves with stating that the sum of a power series in x is differentiable over the

range of values of x for which the series is convergent, and its differential coefficient is then found by using (II.9).)

Since $y = \frac{dy}{dx}$ for all values of x, the right-hand sides of these expressions are identically equal, and we have $a_1 = a_0$, $2a_2 = a_1$, $3a_3 = a_2$, $4a_4 = a_3$, etc. Hence, $a_1 = a_0$, $a_2 = \frac{1}{2}a_1 = \frac{1}{\underline{|2}}a_0$, $a_3 = \frac{1}{3}a_2 = \frac{1}{\underline{|3}}a_0$, $a_4 = \frac{1}{4}a_3 = \frac{1}{\underline{|4}}a_0$, etc.,

$$\therefore y = a_0\left(1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\right)$$

Since $y = e^x$ becomes $e^0 = 1$ when $x = 0$, we have on substitution in the last relation $1 = a_0$, hence, we have

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots$$

or, writing cx for x,

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \frac{c^4x^4}{\underline{|4}} + \ldots$$

which we gave without proof in Art. 9.

The differential coefficient of a^x is found thus

$$\frac{d}{dx}a^x = \underset{\Delta x \to 0}{Lt.}\ \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \underset{\Delta x \to 0}{Lt.}\ \frac{a^{\Delta x} - 1}{\Delta x}$$
$$= a^x \log_e a \text{ (Art. 10)}$$

Hence $\quad \frac{d}{dx}a^x = a^x \log_e a \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (II.12)

Again $\quad \frac{d}{dx}e^{kx} = \frac{d}{dx}a^x$, where $a = e^k$

Hence $\quad \frac{d}{dx}e^{kx} = a^x . \log_e a = ka^x$

$\therefore \quad \frac{d}{dx}e^{kx} = ke^{kx} \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (II.13)

EXAMPLE 1

The equation $i = \frac{V}{R}(1 - e^{-\frac{Rt}{L}})$ gives the current i amperes in an electric

circuit at time t seconds, V, R, and L being the voltage resistance, and self-inductance respectively; to find $\frac{di}{dt}$.

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{Rt}{L}}$$

$$\frac{di}{dt} = 0 - \frac{V}{R}\left(-\frac{R}{L} e^{-\frac{Rt}{L}}\right) = \frac{V}{L} e^{-\frac{Rt}{L}}$$

EXAMPLE 2

If $T = T_0 e^{\mu\theta}$, T_0 and μ being constants, show that $\frac{dT}{d\theta} = \mu T$

Here $\frac{dT}{d\theta} = T_0(\mu e^{\mu\theta}) = \mu \,.\, T_0 e^{\mu\theta} = \mu T.$

30. **Differentiation of a Product of Two Functions.** Let $y = uv$, where u and v are functions of x; and let Δu, Δv, Δy be increments of u, v, y corresponding to increment Δx of x.

Then $$y = uv$$

and $$y + \Delta y = (u + \Delta u)(v + \Delta v)$$
$$= uv + u\Delta v + v\Delta u + \Delta u \,.\, \Delta v$$

Subtracting, $\Delta y = u\Delta v + v\Delta u + \Delta u \,.\, \Delta v$. Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v = u\frac{\Delta v}{\Delta x} + (v + \Delta v)\frac{\Delta u}{\Delta x}$$

In the limit when $\Delta x \to 0$, we have by Art. 3

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \qquad . \quad . \quad . \quad \text{(II.14)}$$

Thus, the differential coefficient of the product of two functions = (1st function) $\times$ (diff. co. of 2nd)
+ (2nd function) $\times$ (diff. co. of 1st).

EXAMPLE

To find $\frac{d}{dx}(x^n e^{ax})$.

$$\frac{d}{dx}(x^n e^{ax}) = x^n \,.\, \frac{d}{dx}(e^{ax}) + e^{ax} \,.\, \frac{d}{dx}(x^n) = x^n \,.\, ae^{ax} + e^{ax} \,.\, nx^{n-1}$$
$$= x^{n-1} e^{ax}(ax + n)$$

31. **Differentiation of a Quotient.** Let $y = \frac{u}{v}$; hence $u = vy$.

By product rule $\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{v}\left[\frac{du}{dx} - \frac{u}{v}\cdot\frac{dv}{dx}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad . \quad \text{(II.15)}$$

Thus, the differential coefficient of a quotient

$$= \frac{\text{(denom.)} \times \text{(diff. co. of num.)} - \text{(num.)} \times \text{(diff. co. of denom.)}}{\text{(denom.)}^2}$$

EXAMPLE

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \,.\, (\cos x) - \sin x \,.\, (-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Similarly, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

32. **Differentiation of a Function of a Function.** Let $y = f(u)$, where $u = \phi(x)$, so that u and y are functions of x; and let Δu, Δy be *simultaneous* increments of u, y corresponding to increment Δx of x.

We have $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x} \text{ identically.}$$

In the limit when Δx (and, therefore, Δu) $\to 0$, we have by Art. 3

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \quad . \quad . \quad . \quad . \quad \text{(II.16)}$$

With the same notation as in Art. 23 we have $\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds}$, so that $v\frac{dv}{ds}$ is an alternative symbol to denote acceleration. The angular acceleration $\frac{d\omega}{dt}$ of a rigid body rotating about an axis can also be denoted by $\omega\frac{d\omega}{d\theta}$, θ being the angle through which the body has turned and ω its angular velocity at time t.

EXAMPLE 1

Let $y = \sec x = (\cos x)^{-1}$

Put $u = \cos x$; then $y = u^{-1}$

$$\therefore \qquad \frac{du}{dx} = -\sin x \text{ and } \frac{dy}{du} = -\frac{1}{u^2} = -\frac{1}{\cos^2 x}$$

Hence, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \left(-\dfrac{1}{\cos^2 x}\right) \times (-\sin x) = \dfrac{\sin x}{\cos^2 x}$ *or* $\tan x \,.\, \sec x$

Similarly, if $y = \operatorname{cosec} x$, $\dfrac{dy}{dx} = -\cot x \,.\, \operatorname{cosec} x$

EXAMPLE 2

Let $\qquad y = \sqrt{a^2 + x^2}$. Put $u = a^2 + x^2$, so that $y = u^{\frac{1}{2}}$.

$$\therefore \qquad \frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{a^2 + x^2}}$$

$$\text{Hence,} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{x}{\sqrt{a^2 + x^2}}$$

When he has had a little practice in differentiating functions of functions, the reader will be able to omit the actual substitutions. In differentiating $\sin^2(2\pi nt + \alpha)$, for example, he will proceed mentally as follows—

$$\frac{d}{dt} \sin^2 (2\pi nt + \alpha)$$

$$= \frac{d[\sin^2 (2\pi nt + \alpha)]}{d[\sin (2\pi nt + \alpha)]} \cdot \frac{d[\sin (2\pi nt + \alpha)]}{d(2\pi nt + \alpha)} \cdot \frac{d(2\pi nt + \alpha)}{dt}$$

$$= 2 \sin (2\pi nt + \alpha) \,.\, \cos (2\pi nt + \alpha) \,.\, 2\pi n$$

$$= 4\pi n \sin (2\pi nt + \alpha) \,.\, \cos (2\pi nt + \alpha)$$

EXAMPLE 3

Assuming the results (II.10) and (II.11), we have the following—

$$\frac{d}{dx} \sin (ax + b) = \cos (ax + b) \frac{d}{dx} (ax + b) = a \cos (ax + b) \qquad \text{(II.17)}$$

$$\frac{d}{dx} \cos (ax + b) = -\sin (ax + b) \frac{d}{dx} (ax + b) = -a \sin (ax + b) \qquad \text{(II.18)}$$

33. Differentiation of Hyperbolic Functions.

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \qquad \text{(II.19)}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x \qquad (II.20)$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x}$$

$$\therefore \quad \frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \qquad (II.21)$$

We leave the reader to prove that

$$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x; \quad \frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{cosech} x = -\coth x \operatorname{cosech} x$$

EXAMPLE

The Cartesian equation of the common catenary is $y = c\cosh\frac{x}{c}$. Its slope ψ at any point is given by

$$\text{Slope} = \tan\psi = \frac{dy}{dx} = c\,.\left(\sinh\frac{x}{c}\right)\left(\frac{1}{c}\right) = \sinh\frac{x}{c}$$

34. **Differentiation of Inverse Functions.** In general, if y is a continuous function of x, x is a continuous function of y, and if y can be differentiated with respect to x within any range, x can, in general, be differentiated with respect to y.

(1) $y = \sin^{-1}\frac{x}{a}$

$$\therefore \quad x = a\sin y$$

$$\therefore \quad \frac{d}{dx}(x) = \frac{d}{dx}(a\sin y) = \frac{d}{dy}(a\sin y)\,.\,\frac{dy}{dx} \quad \text{(Art. 32)}$$

i.e. $$1 = a\cos y\,\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a\cos y} = \pm\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}} = \pm\frac{1}{\sqrt{a^2-x^2}}$$

From the definition of $\sin^{-1} x$ given in Art. 21, we see that $\sin^{-1} x$ increases with x, and so we take the upper sign, and write

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}} \quad . \quad . \qquad \text{(II.22)}$$

(2) $y = \cos^{-1}\dfrac{x}{a}$

$$\therefore \qquad x = a \cos y$$

$$\therefore \qquad \frac{d}{dx}(x) = \frac{d}{dx}(a \cos y) = -a \sin y \frac{dy}{dx}$$

i.e.
$$1 = -a \sin y \frac{dy}{dx}$$

$$\therefore \qquad \frac{dy}{dx} = \mp \frac{1}{\sqrt{a^2 - x^2}}$$

From the definition of $\cos^{-1} x$ (Art. 21) we see that $\cos^{-1} x$ decreases as x increases, and so we take the upper sign, and write

$$\frac{d}{dx}\cos^{-1}\frac{x}{a} = -\frac{1}{\sqrt{a^2 - x^2}} \quad . \qquad \text{(II.23)}$$

(3) $y = \tan^{-1}\dfrac{x}{a}$

$$\therefore \qquad x = a \tan y$$

and
$$\frac{d(x)}{dx} = a \sec^2 y \frac{dy}{dx}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{a \sec^2 y} = \frac{1}{a\left(1 + \dfrac{x^2}{a^2}\right)} = \frac{a}{a^2 + x^2} \quad . \qquad \text{(II.24)}$$

There is here no ambiguity in sign as $\tan^{-1} x$ increases as x increases. (Art. 21.)

(4) $y = \log_a x$.

$$\therefore \qquad x = a^y$$

and
$$\frac{d(x)}{dx} = \frac{d(a^y)}{dx} = a^y \log_e a \frac{dy}{dx}$$

i.e. $$1 = y \log_e a \frac{dy}{dx}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{y \log_e a} \qquad \text{(II.25)}$$

If $$a = e,\ y = \log_e x \text{ and } \frac{dy}{dx} = \frac{1}{x} \qquad \text{(II.26)}$$

It is useful to note that $\frac{d}{dx} \log y = \frac{1}{y} \frac{dy}{dx}$

From the above

$$\frac{d}{dx} \log_e (ax + b) = \frac{1}{ax + b} \frac{d}{dx} (ax + b) = \frac{a}{ax + b} \qquad \text{(II.27)}$$

EXAMPLE 1

Let $y = \tan^{-1} \frac{x + a}{b} = \tan^{-1} u$, where $u = \frac{x + a}{b}$

Then $\frac{du}{dx} = \frac{1}{b}$ and $\frac{dy}{du} = \frac{1}{1 + u^2} = \frac{1}{1 + \left(\frac{x + a}{b}\right)^2} = \frac{b^2}{b^2 + (x + a)^2}$

Hence, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{b}{(b^2 + (x + a)^2)}$

EXAMPLE 2

Let $y = \sin^{-1} \frac{1 - x^2}{1 + x^2} = \sin^{-1} u$, where $u = \frac{1 - x^2}{1 + x^2}$

Then $\frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$ and $\frac{du}{dx} = \frac{(1 + x^2)(-2x) - (1 - x^2)(2x)}{(1 + x^2)^2} = -\frac{4x}{(1 + x^2)^2}$

Now $1 - u^2 = (1 + u)(1 - u) = \left(1 + \frac{1 - x^2}{1 + x^2}\right)\left(1 - \frac{1 - x^2}{1 + x^2}\right)$

$$= \frac{2}{1 + x^2} \cdot \frac{2x^2}{1 + x^2} = \frac{4x^2}{(1 + x^2)^2}$$

$$\therefore \qquad \frac{dy}{du} = \frac{1 + x^2}{2x}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1 + x^2}{2x} \cdot \frac{-4x}{(1 + x^2)^2} = -\frac{2}{1 + x^2}$$

EXAMPLE 3

Let $\quad y = \log_e \dfrac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x}$

$$= \log_e (\sqrt{x^2 + a^2} + x) - \log_e (\sqrt{x^2 + a^2} - x)$$

$$= \log_e u_1 - \log_e u_2$$

$$\frac{dy}{dx} = \frac{1}{u_1} \frac{du_1}{dx} - \frac{1}{u_2} \frac{du_2}{dx}$$

where $\quad u_1 = \sqrt{x^2 + a^2} + x, \; u_2 = \sqrt{x^2 + a^2} - x$

and $\quad \dfrac{du_1}{dx} = \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} . (2x) + 1 = \dfrac{x}{\sqrt{x^2 + a^2}} + 1 = \dfrac{u_1}{\sqrt{x^2 + a^2}}$

$$\frac{du_2}{dx} = \frac{x}{\sqrt{x^2 + a^2}} - 1 = -\frac{u_2}{\sqrt{x^2 + a^2}}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{u_1} \dfrac{u_1}{\sqrt{x^2 + a^2}} + \dfrac{1}{u_2} \dfrac{u_2}{\sqrt{x^2 + a^2}} = \dfrac{2}{\sqrt{x^2 + a^2}}$

(5) $y = \sinh^{-1} \dfrac{x}{a}$

$\therefore \quad x = a \sinh y$ and $\dfrac{d}{dx}(x) = a \cosh y . \dfrac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a \cosh y} = \frac{1}{a\sqrt{1 + \frac{x^2}{a^2}}}$$

i.e. $\quad \dfrac{d}{dx}\left(\sinh^{-1} \dfrac{x}{a}\right) = \dfrac{1}{\sqrt{x^2 + a^2}}$ . . . (II.28)

Since $\cosh y$ is essentially positive, there is no ambiguity in sign here.

(6) $y = \cosh^{-1} \dfrac{x}{a}$

$\therefore \quad x = a \cosh y$ and $\dfrac{d}{dx}(x) = a \sinh y \dfrac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a \sinh y} = \pm \frac{1}{a\sqrt{\frac{x^2}{a^2} - 1}}$$

i.e. $\quad \dfrac{d}{dx}\left(\cosh^{-1} \dfrac{x}{a}\right) = \pm \dfrac{1}{\sqrt{x^2 - a^2}}$ (II.29)

There are two values of y which correspond to a given value of x and $\frac{dy}{dx}$ has opposite signs for these two values $[x > a]$.

(7) $y = \tanh^{-1}\frac{x}{a}$

$$\therefore \quad x = a\tanh y \text{ and } \frac{d}{dx}(x) = a\,\text{sech}^2 y \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a\,\text{sech}^2 y} = \frac{1}{a\left(1 - \frac{x^2}{a^2}\right)} = \frac{a}{a^2 - x^2}$$

$$\text{i.e.} \qquad \frac{d}{dx}\left(\tanh^{-1}\frac{x}{a}\right) = \frac{a}{a^2 - x^2} \quad . \quad . \qquad \text{(II.30)}$$

$[x^2 < a^2$ if y is real.]

(8) $y = \coth^{-1}\frac{x}{a}$. By a similar method we find that

$$\frac{d}{dx}\left(\coth^{-1}\frac{x}{a}\right) = -\frac{a}{x^2 - a^2} \quad . \quad . \qquad \text{(II.31)}$$

$[x^2 > a^2$ if y is real.]

Each of these inverse hyperbolic functions can be expressed in logarithmic form, as shown in Art. 12.

35. **Logarithmic Differentiation.** The work involved in differentiation can often be greatly simplified by taking logarithms of both sides of a given equation before differentiating. This method is especially useful in the case of a function which consists of a number of factors. The following examples will illustrate the process—

EXAMPLE 1

Let $\qquad y = x^n \cdot e^{kx} \cdot \sin^2 \beta x$

Taking logs, $\qquad \log y = n \log x + kx + 2 \log \sin \beta x$

Differentiating with respect to x

$$\frac{1}{y} \cdot \frac{dy}{dx} = n \cdot \frac{1}{x} + k + 2 \cdot \frac{1}{\sin \beta x} \cdot (\cos \beta x) \cdot (\beta)$$

$$\therefore \qquad \frac{dy}{dx} = x^n e^{kx} \sin^2 \beta x \left[\frac{n}{x} + k + 2\beta \cot \beta x\right]$$

differentiated gives the third differential coefficient $\frac{d^3y}{dx^3}$, and so on. The second differential coefficient may also be written as $\left(\frac{d}{dx}\right)^2 y$, $f''(x)$, y_2, or $D_x{}^2 y$, and in the same way the nth differential coefficient may be written as $\frac{d^ny}{dx^n}$, $\left(\frac{d}{dx}\right)^n y$, $f^{(n)}(x)$, y_n, or $D_x^n y$. The second differential coefficient is of special importance to engineering students owing to its constant recurrence in dynamical problems. Linear acceleration is denoted by $\frac{d^2s}{dt^2}$ $\left[\text{i.e. } \frac{d}{dt}\left(\frac{ds}{dt}\right)\right]$ and angular acceleration by $\frac{d^2\theta}{dt^2}$ $\left[\text{i.e. } \frac{d}{dt}\left(\frac{d\theta}{dt}\right)\right]$, the letters s, θ, t having their usual meanings.

In comparatively few cases is it possible to deduce a formula for the nth differential coefficient of a given function. Some of the more important of these cases are given below.

It will be an easy exercise for the reader to establish the following results—

$$\left(\frac{d}{dx}\right)^n x^n = |\underline{n}, \ (n \text{ being a positive integer})$$

$$\left(\frac{d}{dx}\right)^n e^{cx} = c^n e^{cx} \quad : \quad \left(\frac{d}{dx}\right)^n \log_e x = \frac{(-1)^{n-1}\,|\underline{n-1}}{x^n}$$

$$\left(\frac{d}{dx}\right)^n a^x = a^x (\log_e a)^n \quad : \quad \left(\frac{d}{dx}\right)^n \frac{1}{ax+b} = \frac{(-1)^n \cdot a^n \cdot |\underline{n}}{(ax+b)^{n+1}}$$

EXAMPLE 1

Find y_n when (1) $y = \sin kx$, (2) $y = e^{ax} \sin kx$.

(1) $$y_1 = k\cos kx = k\sin\left(kx + \frac{\pi}{2}\right)$$

$$\therefore \quad y_2 = k^2\cos\left(kx + \frac{\pi}{2}\right) = k^2\sin\left(kx + 2\cdot\frac{\pi}{2}\right)$$

$$\therefore \quad y_3 = k^3\sin\left(kx + 3\cdot\frac{\pi}{2}\right)$$

. . . .

$$\therefore \quad y_n = k^n\sin\left(kx + n\cdot\frac{\pi}{2}\right)$$

(2) $y_1 = ce^{cx} \cdot \sin kx + e^{cx} \cdot k \cos kx = e^{cx}(c \sin kx + k \cos kx)$

Let $c = r \cos \alpha$ and $k = r \sin \alpha$, so that $r = \sqrt{c^2 + k^2}$ and $\alpha = \tan^{-1}\dfrac{k}{c}$

Then $y_1 = re^{cx} \sin (kx + \alpha)$;

hence $y_2 = r^2 e^{cx} \sin (kx + 2\alpha)$

$y_3 = r^3 e^{cx} \sin (kx + 3\alpha)$

. . . . . .

$$y_n = r^n e^{cx} \sin (kx + n\alpha) = (c^2 + k^2)^{\frac{n}{2}} e^{cx} \sin \left(kx + n \tan^{-1}\frac{k}{c}\right)$$

We can prove similarly that when $y = \cos kx$, $y_n = k^n \cos\left(kx + n\dfrac{\pi}{2}\right)$

and when $y = e^{cx} \cos kx$, $y_n = (c^2 + k^2)^{\frac{n}{2}} e^{cx} \cos \left(kx + n \tan^{-1} \dfrac{k}{c}\right)$.

EXAMPLE 2

Find y_n when $y = \dfrac{3}{(x+1)(2x-1)}$.

Resolving into partial fractions we obtain $y = \dfrac{2}{2x-1} - \dfrac{1}{x+1}$

$$\therefore \qquad y_n = 2 \cdot \frac{(-1)^n 2^n \underline{|n}}{(2x-1)^{n+1}} - \frac{(-1)^n \underline{|n}}{(x+1)^{n+1}}$$

$$= (-1)^n \underline{|n} \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right]$$

EXAMPLE 3

If $\sqrt{y+x} + \sqrt{y-x} = c$, show that $\dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{\dfrac{y^2}{x^2} - 1}$, and find the value of $\dfrac{d^2y}{dx^2}$. (U.L.)

This example can be done by differentiating at once; otherwise thus—

Squaring, $y + x + y - x + 2\sqrt{y^2 - x^2} = c^2$, i.e. $2\sqrt{y^2 - x^2} = c^2 - 2y$

Squaring again, $4y^2 - 4x^2 = c^4 - 4c^2y + 4y^2$;

whence $4c^2y = c^4 + 4x^2$

Differentiating, $4c^2 \dfrac{dy}{dx} = 8x$; so that $\dfrac{dy}{dx} = \dfrac{2x}{c^2} = \dfrac{x}{y + \sqrt{y^2 - x^2}}$

$$= \frac{x(y - \sqrt{y^2 - x^2})}{x^2}$$

$$= \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$$

Again, since $\dfrac{dy}{dx} = \dfrac{2x}{c^2}$, we have $\dfrac{d^2y}{dx^2} = \dfrac{2}{c^2}$.

37. **Leibnitz' Theorem.** *If $y = uv$, where u and v are functions of x, then $y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_{r-1} u_{n-r+1} v_{r-1} + {}^nC_r u_{n-r} v_r + \ldots + uv_n$, the coefficients ${}^nC_1$, ${}^nC_2$, etc., being those of the binomial theorem.*

Assume the theorem true for n differentiations, and differentiate again.

$$\begin{aligned}
\therefore\ y_{n+1} &= (u_{n+1} v + u_n v_1) + {}^nC_1(u_n v_1 + u_{n-1} v_2) + \ldots \\
&\qquad + {}^nC_{r-1}(u_{n-r+2} v_{r-1} + u_{n-r+1} v_r) \\
&\qquad + {}^nC_r(u_{n-r+1} v_r + u_{n-r} v_{r+1}) + \ldots + (u_1 v_n + uv_{n+1}) \\
&= u_{n+1} v + (1 + {}^nC_1) u_n v_1 + ({}^nC_1 + {}^nC_2) u_{n-1} v_2 + \ldots \\
&\qquad + ({}^nC_{r-1} + {}^nC_r) u_{n-r+1} v_r + \ldots + uv_{n+1}
\end{aligned}$$

Now

$$\begin{aligned}
{}^nC_{r-1} + {}^nC_r &= \frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}} + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \\
&= \frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r}}\left[\frac{1}{n-r+1} + \frac{1}{r}\right] \\
&= \frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r}}\left(\frac{n+1}{r(n-r+1)}\right) \\
&= \frac{\underline{|n+1}}{\underline{|r}\,.\,\underline{|n+1-r}} = {}^{n+1}C_r
\end{aligned}$$

$$\begin{aligned}
\therefore\ y_{n+1} &= u_{n+1} v + {}^{n+1}C_1 u_n v_1 + {}^{n+1}C_2 u_{n-1} v_2 + \ldots \\
&\qquad + {}^{n+1}C_r u_{n-r+1} v_r + \ldots + uv_{n+1} \qquad \text{(II.32)}
\end{aligned}$$

Hence, if the theorem is true for n differentiations, it is also true for $n + 1$ differentiations. But it is true for $n = 1$ and $n = 2$, since $y_1 = u_1 v + uv_1$, and $y_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2) = u_2 v + {}^2C_1\, u_1 v_1 + uv_2$; so that the theorem is true for $n = 3$, and hence for $n = 4$, and so on. This establishes the theorem generally.

EXAMPLE 1

If $\quad y = x^2 \log x$, prove that $y_n = \dfrac{(-1)^{n-1}\,.\,2\,\underline{|n-3}}{x^{n-2}}$

Let $\quad u = \log x$, and $v = x^2$

$\therefore\quad u_n = \dfrac{(-1)^{n-1}\,\underline{|n-1}}{x^n}$ (Art. 36); $v_1 = 2x$; $v_2 = 2$; v_3, v_4, etc., all zero.

Then $y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{|2} . u_{n-2} v_2$

$$= \frac{(-1)^{n-1}|n-1}{x^n} . x^2 + n . \frac{(-1)^{n-2}|n-2}{x^{n-1}} . 2x . + \frac{n(n-1)}{|2} . \frac{(-1)^{n-3}|n-3}{x^{n-2}} . 2$$

$$= \frac{(-1)^{n-1}|n-3}{x^{n-2}} [(n-1)(n-2) - 2n(n-2) + n(n-1)]$$

$$= \frac{(-1)^{n-1} . 2 |n-3}{x^{n-2}}$$

EXAMPLE 2

If $f(\theta) = \left(\frac{\sin\theta}{\theta}\right)^2$, show that $\theta^2 f''(\theta) + 4\theta f'(\theta) = 2[1 - f(\theta) - 2\theta^2 f(\theta)]$

We have $\sin^2\theta = \theta^2 f(\theta)$

By Leibnitz' theorem, $\frac{d^2}{d\theta^2}(\sin^2\theta) = f''(\theta) . \theta^2 + 2 . f'(\theta) . 2\theta + f(\theta) . 2$

Now $\frac{d}{d\theta}(\sin^2\theta) = 2\sin\theta . \cos\theta = \sin 2\theta$

$\therefore \frac{d^2}{d\theta^2}(\sin^2\theta) = 2\cos 2\theta = 2(1 - 2\sin^2\theta) = 2[1 - 2\theta^2 . f(\theta)]$

Hence, $2 - 4\theta^2 f(\theta) = \theta^2 f''(\theta) + 4\theta f'(\theta) + 2f(\theta)$

i.e. $\theta^2 f''(\theta) + 4\theta f'(\theta) = 2[1 - f(\theta) - 2\theta^2 f(\theta)]$

EXAMPLE 3

Show that $\frac{d}{dx}(e^{ax}\cos bx) = Re^{ax}\cos(bx + \alpha)$ where $R\cos\alpha = a$, $R\sin\alpha = b$, $R > 0$.

Find the value of $\left(\frac{d}{dx}\right)^n (c^x e^{ax}\cos bx)$ and hence, or otherwise, show that

$$S^n \cos n\beta = (\log c)^n + n(\log c)^{n-1} R\cos\alpha + \frac{n(n-1)}{2!}(\log c)^{n-2} R^2\cos 2\alpha + \dots + R^n \cos n\alpha$$

where $S\cos\beta = a + \log c$, $S\sin\beta = b$, $S > 0$

(U.L. General Science.)

We have $\frac{d}{dx}(e^{ax}\cos bx) = ae^{ax}\cos bx - be^{ax}\sin bx$

$= e^{ax}[R\cos\alpha . \cos bx - R\sin\alpha . \sin bx]$

(where $a = R\cos\alpha$, $b = R\sin\alpha$)

$$a = \frac{d^2x}{dt^2} = -\omega^2 r \cos(\omega t + \alpha) \quad . \quad . \quad . \quad (II.35)$$

i.e.

$$a = -\omega^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.36)$$

or

$$\frac{\text{Acceleration of } M}{\text{Displacement of } M} = -\omega^2$$

The minus sign in this relation indicates that the acceleration and the displacement always differ in sign, i.e. that the acceleration is always directed towards the point O. If the reader remembers this he may omit the minus sign from (II.36) and write it in the form

$$\frac{\text{Acceleration}}{\text{Displacement}} = \text{constant} \quad . \quad . \quad . \quad (II.37)$$

This is the test for simple harmonic motion. If a particle or point in a body moves so as to satisfy (II.37), and the acceleration is always directed towards a fixed point in its path, the motion is simple harmonic and the above relations apply to the motion. Since M makes a complete oscillation every time OP makes a complete revolution, the time of a complete oscillation is given by $t = \frac{2\pi}{\omega}$ seconds, or

$$t = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} \quad . \quad . \quad . \quad (II.38)$$

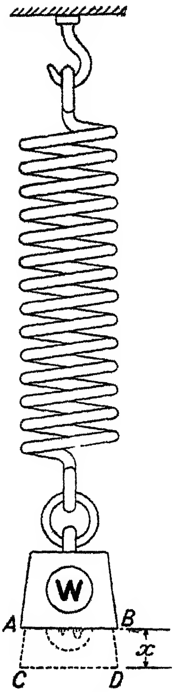


FIG. 19

EXAMPLE

A mass of W lb weight is attached to the lower end of a vertical spiral spring which has its upper end fixed (Fig. 19). The spring stretches a distance a ft under a tension Ea lb when a is small. If the mass is set oscillating in a vertical direction, find the time of a complete oscillation.

Suppose the mass to be pulled downward and then released. It will then oscillate along the axis of the spring. Let CD be the position of the lower end of the mass after t seconds, and let AB be its initial position. Then if x is the distance between AB and CD, the additional tension in the spring due to the stretch x is Ex lb, which is therefore the unbalanced force.

Hence, since Force = Mass $\times$ Acceleration

$$-Ex = \frac{W}{g} \times \frac{d^2x}{dt^2}$$

the negative sign indicating that x and $\frac{d^2x}{dt^2}$ are opposite in sign. Hence, we have

$$\frac{\text{Acceleration}}{\text{Displacement}} = \frac{d^2x}{dt^2}\Big/ x$$

or omitting the sign $\qquad \dfrac{\text{Acceleration}}{\text{Displacement}} = \dfrac{gE}{W} = \text{constant}$

and by (II.37) the motion is simple harmonic. Hence, by (II.38)

$$t = \text{time of oscillation} = 2\pi\sqrt{\frac{W}{gE}}$$

39. Motion of Crosshead and Piston of the Simple Engine Mechanism. In Example 4, Art. 8 (see Fig. 2), we obtained the expressions

$$x = r\cos(\omega t + \alpha) + l\left(1 - \frac{r^2}{l^2}\sin^2(\omega t + \alpha)\right)^{\frac{1}{2}} \quad . \quad \text{(II.39)}$$

and $\quad x = r\cos(\omega t + \alpha) + l\left(1 - \dfrac{r^2}{4l^2}\right) + \dfrac{r^2}{4l}\cos(2\omega t + 2\alpha) \quad$ (II.40)

for the distance from the crankshaft to the crosshead, (II.39) being an exact expression and (II.40) an approximate expression for x. With the meanings attached to the quantities in that example we have

$$v = \text{speed of piston (or crosshead)} = \frac{dx}{dt} \text{ ft per sec}$$

and $\qquad a = \text{acceleration of piston} = \dfrac{d^2x}{dt^2}$ ft per sec per sec.

As found in Example 4, Art. 35, we have

$$\left.\begin{aligned} v = \frac{dx}{dt} &= -\omega r\sin(\omega t + \alpha) \\ &\quad - \frac{\omega r^2}{2} \cdot \frac{\sin 2(\omega t + \alpha)}{\sqrt{l^2 - r^2\sin^2(\omega t + \alpha)}} \end{aligned}\right\} \quad . \quad \text{(II.41)}$$

an exact expression for the velocity.

To find the acceleration we obtain $\frac{dv}{dt}$ from (II.41). This expression is, however, rather cumbersome and as an approximate expression

is usually sufficient, we find approximate expressions for the velocity and acceleration. Thus, from the expression (II.40), we have

$$v = \frac{dx}{dt} = -\omega r \sin(\omega t + \alpha) - \frac{\omega r^2}{2l}\sin(2\omega t + 2\alpha) \quad . \quad \text{(II.42)}$$

and $$a = \frac{d^2x}{dt^2} = -\omega^2 r\cos(\omega t + \alpha) - \frac{\omega^2 r^2}{l}\cos(2\omega t + 2\alpha) \quad \text{(II.43)}$$

The reader will note that each term on the right of (II.43) is of the same type as that on the right-hand side of (II.35), the first term being the same as that of (II.35). The second term corresponds if we put $\frac{r^2}{l}$ for r and $2\omega t$ for ωt. Thus, the motion of the piston of the steam engine is approximately the sum of two simple harmonic motions of amplitudes r and $\frac{r^2}{l}$ respectively, and whose times of oscillation are

$$\frac{2\pi}{\omega} \text{ and } \tfrac{1}{2} \cdot \frac{2\pi}{\omega} = \frac{\pi}{\omega} \text{ respectively.}$$

40. **Change in Value of $f(x)$ due to a Small Change in x. Rates.** Referring again to Fig. 16, we see that if the given curve is the graph of $y = f(x)$, a small change Δx in the value of x produces a small change Δy in the value of y or $f(x)$. In order to obtain a value for Δy, we have

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt.} \frac{\Delta y}{\Delta x}$$

The relation $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ is approximately true for small values of Δx, as the difference between the two sides can be made as small as we please by taking Δx sufficiently small. Hence,

$$\Delta y = \frac{dy}{dx}\Delta x \quad . \quad . \quad . \quad . \quad \text{(II.44)}$$

may be taken as approximately true, the closeness of the value of Δy to its true value being greater the smaller Δx is made.

EXAMPLE 1

Find the changes in (1) $\sin x$, (2) $\tan x$, (3) x^3, and (4) $\cosh x$, caused by a small change Δx in the value of x.

(1) $y = \sin x$, $\frac{dy}{dx} = \cos x$. Hence, $\Delta y = \frac{dy}{dx} . \Delta x = \cos x . \Delta x$

(2) $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$. Hence, $\Delta y = \dfrac{dy}{dx} \cdot \Delta x = \sec^2 x \cdot \Delta x$

(3) $y = x^3$, $\dfrac{dy}{dx} = 3x^2$. Hence $\Delta y = \dfrac{dy}{dx} \cdot \Delta x = 3x^2 \cdot \Delta x$

(4) $y = \cosh x$, $\dfrac{dy}{dx} = \sinh x$. Hence $\Delta y = \dfrac{dy}{dx} \cdot \Delta x = \sinh x \cdot \Delta x$

EXAMPLE 2

Find the percentage errors caused in $\tan x$ and x^3 by a small error of r per cent in the value of x. From Ex 1 (2) and (3) we have

If $y = \tan x$, $\Delta y = \sec^2 x \cdot \Delta x$ and if e is the percentage error $e = 100 \times \dfrac{\Delta y}{y} = \dfrac{100 \sec^2 x \cdot \Delta x}{\tan x}$ and since $\Delta x = \dfrac{r}{100} \cdot x$, $e = \dfrac{\sec^2 x}{\tan x} \cdot rx$, or the percentage error is proportional to $\dfrac{\sec^2 x}{\tan x} \cdot x$.

If $y = x^3$, $\Delta y = 3x^2 \Delta x$ and $e = 100 \dfrac{\Delta y}{y} = \dfrac{300x^2 \Delta x}{x^3} = \dfrac{300 \Delta x}{x}$ and since $\Delta x = \dfrac{r}{100} \cdot x$, $e = 3r$, or the percentage error in x^3 is three times that in x.

Dividing each side of (II 44) by Δt, and making Δt approach the limit zero, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \text{ [See (II 16)]} \qquad \text{(II 45)}$$

By means of this relation, we are able to compare the rates of increase with respect to time (or any other variable) of two related quantities.

EXAMPLE 3

Find the ratio of the rates of increase of y and x with respect to time, if $y = \sin x$.

Applying (II 45) we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or

$$\frac{dy}{dt} = \cos x \cdot \frac{dx}{dt} \quad \text{i.e.} \quad \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \cos x$$

EXAMPLE 4

An airship is flying horizontally at a height of 1 000 ft with a speed relative to the land of 10 miles an hour. Find an expression for the rate at which it is receding from a fixed point on the ground over which it passed t hours ago, and find the rate when $t = \frac{1}{4}$.

Let y = distance in miles from point on ground to airship at any time. Then if x is the horizontal projection of y

$$x = 10t \text{ and}$$

$$y = \sqrt{x^2 + \left(\frac{1\,000}{5\,280}\right)^2}$$

and by (II 45)

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{x}{\sqrt{x^2 + \left(\frac{100}{528}\right)^2}} \cdot \frac{dx}{dt}$$

But $\frac{dy}{dt}$ is the required rate and $\frac{dx}{dt}$ is the speed of the airship Hence

$$\text{Required rate} = \frac{dy}{dt} = \frac{10x}{\sqrt{x^2 + \left(\frac{100}{528}\right)^2}} \text{ miles per hour}$$

$$= \frac{10t}{\sqrt{t^2 + \frac{100}{528^2}}} \text{ miles per hour}$$

When $t = \frac{1}{6}$,

$$\frac{dy}{dt} = \frac{10}{6\sqrt{\frac{1}{36} + \frac{100}{(528)^2}}}$$

$$= 9\,936$$

and the rate is 9 936 miles per hour

41 **Virtual Work or Virtual Velocities.** If a particle or a body is under the action of an unbalanced force F, or of a system of forces whose resultant is F the work done by F in any small displacement Δs is $F \cos\theta \cdot \Delta s$, where θ is the angle between the line of action of F and the direction of the displacement Δs The quantity $F \cos\theta \cdot \Delta s$ is called the *virtual work* of the force F If P is any one of the system of forces acting on the body, and its point of application moves through a small distance Δp in a direction making an angle ϕ with the line of action of P, the virtual work of P is $P \cos\phi \cdot \Delta p$, and it is proved in textbooks on mechanics that

$$F \cos\theta \cdot \Delta s = \Sigma P \cos\phi \cdot \Delta p,$$

the right-hand side representing the sum of the virtual works of all the forces acting on the body F is the resultant of the system of forces If the body is in equilibrium $F = 0$, and the above relation becomes

$$\Sigma P \cos\phi \cdot \Delta p = 0 \qquad \text{(II 46)}$$

This is the principle of virtual work which is of use in finding the relations between the forces acting on a body in equilibrium

EXAMPLE 1

Differentiate $\sin\theta$ with respect to θ from first principles. What is the differential coefficient of $\sqrt{1 - 4\sin^2\theta}$ with respect to θ?

$OABC$ is a freely-jointed frame of uniform heavy rods suspended from O. $OA = OC = 2$ ft, $AB = CB = 1$ ft, and AC is a strut of negligible weight, such that $\widehat{AOC} = 2\theta$. Supposing a small variation in θ, determine by virtual work the stress in AC if the frame weighs 20 lb. (U.L.)

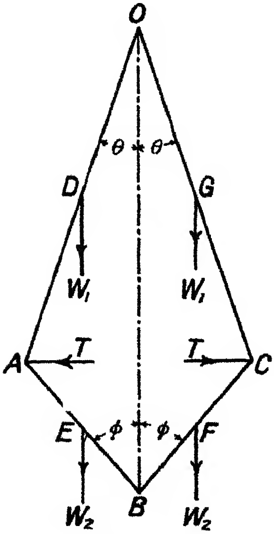


FIG. 20

We leave the first part of the example as an exercise for the reader. The frame is shown in Fig. 20. Since the strut has no weight, we can imagine it removed if we introduce two forces T lb weight, as shown, representing the thrust of its ends on A and C. The weights of the rods are represented by forces W_1 lb at each of the mid-points of OA and OC, and W_2 lb at the mid-points of AB and CB.

The wording of the example, though somewhat vague, indicates that as the weights are proportional to the lengths, $W_1 = 2W_2$. Let $\widehat{ABO} = \widehat{OBC} = \phi$. The external forces acting on $OABC$ are as follows—

(1) The supporting force at O, which does no work, since O is fixed.

(2) The weights W_1 at D and G. The depths of these points below O are $1 \times \cos\theta$ ft, and if θ increases by $\Delta\theta$, these depths increase by $\Delta(\cos\theta)$. Hence, the work of these two forces is $2W_1 . \Delta(\cos\theta)$.

(3) The weights at E and F. The depths of these points below O are $2\cos\theta + 0{\cdot}5\cos\phi$, and the work done by the weights is $2W_2\Delta(2\cos\theta + 0{\cdot}5\cos\phi)$.

(4) The two forces T acting at A and C. The distance of each of these points from OB is $2\sin\theta$, and the virtual work of these forces is therefore $2T\Delta(2\sin\theta)$. Since the system is in equilibrium, we have by (II.46)

$$2W_1 . \Delta(\cos\theta) + 2W_2\Delta(2\cos\theta + 0{\cdot}5\cos\phi) + 2T . \Delta(2\sin\theta) = 0$$

Now, by the sine rule

$$2\sin\theta = \sin\phi \text{ or } \cos\phi = \sqrt{1 - 4\sin^2\theta}$$

Hence, substituting and dividing by 2, we have

$$W_1\Delta(\cos\theta) + W_2\Delta(2\cos\theta + 0{\cdot}5\sqrt{1 - 4\sin^2\theta}) + T\Delta(2\sin\theta) = 0$$

and applying (II.44)

$$W_1\sin\theta . \Delta\theta - 2W_2\sin\theta . \Delta\theta - \frac{2W_2\sin\theta\cos\theta}{\sqrt{1 - 4\sin^2\theta}} . \Delta\theta + 2T\cos\theta . \Delta\theta = 0$$

Dividing out by $\Delta\theta$, and putting $W_1 = 2W_2 = 2W$

$$-2W\sin\theta - 2W\sin\theta - \frac{2W\sin\theta \, . \cos\theta}{\sqrt{1-4\sin^2\theta}} + 2T\cos\theta = 0$$

or
$$T = \frac{10}{3}\left\{2\tan\theta + \frac{\sin\theta}{\sqrt{1-4\sin^2\theta}}\right\} \text{ lb, since } 6W = 20 \text{ lb.}$$

EXAMPLE 2

In the simple engine mechanism of Fig. 2, if R lb is the total pressure of the piston rod on the crosshead, and M lb-ft is the couple resisting the rotation of the crankshaft, prove that, if there is no fluctuation of speed, $Rv = M\omega$, where v is the linear velocity in feet per second of the piston, and ω is the angular speed in radians per second of the crankshaft. Neglecting friction and inertia forces, the only forces acting on OPC are—

(1) The force at O, which does no work, since O is fixed.

(2) The couple M opposing the motion of the crank and crankshaft. This does work of amount $-M\Delta(\omega t + \alpha)$, the minus sign indicating that work is done against the couple.

(3) The pressure of the piston rod. This does work of amount $-R\Delta x$, the minus sign indicating that if R does work on the mechanism Δx is negative.

(4) The reaction of the guide-bar at C, which will act vertically upwards (in the absence of friction). This force does no work, as C does not move in a vertical direction.

The total work done is zero, since there is equilibrium, and therefore

$$-M\Delta(\omega t + \alpha) - R\Delta x = 0$$

$\therefore$
$$-M\omega \, . \Delta t - R\Delta x = 0$$

or
$$-\frac{\Delta x}{\Delta t} = \frac{M\omega}{R}$$

But
$$-\frac{dx}{dt} = \text{velocity of piston}$$
$$= v$$

$\therefore$
$$Rv = M\omega$$

EXAMPLES II

(1) Find from first principles the differential coefficients with regard to x of the following functions—

(i) $\dfrac{1}{\sqrt{x}}$ (ii) $\tan\dfrac{x}{2}$ (iii) $\sec x$ iv $\tan x$

Differentiate the following functions with regard to x—

(2) $\dfrac{3}{x^4} - \dfrac{5}{x} + 2\sqrt[3]{x} - 7x + 8$

(3) $4x^{\frac{2}{3}} - 5x^{-\frac{3}{4}}$

(4) $(x-3)^4$

(5) $\dfrac{6}{(x-1)^2}$

(6) $\sin(3x-4)$

(7) $\tan(2x+1)$

(8) $\sec 5x$

(9) $e^{3x} - \dfrac{3}{e^x}$

(10) $7e^{\frac{1}{3}x} - 2e^{-}$[illegible]

(11) $\log_e (x + a)$

(12) $\sinh 3x - 4 \cosh 2x$

(13) $\frac{1}{2} \tanh \frac{1}{2}x + \frac{1}{3} \coth \frac{1}{3}x$

(14) $2 \sin^{-1} (x - 1)$

(15) $\tan^{-1} (a + x)$

(16) $\tanh^{-1} 8x$

(17) $\cosh^{-1} \frac{1}{4}x$

(18) xe^{2x}

(19) $\dfrac{x^2}{\log_e x}$

(20) $\sqrt{x} \sin x$

(21) $\dfrac{\sin x}{1 + \cos x}$

(22) $(ax + b)(px + q)$

(23) $\dfrac{ax + b}{px + q}$

(24) $\dfrac{2x^2}{3x + 4}$

(25) $\dfrac{x}{\tan x}$

(26) $(2 + x)^3 (3 - x)^4$

(27) $10^x - x \log_e x$

(28) $\sin kx \cos lx$

(29) $\dfrac{\sin kx}{\cos lx}$

(30) $x^n \sin^{-1} px$

(31) $\tan^{-1} \dfrac{x}{\sqrt{1 - x^2}}$

(32) $\sqrt{px^2 + qx + r}$

(33) $\sqrt{\dfrac{a + x}{a - x}}$

(34) $\sqrt[3]{1 + x^2}$

(35) $x\sqrt{1 - x^2}$

(36) $\dfrac{x^2}{\sqrt{1 - x}}$

(37) $\sinh x \cosh x$

(38) a^{3x-4}

(39) $\operatorname{cosec}^{-1} \dfrac{1 + x^2}{2x}$

(40) $(p^2 - x^2)^{\frac{3}{2}}$

(41) $\dfrac{1}{\sqrt[3]{2x^2 - 5}}$

(42) $\sinh^2 x - \cosh^2 x$

(43) $\sinh^{-1} \dfrac{2}{x}$

(44) $e^{-x} \tan \frac{1}{2}x$

(45) $e^{4x} \sin (2x + 7)$

(46) $e^{\sqrt{x}}$

(47) $\log_e \dfrac{x + 1}{x - 1}$

(48) $\log_e \sqrt{5x^2 - 3x + 2}$

(49) $\log_e \sin x$

(50) $\log_e \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

(51) $\sin^3 kx \cos^2 lx$

(52) $\sqrt{p \cos^2 x + q \sin^2 x}$

(53) $\tan^{-1} \dfrac{\sin \alpha . \sin x}{\cos \alpha + \cos x}$

(54) $\dfrac{\cosh x + \cos x}{\sinh x - \sin x}$

(55) $x^4 e^{3x} \sin 2x$

(56) $\dfrac{\sqrt{x^2 + a^2} - x}{\sqrt{x^2 + a^2} + x}$

(57) $(x + 1)(2x - 3)(x - 2)^2$

(58) $\cosh^{-1} 3x$

(59) $\tanh^{-1} (\cot x)$

(60) $\sin^{-1} (\tanh x)$

(61) $e^{5x} \log_e \sqrt{1 + 2x}$

(62) $\log_e (x^2 \tan^{-1} x)$

(63) $x^2 \log_e (\sin^{-1} x)$

(64) $x \sin^{-1} x + \frac{1}{2} \log_e \frac{x+1}{x-1}$

(65) $\log_e (\tan x^2)$

(66) $\frac{x}{\sqrt{1+x^2} - x}$

(67) Find y_n when (1) $y = e^{3x} \sin 4x$. (2) $y = \frac{5}{(x+2)(3x-2)}$

(68) Using Leibnitz' theorem, prove that if $x = \cos pt \cos qt$

$$\frac{d^n x}{dt^n} = \frac{(p+q)^n}{2} \cos \left[(p+q)t + \frac{n\pi}{2}\right] + \frac{(p-q)^n}{2} \cos \left[(p-q)t + \frac{n\pi}{2}\right]$$

(69) Prove that $\frac{d^4}{dx^4}(e^{ax} \sin bx) = r^4 e^{ax} \sin (bx + 4\theta)$ where $r^2 = a^2 + b^2$ and $\tan \theta = \frac{b}{a}$.

Show that $y = A \sin (pt + \alpha) + B \sin (2pt + \beta)$ satisfies the equation $\frac{d^4 y}{dt^4} + 5p^2 \frac{d^2 y}{dt^2} + 4p^4 y = 0$, for all values of the constants A, B, α and β. (U.L.)

(70) Verify that the solution of the equation $\frac{d^2 y}{dt^2} + n^2 y = a \sin pt$ is $y = \frac{a}{n^2 - p^2} \sin pt + K \sin (nt + \alpha)$, where K and α are arbitrary constants. If $n = 2p$, find the values of K and α in the case for which y and $\frac{dy}{dt}$ are both zero for $t = 0$. Draw the graph of y for values of t from 0 to $2\pi/p$. (U.L.)

(71) Define a differential coefficient. If $y = f(z)$ where $z = F(x)$, find from your definition $\frac{dv}{dx}$.

A conical vessel with its axis vertical and vertex upwards is being filled at a uniform rate with liquid through a hole at the vertex. Prove that when the vessel is eight-ninths full the surface of the liquid is rising four times as fast as when the vessel is one-ninth full. (U.L.)

(72) A body moves along a straight line and its displacement x from a fixed point O on the line in time t is given by the equation $x = A \sin pt + B \cos pt$, where A, B, p are constants. Show that the acceleration of the body is directed towards O and is proportional to the displacement from O. Express x in the form $x = R \sin (pt + q)$ and find an expression for the periodic time.

(73) If x and y are both functions of a third variable t, show that $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

A crank OQ rotates round O with constant angular velocity ω, and a connecting-rod QP is hinged to it at one end Q, while the other end P moves along a fixed straight line OX. If PQ meets the perpendicular to OX through O in R, prove that the angular velocity of PQ is proportional to QR and that the velocity of P is proportional to OR. (U.L.)

(74) Differentiate $\sin \frac{\theta}{2}$ with regard to θ from first principles. A heavy uniform rod of weight W is suspended symmetrically and in a horizontal position by two

vertical strings of length l distant $2a$ apart, a couple of moment M is applied twisting the bar about the vertical axis of symmetry through an angle θ; show by virtual work (or otherwise) that

$$M = \frac{Wa^2 \sin\theta}{\sqrt{l^2 - 4a^2 \sin^2 \dfrac{\theta}{2}}} \qquad \text{(U.L.)}$$

(75) Determine the differential coefficient of $\cot\theta$ from first principles.

A frame ABC consists of three light rods, of which AB, AC, are each of length $3a$, and BC of length $4a$, freely jointed together; it rests with BC horizontal, A below BC, and the rods AB, AC, over two smooth pegs D and E, in the same horizontal line at distance $3a$ apart. A weight W is suspended from A. If $BAC = 2\theta$, determine the stress in BC by supposing a small variation in θ. (U.L.)

(76) Differentiate with respect to θ the function $\dfrac{4\sin\theta}{2 + \cos\theta} - \theta$, and deduce that in the range $\theta = 0$ to $\theta = \dfrac{\pi}{2}$ the function increases with θ. Show also that the function is positive in this range, and does not exceed 1 in value.

(77) If $y = \dfrac{1}{\sqrt{1 - x^2}} \sin^{-1} x$, prove that $(1 - x^2)\dfrac{dy}{dx} = xy + 1$.

By applying Leibnitz' theorem show that

$$(1 - x^2)\frac{d^{n+1}y}{dx^{n+1}} - (2n + 1)x\frac{d^n y}{dx^n} - n^2 \frac{d^{n-1}y}{dx^{n-1}} = 0$$

Hence, find the value of $\dfrac{d^{n+1}y}{dx^{n+1}}$ when $x = 0$. (U.L.)

(78) Six equal rods each of weight W, freely hinged at the ends, form a regular hexagon $ABCDEF$, which when suspended by the point A, is kept from altering its shape by two light rods BF, CE. Employ the principle of virtual work to find the thrusts in the rods BF, CE. (U.L.)

(79) Four equal rods, each of weight W, freely hinged at the ends, form a rhombus $ABCD$. The frame is suspended from A, and is kept from altering its shape by a light rod which joins E and F, the mid-points of AB and AD respectively. Find the thrust in the rod EF when a weight $3W$ is hung from C, given that the angle $BAD = 2\theta$.

(80) The angular elevation of the top of a vertical flagstaff at a point on the ground d ft from the foot of the flagstaff is θ. Find the error in the height h of the flagstaff due to an error $\Delta\theta$ in the angular elevation.

(81) Two sides a and b and the included angle C of a triangle ABC are measured. Find the error in the area as calculated from these measurements if there is an error ΔC in the angle C.

(82) If $y = \sinh x$, find the error in y due to an error Δx in x. Hence, find $\sinh 2{\cdot}705$, given that $\sinh 2{\cdot}7 = 7{\cdot}4063$.

(83) A rod OP rotates about O in one plane with constant angular velocity ω radians per second. If OX is a fixed straight line through O in the plane of

motion and angle $XOP = \theta$, compare the rates of increase of $\tan\theta$ and $\sin\theta$ when (1) $\theta = 0$, (2) $\theta = 60^\circ$, (3) $\theta = 120^\circ$.

(84) A train travelling at 50 miles per hour reaches a bridge at the same time as a motor-car travelling at 30 miles per hour on the road over the bridge. Assuming that the road and the rail track are straight and at right angles to each other, find the rate at which the train and the motor-car are separating 6 minutes after leaving the bridge. [Neglect difference of levels.]

INTEGRATION

42. Integration the Converse of Differentiation. In Chapter II we were concerned with finding the differential coefficient of a given function $f(x)$. We have now to consider the converse problem, which we may state as follows—

Let $F(x)$ be a given function. It is required to determine a function $f(x)$ such that $f'(x) = F(x)$.

There will be a solution to this problem provided that $F(x)$ is continuous. The process of forming $f(x)$ from $F(x)$ is called *integration*, and $f(x)$ is called the *indefinite integral* of $F(x)$. Symbolically, we write $\int F(x)\,dx = f(x)$. The reader is advised to regard $\int \ . \ . \ . \ dx$, for the present at any rate, as merely a symbol denoting the process of integrating with respect to x.

Working with the differential coefficients tabulated in Art. 35 we can build up a list of standard integrals. Thus, for example, $\frac{d}{dx}(x^n) = nx^{n-1}$, and hence $\int nx^{n-1}\,dx = x^n$; but in this case we obtain a more convenient form by considering $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, which gives us $\int x^n dx = \frac{x^{n+1}}{n+1}$. The addition of an arbitrary constant C to $\frac{x^{n+1}}{n+1}$ does not alter the differential coefficient x^n, so that to be quite general we should write $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. This constant of integration, as it is termed, should appear in every indefinite integral, although as a matter of convenience we shall usually omit it. There is an exceptional case to be considered with regard to the formula just obtained for $\int x^n dx$, that in which $n = -1$, for with this value of n, $x^{n+1}/(n+1)$ gives $x^0/0$, a result which is incorrect. We saw in Art. 26 why the above rule does not apply in this case.

We have already found, however, that $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, and therefore $\int \frac{1}{x}\,dx = \log_e x$.

Our first formulae in the standard list will, then, be as follows:

EXAMPLES

$$\int \frac{6x^2 - 7}{2x^3 - 7x + 4}\,dx = \log_e (2x^3 - 7x + 4)$$

$$\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\log_e \cos x = \log_e \sec x$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log_e \sin x$$

$$\int \frac{8x - 3}{\sqrt{4x^2 - 3x + 9}}\,dx = 2\sqrt{4x^2 - 3x + 9}$$

44. Use of Partial Fractions in Integrating Rational Algebraic Fractions. For a detailed discussion of the principles underlying partial fractions the reader is referred to one of the standard textbooks on algebra. It will be sufficient here to give a brief sketch of the methods adopted in the more important cases that arise.

Let $\dfrac{\psi(x)}{\phi(x)}$ be a rational algebraic fraction, the degree of $\psi(x)$ being assumed less than that of $\phi(x)$. Then, corresponding to any linear factor of $\phi(x)$, we have a partial fraction of the form $\dfrac{A}{x-a}$; corresponding to any repeated linear factor of $\phi(x)$, say $(x-e)^m$, we have the sum of the partial fractions $\dfrac{E_1}{x-e} + \dfrac{E_2}{(x-e)^2} + \ldots + \dfrac{E_m}{(x-e)^m}$; corresponding to any quadratic factor, say $x^2 + px + q$, we have a partial fraction of the form $\dfrac{Px+Q}{x^2+px+q}$.

Thus, if $\phi(x) \equiv (x-a)(x-e)^m(x^2+px+q)\ldots$, then

$$\frac{\psi(x)}{\phi(x)} \equiv \frac{A}{x-a} + \frac{E_1}{x-e} + \frac{E_2}{(x-e)^2} + \ldots + \frac{E_m}{(x-e)^m} + \frac{Px+Q}{x^2+px+q} + \ldots \qquad \text{(III.2)}$$

Now $\displaystyle\int \frac{A}{x-a}\,dx = A\log_e(x-a)$; $\displaystyle\int \frac{E_1}{x-e}\,dx = E_1\log_e(x-e)$;

$$\int \frac{E_r}{(x-e)^r}\,dx \text{ (when } r > 1) = E_r\int (x-e)^{-r}\,dx = E_r\frac{(x-e)^{-r+1}}{-r+1}.$$

We shall consider the form $\displaystyle\int \frac{Px+Q}{x^2+px+q}\,dx$ in the next article.

EXAMPLE

Find (1) $\displaystyle\int \frac{x^2}{(x-1)(x+2)(2x-3)}\,dx.$ (2) $\displaystyle\int \frac{x^3-x-1}{x(x+3)^2}\,dx.$

(1) Let $$\frac{x^2}{(x-1)(x+2)(2x-3)} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{2x-3}$$

To find A, put $x = 1$ in the expression on the left, neglecting the factor $x - 1$.

$$\therefore \qquad A = \frac{1}{(3)(-1)} = -\frac{1}{3}$$

By substituting $x = -2$ and $x = \frac{3}{2}$ similarly, we obtain $B = \dfrac{4}{(-3)(-7)} = \dfrac{4}{21}$; $C = \dfrac{\frac{9}{4}}{(\frac{1}{2})(\frac{7}{2})} = \dfrac{9}{7}$.

$$\therefore \text{ Integral} = -\frac{1}{3}\int\frac{dx}{x-1} + \frac{4}{21}\int\frac{dx}{x+2} + \frac{9}{7}\int\frac{dx}{2x-3}$$

$$= -\frac{1}{3}\log_e(x-1) + \frac{4}{21}\log_e(x+2) + \frac{9}{14}\log_e(2x-3) + \text{constant}$$

(2) Since the numerator is not of lower degree than the denominator we divide out and obtain

$$\frac{x^3-x-1}{x(x+3)^2} = 1 - \frac{6x^2+10x+1}{x(x+3)^2}$$

Let $$\frac{6x^2+10x+1}{x(x+3)^2} \equiv \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\therefore \qquad 6x^2 + 10x + 1 \equiv A(x+3)^2 + Bx(x+3) + Cx$$

Since this is an identity we can equate the coefficients of like powers of x on either side, or give x any convenient values. Using the latter method—

Let $x = 0$, $\quad \therefore 1 = 9A$, $\quad \therefore A = \dfrac{1}{9}$

Let $x = -3$, $\quad \therefore 54 - 30 + 1 = -3C$, $\quad \therefore C = -\dfrac{25}{3}$

Let $x = -1$, $\quad \therefore 6 - 10 + 1 = 4A - 2B - C$.

$$\therefore \qquad 2B = \frac{4}{9} + \frac{25}{3} + 3 = \frac{106}{9}, \qquad \therefore B = \frac{53}{9}$$

$$\therefore \text{ Integral} = \int 1\,dx - \frac{1}{9}\int\frac{dx}{x} - \frac{53}{9}\int\frac{dx}{x+3} + \frac{25}{3}\int\frac{dx}{(x+3)^2}$$

$$= x - \frac{1}{9}\log_e x - \frac{53}{9}\log_e(x+3) - \frac{25}{3(x+3)} + \text{constant}$$

45. Integrals of the Types

$$(a)\ \int\frac{Px+Q}{x^2+px+q}\,dx \qquad (b)\ \int\frac{Px+Q}{\sqrt{x^2+px+q}}\,dx$$

$$(c) \int \frac{Px + Q}{\sqrt{q + px - x^2}}\, dx$$

(a) $$\int \frac{2x + p}{x^2 + px + q}\, dx = \log_e (x^2 + px + q) \text{ [by (III.1)]}$$

Accordingly we write

$$\int \frac{Px + Q}{x^2 + px + q}\, dx = \int \frac{\frac{P}{2}(2x + p) + Q - \frac{Pp}{2}}{x^2 + px + q}\, dx$$

$$= \frac{P}{2} \int \frac{2x + p}{x^2 + px + q}\, dx + \left(Q - \frac{Pp}{2}\right) \int \frac{dx}{x^2 + px + q} \quad \text{(III.3)}$$

The first integral on the right is known; we have now to consider

$$\int \frac{dx}{x^2 + px + q}$$

By "completing the square" we can express $x^2 + px + q$ in the form $(x + a)^2 + \beta^2$ or $(x + a)^2 - \beta^2$.

Let $\qquad z = x + a, \quad \therefore x = z - a$ and $\dfrac{dx}{dz} = 1$

[If $I = \int f(x)\, dx$ so that $\dfrac{dI}{dx} = f(x)$ and the variable is changed from x to z, so that $f(x)$ becomes $\phi(z)$, then

$$\frac{dI}{dz} = \frac{dI}{dx}\frac{dx}{dz} = f(x)\frac{dx}{dz} = \phi(z)\frac{dx}{dz}$$

Integrating with respect to z, we have

$$I = \int \phi(z) \frac{dx}{dz}\, dz$$

Hence, when changing the variable from x to z, we replace $f(x)$ by $\phi(z)$ and dx by $\dfrac{dx}{dz}\, dz$. This *method of substitution* of a new variable is frequently used in order to simplify integration.]

Hence, $$\int \frac{dx}{(x + a)^2 + \beta^2} = \int \frac{dz}{z^2 + \beta^2}$$

$$= \frac{1}{\beta} \tan^{-1} \frac{z}{\beta} \text{ (see standard integrals)}$$

$$= \frac{1}{\beta} \tan^{-1} \frac{x + a}{\beta} \quad . \quad . \quad . \quad \text{(III.4)}$$

Also
$$\int \frac{dx}{(x+a)^2-\beta^2} = \int \frac{dz}{z^2-\beta^2} = \frac{1}{2\beta}\log_e \frac{z-\beta}{z+\beta}$$
$$= \frac{1}{2\beta}\log_e \frac{x+a-\beta}{x+a+\beta} \qquad \text{(III 5)}$$

NOTE If x^2+px+q is resolvable into two linear factors, we use the method of partial fractions as in Art 44

(*b*)
$$\int \frac{2x+p}{\sqrt{x^2+px+q}}\,dx = 2\sqrt{x^2+px+q} \text{ [by Art 43]}$$

Accordingly we write
$$\int \frac{Px+Q}{\sqrt{x^2+px+q}}\,dx = \int \frac{\frac{P}{2}(2x+p)+Q-\frac{Pp}{2}}{\sqrt{x^2+px+q}}\,dx$$
$$= \frac{P}{2}\int \frac{2x+p}{\sqrt{x^2+px+q}}\,dx + \left(Q-\frac{Pp}{2}\right)\int \frac{dx}{\sqrt{x^2+px+q}} \qquad \text{(III 6)}$$

The first integral on the right is known, the second we have now to consider

As already indicated $x^2+px+q = (x+a)^2 \pm \beta^2 = z^2 \pm \beta^2$, where $z = x+a$

Now
$$\int \frac{dz}{\sqrt{z^2+\beta^2}} = \sinh^{-1}\frac{z}{\beta} = \sinh^{-1}\frac{x+a}{\beta}$$

or
$$\log_e \frac{x+a+\sqrt{(x+a)^2+\beta^2}}{\beta} \qquad \text{(III 7)}$$

and
$$\int \frac{dz}{\sqrt{z^2-\beta^2}} = \cosh^{-1}\frac{z}{\beta} = \cosh^{-1}\frac{x+a}{\beta}$$

or
$$\log_e \frac{x+a+\sqrt{(x+a)^2-\beta^2}}{\beta} \qquad \text{(III 8)}$$

(*c*) By similar reasoning we write
$$\int \frac{Px+Q}{\sqrt{q+px-x^2}}\,dx = \int \frac{-\frac{P}{2}(-2x+p)+Q+\frac{Pp}{2}}{\sqrt{q+px-x^2}}\,dx$$
$$= -\frac{P}{2}\int \frac{p-2x}{\sqrt{q+px-x^2}}\,dx + \left(Q+\frac{Pp}{2}\right)\int \frac{dx}{\sqrt{q+px-x^2}}$$
$$= -P\sqrt{q+px-x^2} + \left(Q+\frac{Pp}{2}\right)\int \frac{dx}{\sqrt{q+px-x^2}} \qquad \text{(III.9)}$$

To find $\int \sqrt{\beta^2 - z^2}\, dz$

Let $z = \beta \sin\theta, \qquad dz = \beta \cos\theta\, d\theta$

Integral $= \int \beta\cos\theta \cdot \beta\cos\theta\, d\theta$

$$= \frac{\beta^2}{2}\int 2\cos^2\theta\, d\theta$$

$$= \frac{\beta^2}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{\beta^2}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]$$

$$= \frac{\beta^2}{2}\left[\sin^{-1}\frac{z}{\beta} + \frac{z\sqrt{\beta^2 - z^2}}{\beta^2}\right] \qquad \text{(III 13)}$$

EXAMPLE

Find (1) $\int \sqrt{3x^2 - 4x + 5}\, dx$ (2) $\int \sqrt{6 - 3x - x^2}\, dx$

(1) $3x^2 - 4x + 5 = 3(x^2 - \frac{4}{3}x + \frac{5}{3}) = 3[(x - \frac{2}{3})^2 + \frac{11}{9}]$

Let $x - \frac{2}{3} = \frac{\sqrt{11}}{3}\sinh\theta, \qquad dx = \frac{\sqrt{11}}{3}\cosh\theta\, d\theta$

Integral $= \sqrt{3}\int \frac{\sqrt{11}}{3}\cosh\theta \cdot \frac{\sqrt{11}}{3}\cosh\theta\, d\theta$

$$= \frac{11\sqrt{3}}{18}\int 2\cosh^2\theta\, d\theta$$

$$= \frac{11\sqrt{3}}{18}\left[\sinh^{-1}\frac{x - \frac{2}{3}}{\frac{\sqrt{11}}{3}} + \frac{(x - \frac{2}{3})\sqrt{x^2 - \frac{4}{3}x + \frac{5}{3}}}{\frac{11}{9}}\right]$$

$$= \frac{11\sqrt{3}}{18}\left[\sinh^{-1}\frac{3x - 2}{\sqrt{11}} + \frac{\sqrt{3}}{11}(3x - 2)\sqrt{3x^2 - 4x + 5}\right]$$

(2) $6 - 3x - x^2 = \frac{33}{4} - \left(x + \frac{3}{2}\right)^2$

Let $x + \frac{3}{2} = \frac{\sqrt{33}}{2}\sin\theta \qquad dx = \frac{\sqrt{33}}{2}\cos\theta\, d\theta$

Integral $= \int \frac{\sqrt{33}}{2}\cos\theta \cdot \frac{\sqrt{33}}{2}\cos\theta\, d\theta$

$$= \frac{33}{8}\int 2\cos^2\theta\, d\theta$$

$$= \frac{33}{8}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]$$

$$= \frac{33}{8}\left[\sin^{-1}\frac{2x + 3}{\sqrt{33}} + \frac{2(2x + 3)\sqrt{6 - 3x - x^2}}{33}\right]$$

47. Definite Integrals. Fig. 21 shows part of the graph of the continuous function $y = f(x)$. We consider the area included by the curve, the x-axis, and the ordinates AK, BL at $x = a$, $x = b$ respectively. Ordinates PM and QN are drawn through any point $P(x, y)$ on the curve between A and B, and a neighbouring point $Q(x + \Delta x, y + \Delta y)$. Let the area $AKMP$ be denoted by S, and hence the area $PMNQ$ by ΔS. Since we assume Δx small, we can regard $PMNQ$ as a trapezium.

$$\therefore \qquad \Delta S = \tfrac{1}{2}(y + y + \Delta y) \,.\, \Delta x$$

$$\therefore \qquad \frac{\Delta S}{\Delta x} = y + \tfrac{1}{2}\Delta y$$

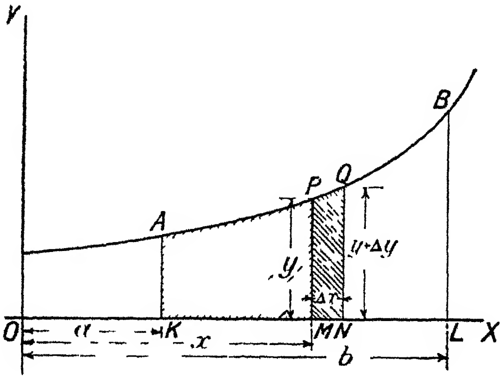


FIG. 21

In the limit when Δx (and, therefore, Δy) tends to zero, we have $\dfrac{dS}{dx} = y = f(x)$.

Integrating, $S = \int f(x)dx = \psi(x) + C$, where $\dfrac{d}{dx}\psi(x) = f(x)$ and C is a constant.

To find C we note that when P is at A, i.e. when $x = a$, $S = 0$; hence, $0 = \psi(a) + C$, $\therefore\ C = -\psi(a)$.

$$\therefore \qquad S = \psi(x) - \psi(a)$$

Suppose now that P moves up to B; then x becomes equal to b. Hence,

$$\text{Area} = S = \psi(b) - \psi(a) \qquad . \qquad . \ \text{(III.14)}$$

The value of $\int f(x)dx$ between $x = a$ and $x = b$ is then $\psi(b) - \psi(a)$, where $\psi(x)$ is the indefinite integral of $f(x)$. We write

$$\int_a^b f(x)dx = \psi(b) - \psi(a) \qquad . \qquad . \text{(III.15)}$$

the integral being now *definite*, as there is no arbitrary constant in the result.

It follows that in order to evaluate a definite integral $\int_a^b f(x)dx$ we must find $\psi(x)$ and then substitute in order in $\psi(x)$ the values $x = b$ (the upper limit) and $x = a$ (the lower limit), and finally we must subtract the second result from the first. No arbitrary constant appears in this result.

If when b is increased indefinitely (or a is decreased indefinitely) the value of $\int_a^b f(x)dx$ tends to some definite limit, the function $f(x)$ remaining finite and continuous over the range $x = a$ to $x = b$, then we denote this limit by $\int_a^\infty f(x)dx \left(\text{or } \int_{-\infty}^b f(x)dx\right)$.

Fundamentally, every definite integral denotes an area, although the notion of area may not be present in the problem which gives rise to the integral.

EXAMPLE 1

Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^2+9}$ and $\displaystyle\int_2^3 \frac{x^2-2}{x(x-1)}dx$

$$\int_0^\infty \frac{dx}{x^2+3^2} = \frac{1}{3}\left(\tan^{-1}\frac{x}{3}\right)_0^\infty = \frac{1}{3}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{1}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{6}$$

$$\int_2^3 \frac{x^2-2}{x(x-1)}dx = \int_2^3 \frac{x^2 - x + x - 2}{x(x-1)}dx = \int_2^3 1dx + \int_2^3 \frac{x-2}{x(x-1)}dx$$

$$= \Big[x\Big]_2^3 + \int_2^3 \left[\frac{2}{x} - \frac{1}{x-1}\right]dx$$

$$= (3-2) + \Big[2\log_e x - \log_e (x-1)\Big]_2^3$$

$$= 1 + \left[\log_e \frac{x^2}{x-1}\right]_2^3$$

$$= 1 + \log_e 4{\cdot}5 - \log_e 4$$

$$= 1 + \log_e 1{\cdot}125 = 1{\cdot}1178$$

EXAMPLE 2

Evaluate $\int_0^{2\pi} \sin nx\,dx$ and $\int_0^{2\pi} \cos nx\,dx$ (n being an integer)

$$\int_0^{2\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_0^{2\pi} = -\frac{1}{n}(\cos 2n\pi - \cos 0)$$

$$= -\frac{1}{n}(1-1) = 0 \quad . \quad . \quad \text{(III.16)}$$

$$\int_0^{2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^{2\pi} = \frac{1}{n}(\sin 2n\pi - \sin 0)$$

$$= \frac{1}{n}(0-0) = 0 \quad . \quad . \quad \text{(III.17)}$$

48. **Integration of Trigonometrical Functions.** The integrals of $\sin x$, $\cos x$, $\tan x$, and $\cot x$ have already been found; there remain cosec x and $\sec x$ to be considered.

Since $\sin x = 2 \sin\frac{x}{2}\cdot\cos\frac{x}{2} = 2\tan\frac{x}{2}\cdot\cos^2\frac{x}{2} = \dfrac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$, and $d\left(\tan\frac{x}{2}\right) = \frac{1}{2}\sec^2\frac{x}{2}\,dx$, then

$$\int \operatorname{cosec} x\,dx = \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}}\,dx = \int \frac{d\left(\tan\frac{x}{2}\right)}{\tan\frac{x}{2}} = \log_e\left(\tan\frac{x}{2}\right) \quad \text{(III.18)}$$

Also $\int \sec x dx = \int \operatorname{cosec}\left(\frac{\pi}{2} + x\right) dx = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ (III.19)

As in Ex. 3, Art. 37, we express $a \sin x + b \cos x$ in the form $R \sin(x + \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$; hence,

$$\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{R}\int \operatorname{cosec}(x+\alpha)\,dx$$

$$= \frac{1}{R}\log_e \tan\frac{x+\alpha}{2}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log_e \tan \tfrac{1}{2}\left(x + \tan^{-1}\frac{b}{a}\right) \quad \text{(III.20)}$$

EXAMPLE

$$\int \frac{dx}{2\sin x - \cos x} = \frac{1}{\sqrt{4+1}}\int \operatorname{cosec}(x - \tan^{-1}\tfrac{1}{2})\,dx$$

$$= \frac{1}{\sqrt{5}}\log_e \tan \tfrac{1}{2}(x - \tan^{-1}\tfrac{1}{2})$$

A similar method gives

$$\int \frac{dx}{(a \sin x \pm b \cos x)^2} = \frac{1}{R^2} \int \frac{dx}{\sin^2 (x \pm \alpha)}$$
$$= \frac{1}{R^2} \int \operatorname{cosec}^2 (x \pm \alpha)\, dx$$
$$= -\frac{\cot\left(x \pm \tan^{-1} \dfrac{b}{a}\right)}{a^2 + b^2} \quad . \quad . \quad \text{(III.21)}$$

The integrals $\int \frac{dx}{a \sin x + b}$ and $\int \frac{dx}{a \cos x + b}$ can be found as follows—

Let $t = \tan \frac{x}{2}$, then, as above, $\sin x = \dfrac{2 \tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = \dfrac{2t}{1 + t^2}$,

and $\cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{1 + t^2} - 1 = \frac{1 - t^2}{1 + t^2}$. Also $dt = \frac{1}{2}(1 + t^2)dx$

Hence, $$\int \frac{dx}{a \sin x + b} = \int \frac{\dfrac{2dt}{1 + t^2}}{\dfrac{2at}{1 + t^2} + b} = 2 \int \frac{dt}{bt^2 + 2at + b} \quad . \quad . \quad \text{(III.22)}$$

and $$\int \frac{dx}{a \cos x + b} = \int \frac{\dfrac{2dt}{1 + t^2}}{\dfrac{a(1 - t^2)}{1 + t^2} + b} = 2 \int \frac{dt}{(b - a)t^2 + (b + a)} \quad \text{(III.23)}$$

We have thus reduced the integrals to forms already dealt with. The actual evaluation will depend on the relative magnitudes of a and b.

EXAMPLE

Find $\int \frac{dx}{3 \sin x + 4}$ and $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{4 - 5 \cos x}$

Putting $t = \tan \frac{x}{2}$, $dt = \frac{1}{2} (1 + t^2)\, dx$, we have

$$\int \frac{dx}{3 \sin x + 4} = 2 \int \frac{dt}{4t^2 + 6t + 4} = \frac{1}{2} \int \frac{dt}{(t + \frac{3}{4})^2 + \frac{7}{16}}$$
$$= \frac{1}{2} \frac{4}{\sqrt{7}} \tan^{-1} \left[\frac{4(t + \frac{3}{4})}{\sqrt{7}}\right]$$
$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{x}{2} + 3}{\sqrt{7}}\right)$$

With the same substitution,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{4 - 5\cos x} = 2\int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{9t^2 - 1} \left(\text{since } t = \frac{1}{\sqrt{3}}, 1 \text{ when } x = \frac{\pi}{3}, \frac{\pi}{2}\right)$$

$$= \frac{2}{3}\int_{\frac{1}{\sqrt{3}}}^{1} \frac{d(3t)}{(3t)^2 - 1} = \frac{2}{3} \cdot \frac{1}{2}\left[\log \frac{3t - 1}{3t + 1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \frac{1}{3}\left[\log \frac{2}{4} - \log \frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right]$$

$$= \frac{1}{3}\log \frac{\sqrt{3} + 1}{2(\sqrt{3} - 1)}$$

$$= \frac{1}{3}\log_e\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1}{3}\log_e 1.866$$

$$= 0.208 \text{ nearly}$$

To obtain $\displaystyle\int \frac{dx}{a^2 \sin^2 x \pm b^2 \cos^2 x}$ we change the variable from x to $a \tan x$ as follows

Let
$$I = \int \frac{dx}{a^2 \sin^2 x \pm b^2 \cos^2 x} = \int \frac{\sec^2 x \, dx}{a^2 \tan^2 x \pm b^2}$$

$$= \frac{1}{a}\int \frac{d(a \tan x)}{(a \tan x)^2 \pm b^2}$$

With the plus sign before b^2, we obtain

$$I = \frac{1}{ab}\tan^{-1}\left(\frac{a \tan x}{b}\right) \tag{III.24}$$

and with the minus sign

$$I = \frac{1}{2ab}\log_e\left(\frac{a \tan x - b}{a \tan x + b}\right) \tag{III.25}$$

EXAMPLE

$$\int_0^{\pi} \frac{dx}{\sin^2 x + 4\cos^2 x} = \int_0^{\pi} \frac{d(\tan x)}{\tan^2 x + 4}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{\tan x}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{\tan \pi}{2}\right) - \tan^{-1}\left(\frac{\tan 0}{2}\right)\right]$$

$$= \frac{1}{2}[\pi - 0] = \frac{\pi}{2}$$

49 Integration of Powers of Sin x and Cos x. Since

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

we have—

$$\int \sin^2 x\, dx = \tfrac{1}{2}\int (1 - \cos 2x)dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) \qquad \text{(III 26)}$$

$$\int \cos^2 x\, dx = \tfrac{1}{2}\int (1 + \cos 2x)dx = \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x) \qquad \text{(III 27)}$$

EXAMPLE

Find $\int_0^{\pi} \sin^2 pt\, dt$ and $\int_0^{\pi} \cos^2 pt\, dt$ where p is an integer

$$\int_0^{\pi} \sin^2 pt\, dt = \tfrac{1}{2}\left[t - \frac{1}{2p}\sin 2pt\right]_0^{\pi} = \frac{\pi}{2} \qquad \text{(III 28)}$$

$$\int_0^{\pi} \cos^2 pt\, dt = \int_0^{\pi} (1 - \sin^2 pt)\, dt = \Big[t\Big]_0^{\pi} - \frac{\pi}{2} = \pi - \frac{\pi}{2} = \frac{\pi}{2} \qquad \text{(III 29)}$$

In Art. 54 we establish reduction formulae for $\int \sin^m x \cos^n x\, dx$, where m and n are positive integers, but when m and n are not inconveniently large, we can proceed as follows—

(1) If either m or n is odd, then, whether the other be odd or even, we can express $\int \sin^m x \,.\, \cos^n x\, dx$ as

$$\int \sin^{m-1} x \,.\, \cos^n x \,.\, \sin x\, dx$$

i.e.
$$-\int (1 - \cos^2 x)^{\frac{m-1}{2}} \cos^n x\, d(\cos x),$$

where m is odd; and as

$$\int \sin^m x \,.\, \cos^{n-1} x \,.\, \cos x\, dx$$

i.e.
$$\int \sin^m x \,.\, (1 - \sin^2 x)^{\frac{n-1}{2}}\, d(\sin x),$$

where n is odd.

EXAMPLE

Find $\int \sin^3 x \cos^2 x\, dx$ and $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$

$$\int \sin^3 x \cos^2 x\, dx = \int (1 - \cos^2 x) \cos^2 x \sin x\, dx$$
$$= -\int (\cos^2 x - \cos^4 x) d(\cos x)$$
$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)^2 \cos x\, dx$$
$$= \int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x + \sin^4 x)\, d(\sin x)$$

$$= \left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_0^{\frac{\pi}{2}}$$

$$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$$

(2) If m and n are both even integers, we transform $\sin^m x \cdot \cos^n x$ into an expression containing sines and cosines of multiple angles. The examples worked out below will illustrate the method.

EXAMPLE

Find $\int_0^\pi \sin^2 x \cos^4 x \, dx$ and $\int \sin^4 x \, dx$

$$\int_0^\pi \sin^2 x \cos^4 x \, dx = \int_0^\pi (\sin x \cdot \cos x)^2 \cos^2 x \, dx$$

$$= \int_0^\pi \frac{\sin^2 2x}{4} \cdot \frac{1 + \cos 2x}{2} dx$$

$$= \frac{1}{8}\left[\int_0^\pi \sin^2 2x \, dx + \frac{1}{2}\int_0^\pi \sin^2 2x \, d(\sin 2x)\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}\int_0^\pi (1 - \cos 4x)\, dx + \frac{1}{6}\left[\sin^3 2x\right]_0^\pi\right]$$

$$= \frac{1}{16}\left[x - \frac{1}{4}\sin 4x\right]_0^\pi + 0$$

$$= \frac{\pi}{16}$$

$$\int \sin^4 x \, dx = \int\left(\frac{1 - \cos 2x}{2}\right)^2 dx = \tfrac{1}{4}\int(1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \tfrac{1}{4}(x - \sin 2x) + \tfrac{1}{8}\int(1 + \cos 4x)\, dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x$$

$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x$$

50. Integration of the Products of Sines and Cosines of Multiple Angles.

$$\sin px \cdot \cos qx = \tfrac{1}{2}[\sin(p+q)x + \sin(p-q)x]$$
$$\sin px \cdot \sin qx = -\tfrac{1}{2}[\cos(p+q)x - \cos(p-q)x]$$
$$\cos px \cdot \cos qx = \tfrac{1}{2}[\cos(p+q)x + \cos(p-q)x]$$

Hence,

$$\int \sin px \cdot \cos qx \, dx = \tfrac{1}{2}\left[\int \sin(p+q)x \, dx + \int \sin(p-q)x \, dx\right]$$

$$= -\frac{\cos(p+q)x}{2(p+q)} - \frac{\cos(p-q)x}{2(p-q)} \qquad \text{(III.30)}$$

Similarly,

$$\int \sin px \,.\, \sin qx \, dx = -\frac{\sin(p+q)x}{2(p+q)} + \frac{\sin(p-q)x}{2(p-q)} \qquad \text{(III.31)}$$

and

$$\int \cos px \,.\, \cos qx \, dx = \frac{\sin(p+q)x}{2(p+q)} + \frac{\sin(p-q)x}{2(p-q)} \qquad \text{(III.32)}$$

We leave the reader to deduce the important results that, when p and q are integers and $p \neq q$, $\int_0^{2\pi} \sin px \cos qx \, dx$

$$= \int_0^{2\pi} \sin px \sin qx \, dx = \int_0^{2\pi} \cos px \cos qx \, dx = 0 \qquad \text{(III.33)}$$

EXAMPLE

Find $\int \sin 3pt \,.\, \sin pt \, dt$ and $\int \cos (pt + \alpha) \,.\, \cos (pt + \beta) \, dt$

$$\int \sin 3pt \sin pt \, dt = -\tfrac{1}{2}\int(\cos 4pt - \cos 2pt)\, dt = -\frac{\sin 4pt}{8p} + \frac{\sin 2pt}{4p}$$

$$\int \cos (pt + \alpha) \,.\, \cos (pt + \beta)\, dt = \tfrac{1}{2}\int[\cos (2pt + \alpha + \beta) + \cos (\alpha - \beta)]\, dt$$

$$= \frac{\sin (2pt + \alpha + \beta)}{4p} + \frac{\cos(\alpha - \beta)}{2}\, t$$

51. **Integration of Powers of Tan x.** Let

$$u_n = \int \tan^n x \, dx, \text{ where } n \text{ is an integer} > 1.$$

Then
$$u_n = \int \tan^{n-2} x \,.\, \tan^2 x \, dx$$
$$= \int \tan^{n-2} x \,(\sec^2 x - 1)\, dx$$
$$= \int \tan^{n-2} x \,.\, d(\tan x) - \int \tan^{n-2} x \, dx$$

$$\therefore \qquad u_n = \frac{\tan^{n-1} x}{n-1} - u_{n-2} \quad . \quad . \quad . \qquad \text{(III.34)}$$

Similarly, $\quad u_{n-2} = \dfrac{\tan^{n-3} x}{n-3} - u_{n-4}$; and so on.

If n is odd, we arrive ultimately at $u_1 = \int \tan x \, dx = \log \sec x$, and, if n is even, at $u_0 = \int 1 dx = x$.

If $u_n = \int \cot^n x \, dx$, we obtain in like manner

$$u_n = -\frac{\cot^{n-1} x}{n-1} - u_{n-2} \quad . \qquad \text{(III.35)}$$

EXAMPLE

Find $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$ and $\int \tan^6 x \, dx$

$$\int_0^{\frac{\pi}{4}} \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) dx = [\tan x - x]_0^{\frac{\pi}{4}} = \tan \frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}$$

$$u_5 = \int \tan^5 x \, dx = \frac{\tan^4 x}{4} - u_3 = \frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - u_1\right)$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log_e \sec x$$

52. **Useful Substitutions.** In order to change an expression into a form readily integrable we have often to substitute a new variable—a method already illustrated in some of the examples worked out in previous articles. The student will learn from experience which substitution, if any, will enable him to integrate a given expression. We give below a list of substitutions applicable to certain types of expressions.

Type of Expression to be Integrated	Substitution Suggested
Expression containing $\sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $x = a \tanh \theta$
,, ,, $\sqrt{x^2 + a^2}$	$x = a \tan \theta$ or $x = a \sinh \theta$
,, ,, $\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $x = a \cosh \theta$
$\dfrac{1}{(x + a)\sqrt{px^2 + qx + r}}$	$x + a = \dfrac{1}{z}$
$\dfrac{1}{x(px^m + q)}$	$x^m = \dfrac{1}{z}$
Expression containing fractional powers of x	$x = z^r$, where r is the L.C.M. of the denominators of the fractional indices
(Function of x^2) . x	$x^2 = z$,
Function of x and $\sqrt{px + q}$	$px + q = z^2$
,, ,, $\sqrt{(x - a)(x - b)}$	$x - b = (x - a)z^2$
,, ,, $\sqrt{(x - a)(b - x)}$	$b - x = (x - a)z^2$
,, ,, $\sqrt{x^2 + px + q}$	$x + \sqrt{x^2 + px + q} = z$

EXAMPLE 1

Evaluate the integrals $\int \dfrac{dx}{(1 - x^2)^{\frac{3}{2}}}$ and $\int x(1 - x^2)^{\frac{1}{2}} \, dx$

Prove that (1) $\int_0^1 (1-x^2)^{\frac{1}{2}} dx = \frac{\pi}{4}$; (2) $\int_1^2 \frac{dx}{(x+1)(x+2)} = 0{\cdot}11778$. (U.L.)

The first integral is $\sin^{-1} x$. We shall, however, work out the result from the substitution $x = \sin\theta$; then $dx = \cos\theta\, d\theta$.

$$\therefore \text{ Integral } = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta = \theta = \sin^{-1} x$$

The second integral $= \frac{1}{2}\int(1-x^2)^{\frac{1}{2}}\, d(x^2) = \frac{1}{2}\int(1-z)^{\frac{1}{2}}\, dz$, where $z = x^2$

$$= -\frac{1}{2}\cdot\frac{2}{3}(1-z)^{\frac{3}{2}} = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$$

With the substitution $x = \sin\theta$, $dx = \cos\theta\, d\theta$,

$$\int_0^1 (1-x^2)^{\frac{1}{2}}\, dx = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta, \text{ since } \theta = 0, \frac{\pi}{2} \text{ when } x = 0, 1$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

Using partial fractions, we have

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = \int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \Big[\log_e(x+1) - \log_e(x+2)\Big]_1^2$$

$$= \left[\log_e \frac{x+1}{x+2}\right]_1^2$$

$$= \log_e \frac{3}{4} - \log_e \frac{2}{3}$$

$$= \log_e \left(\frac{3}{4}\cdot\frac{3}{2}\right) = \log_e \frac{9}{8} = 0{\cdot}11778$$

EXAMPLE 2

Find $\int \frac{dx}{\sqrt{x+a} - \sqrt{x}}$ and $\int_0^a \sqrt{2ax - x^2}\, dx$

Rationalizing the denominator, we have

$$\int \frac{dx}{\sqrt{x+a} - \sqrt{x}} = \int \frac{\sqrt{x+a} + \sqrt{x}}{(x+a) - x}\, dx = \frac{1}{a}\int(\sqrt{x+a} + \sqrt{x})\, dx$$

$$= \frac{2}{3a}\left[(x+a)^{\frac{3}{2}} + x^{\frac{3}{2}}\right]$$

$$\int_0^a \sqrt{2ax - x^2}\, dx = \int_0^a \sqrt{a^2 - (x-a)^2}\, dx$$

Let $x - a = a\sin\theta$. $\therefore dx = a\cos\theta\, d\theta$. Also $\theta = -\frac{\pi}{2}$, 0 when $x = 0$, a

$$\therefore \quad \text{Integral} = \int_{-\frac{\pi}{2}}^{0} a\cos\theta \,.\, a\cos\theta \,.\, d\theta = a^2\int_{-\frac{\pi}{2}}^{0}\cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0}$$

$$= \frac{a^2}{2}\left[0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi a^2}{4}$$

EXAMPLE 3

Find: (1) $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}+2}\,dx$ (2) $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x-2}}$ (3) $\displaystyle\int_3^8 \frac{x+2}{x\sqrt{x+1}}\,dx$

(1) Let $x = z^2$; then $dx = 2z\,dz$

$$\therefore \text{Integral} = \int \frac{z}{z+2}\cdot 2z\,dz = 2\int \frac{z^2}{z+2}\,dz = 2\int\left(z - 2 + \frac{4}{z+2}\right)dz$$

$$= 2\left[\frac{z^2}{2} - 2z + 4\log_e(z+2)\right]$$

$$= x - 4\sqrt{x} + 8\log_e(\sqrt{x}+2)$$

(2) Let $x + \sqrt{x^2+2x-2} = z$; then $x^2 + 2x - 2 = (z-x)^2$, which gives $x = \dfrac{z^2+2}{2(z+1)}$

Also $\left(1 + \dfrac{x+1}{\sqrt{x^2+2x-2}}\right)dx = dz$, i.e $\dfrac{dx}{\sqrt{x^2+2x-2}} = \dfrac{dz}{z+1}$

$$\therefore \text{Integral} = \int \frac{2(z+1)}{z^2+2}\cdot\frac{dz}{z+1} = 2\int\frac{dz}{z^2+2} = \frac{2}{\sqrt{2}}\cdot\tan^{-1}\frac{z}{\sqrt{2}}$$

$$= \sqrt{2}\,.\tan^{-1}\left(\frac{x+\sqrt{x^2+2x-2}}{\sqrt{2}}\right)$$

(3) Let $x + 1 = z^2$; then $dx = 2z\,dz$; also $z = 2, 3$, when $x = 3, 8$.

$$\therefore \text{Integral} = \int_2^3 \frac{z^2+1}{z^2-1}\cdot\frac{2z\,dz}{z} = 2\int_2^3\frac{z^2-1+2}{z^2-1}\,dz$$

$$= 2\int_2^3 dz + 4\int_2^3\frac{dz}{z^2-1}$$

$$= 2\Big[z\Big]_2^3 + 2\left[\log_e\frac{z-1}{z+1}\right]_2^3$$

$$= 2 + 2\left(\log_e\tfrac{1}{2} - \log_e\tfrac{1}{3}\right)$$

$$= 2[1 + \log_e 1{\cdot}5] = 2{\cdot}811.$$

53. **Integration by Parts.** If u and v are functions of a single independent variable x, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (see Art. 30).

$$\text{Integrating, } uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx = \int udv + \int vdu$$

[For, if $I = \int u\frac{dv}{dx}dx$, then $\frac{dI}{dx} = u\frac{dv}{dx}$ and $\frac{dI}{dv} = \frac{dI}{dx}\frac{dx}{dv} = u\frac{dv}{dx}\frac{dx}{dv}$

i.e. $\frac{dI}{dv} = u$, and hence $I = \int u\frac{dv}{dx}dx = \int udv$

Similarly, $\int v\frac{du}{dx}dx = \int vdu$]

$$\therefore \qquad \int udv = uv - \int vdu \qquad . \quad . \quad . \text{(III.36)}$$

A function which cannot be integrated directly or can with difficulty be so integrated, can sometimes be broken up into the product of two functions u and dv, such that $\int vdu$ is readily obtainable; the above formula for "integration by parts" will then give $\int u\,dv$.

EXAMPLE

Find: (1) $\int \sin^{-1} x\,dx$ (2) $\int x^n \log_e x\,dx$ (3) $\int_0^c x\cos\frac{2\pi nx}{c}\,dx$ (n an integer)

Inter 1971 Supplementary

(1) Let $u = \sin^{-1} x$, $v = x$

Then integral $= x\sin^{-1} x - \int xd(\sin^{-1} x)$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx = x\sin^{-1} x + \sqrt{1-x^2}$$

(2) Let $u = \log_e x$, $dv = x^n dx$, so that $v = \frac{x^{n+1}}{n+1}$

$$\text{Then integral} = \int \log_e x d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1}\log_e x - \int \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\log_e x - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}}{n+1}\left[\log_e x - \frac{1}{n+1}\right]$$

[If $n = 1$, $\int x\log_e x\,dx = \frac{x^2}{2}(\log_e x - \frac{1}{2})$; and if $n = 0$, $\int \log_e x\,dx = x(\log_e x - 1)$]

(3) Let $u = x$, $\cos\frac{2\pi nx}{c}\,dx = dv$ so that $v = \frac{c}{2\pi n}\sin\frac{2\pi nx}{c}$

Then integral $= \frac{c}{2\pi n}\int_0^c x\,d\left(\sin\frac{2\pi nx}{c}\right)$

$$= \frac{c}{2\pi n}\left\{\left[x\sin\frac{2\pi nx}{c}\right]_0^c - \int_0^c \sin\frac{2\pi nx}{c}\,dx\right\}$$

$$= \frac{c}{2\pi n}\left\{0 + \frac{c}{2\pi n}\left[\cos\frac{2\pi nx}{c}\right]_0^c\right\}$$

$$= \frac{c^2}{4\pi^2 n^2}\,[\cos 2\pi n - \cos 0]$$

$$= \frac{c^2}{4\pi^2 n^2}\,[1 - 1] = 0$$

We can prove similarly that $\int_0^c x\sin\frac{2\pi nx}{c}\,dx = -\frac{c^2}{2\pi n}$

To find $\int e^{kt}\sin pt\,dt$ *and* $\int e^{kt}\cos pt\,dt$ Let

$$I_1 = \int e^{kt}\sin pt\,dt \text{ and } I_2 = \int e^{kt}\cos pt\,dt$$

Then $I_1 = \int \sin pt\, d\left(\frac{e^{kt}}{k}\right) = \frac{e^{kt}}{k}\sin pt - \int \frac{e^{kt}}{k}\cdot p\cos pt\,dt$

$$= \frac{1}{k}e^{kt}\sin pt - \frac{p}{k}I_2$$

Again $I_2 = \int \cos pt\, d\left(\frac{e^{kt}}{k}\right) = \frac{1}{k}e^{kt}\cdot\cos pt - \int \frac{e^{kt}}{k}(-p\sin pt)\,dt$

$$= \frac{1}{k}e^{kt}\cos pt + \frac{p}{k}I_1$$

Hence, $kI_1 + pI_2 - e^{kt}\sin pt = 0$,

and $\qquad pI_1 - kI_2 + e^{kt}\cos pt = 0$

$$\frac{I_1}{e^{kt}(p\cos pt - k\sin pt)} = \frac{I_2}{e^{kt}(p\sin pt + k\cos pt)} = \frac{1}{-(k^2 + p^2)}$$

i e
$$I_1 = \frac{e^{kt}(k\sin pt - p\cos pt)}{k^2 + p^2} \qquad \text{(III 37)}$$

and
$$I_2 = \frac{e^{kt}(p\sin pt + k\cos pt)}{k^2 + p^2} \qquad \text{(III 38)}$$

Otherwise
$$I_2 + iI_1 = \int e^{kt}(\cos pt + i \sin pt)dt$$
$$= \int e^{kt} e^{ipt} dt = \int e^{(k+ip)t} dt$$
$$= \frac{1}{k+ip} e^{(k+ip)t}$$
$$= \frac{k-ip}{k^2+p^2} e^{kt}(\cos pt + i \sin pt)$$
$$= \frac{e^{kt}}{k^2+p^2}[(k \cos pt + p \sin pt) + i(k \sin pt - p \cos pt)]$$

Equating real and imaginary parts, we obtain I_1 and I_2 as above

EXAMPLE

Evaluate the integral $I = \int e^{\frac{Rt}{L}} \sin \frac{n\pi t}{\tau} dt$ (U.L.)

We can write the result down at once from above, but it is the method of obtaining this result that the student must note

$$I = e^{\frac{Rt}{L}} \frac{\left[\frac{R}{L}\sin\frac{n\pi t}{\tau} - \frac{n\pi}{\tau}\cos\frac{n\pi t}{\tau}\right]}{\left(\frac{R}{L}\right)^2 + \left(\frac{n\pi}{\tau}\right)^2} \qquad \text{(III.39)}$$

We work out below some examples to indicate other types of integrals to which this method of "integration by parts" may be applied

(1) $\int x^3 \sin 2x\, dx = -\frac{1}{2}\int x^3 d(\cos 2x)$
$$= -\tfrac{1}{2}[x^3 \cos 2x - \int \cos 2x \cdot 3x^2 dx]$$
$$= -\tfrac{1}{2} x^3 \cos 2x + \tfrac{3}{2}\int x^2 \cos 2x\, dx$$

Also $\int x^2 \cos 2x\, dx = \frac{1}{2}\int x^2 d(\sin 2x)$
$$= \tfrac{1}{2}[x^2 \sin 2x - \int \sin 2x \cdot 2x\, dx]$$
$$= \tfrac{1}{2} x^2 \sin 2x - \int x \sin 2x\, dx$$

Again $\int x \sin 2x\, dx = -\frac{1}{2}\int x\, d(\cos 2x)$
$$= -\tfrac{1}{2}[x \cos 2x - \int \cos 2x\, dx]$$
$$= -\tfrac{1}{2} x \cos 2x + \tfrac{1}{4} \sin 2x$$

Hence, $\int x^3 \sin 2x\, dx = -\frac{1}{2}x^3 \cos 2x +$
$$\tfrac{3}{2}[\tfrac{1}{2}x^2 \sin 2x + \tfrac{1}{2}x \cos 2x - \tfrac{1}{4} \sin 2x]$$
$$= \frac{3(2x^2 - 1)}{8} \sin 2x - \frac{x(2x^2 - 3)}{4} \cos 2x$$

(2) $$\int x^2 e^{kx}dx = \frac{1}{k}\cdot\int x^2 d(e^{kx}) = \frac{1}{k}[x^2e^{kx} - \int e^{kx}\,.\,2xdx]$$

$$= \frac{1}{k}x^2e^{kx} - \frac{2}{k^2}\int xd\,(e^{kx})$$

$$= \frac{1}{k}x^2e^{kx} - \frac{2}{k^2}[xe^{kx} - \int e^{kx}\,dx]$$

$$= \frac{e^{kx}}{k^3}[k^2x^2 - 2kx + 2]$$

(3) $\int x^3 \cos^5 x\, dx$ and $\int e^{2x} \sin^4 x\, dx$.

Let $\cos x + i \sin x = z$, then $\cos x - i \sin x = \frac{1}{z}$. (Compare Art. 16, Example.)

Also $\cos nx + i \sin nx = z^n$; $\cos nx - i \sin nx = \frac{1}{z^n}$; and $2 \cos x = z + \frac{1}{z}$; $2 i \sin x = z - \frac{1}{z}$; $2 \cos nx = z^n + \frac{1}{z^n}$; $2 i \sin nx = z^n - \frac{1}{z^n}$

$$\therefore\ (2 \cos x)^5 = \left(z+\frac{1}{z}\right)^5 = \left(z^5+\frac{1}{z^5}\right) + 5\left(z^3+\frac{1}{z^3}\right)$$
$$+ 10\left(z+\frac{1}{z}\right) = 2\cos 5x + 10 \cos 3x + 20 \cos x.$$

$$\therefore\qquad \cos^5 x = \tfrac{1}{16}[\cos 5x + 5 \cos 3x + 10 \cos x]$$

Again $$(2 i \sin x)^4 = \left(z-\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right) - 4\left(z^2+\frac{1}{z^2}\right) + 6$$
$$= 2 \cos 4x - 8 \cos 2x + 6$$

$$\therefore\qquad \sin^4 x = \tfrac{1}{8}[\cos 4x - 4 \cos 2x + 3]$$

Hence, $\int x^3 \cos^5 x\, dx = \frac{1}{16}[\int x^3 \cos 5x\, dx + 5\int x^3 \cos 3x\, dx + 10\int x^3 \cos x\, dx]$; and these integrals can be found as in Example (1). Also $\int e^{2x} \sin^4 x\, dx = \frac{1}{8}[\int e^{2x} \cos 4x\, dx - 4\int e^{2x} \cos 2x\, dx + 3\int e^{2x}dx]$.

Now $\int e^{2x}dx = \frac{1}{2}e^{2x}$, and the other two integrals can be found by using II.38).

54. Reduction Formulae. [m and n positive integers.]

$$\int_0^{\frac{\pi}{2}} \sin^m x\,.\,\cos^n x\, dx;\quad \int_0^{\frac{\pi}{2}} \sin^m x\, dx;\quad \int_0^{\frac{\pi}{2}} \cos^n x\, dx.$$

Let $$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^n x\, dx$$

Then $$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^{n-1} x\, d(\sin x)$$

$$= \frac{1}{m+1}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\, d(\sin^{m+1} x)$$

$$= \frac{1}{m+1}\Big\{[\cos^{n-1} x \sin^{m+1} x]_0^{\frac{\pi}{2}}$$

$$- \int_0^{\frac{\pi}{2}} \sin^{m+1} x \,.\, (n-1)\cos^{n-2} x(-\sin x)\, dx\Big\}$$

$$= \frac{1}{m+1}\left[0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x(1-\cos^2 x)dx\right]$$

$$= \frac{n-1}{m+1}\left[\int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^{n-2} x dx - \int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^n x dx\right]$$

$$I_{m,n} = \frac{n-1}{m+1}[I_{m,n-2} - I_{m,n}]$$

$\therefore$ $$(m+1+n-1)I_{m,n} = (n-1)I_{m,n-2}$$

$\therefore$ $$I_{m,n} = \frac{n-1}{m+n} \,.\, I_{m,n-2} \qquad \text{(III.40)}$$

By writing

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^n x \,.\, \sin^{m-1} x\, d(-\cos x)$$

$$= -\frac{1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-1} x\, d(\cos^{n+1} x)$$

we obtain similarly

$$I_{m,n} = \frac{m-1}{m+n} \,.\, I_{m-2,n} \qquad \text{(III.41)}$$

Thus, by applying the results (III.40) and (III.41) in succession, we find that $I_{m,n}$ depends on $I_{0,0} = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$, if m and n are both even integers; on $I_{1,0} = \int_0^{\frac{\pi}{2}} \sin x\, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$, if m is

odd and n even, on $I_{0,1} = \int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = 1$, if m is even and n odd, and on $I_{1,1} = \int_0^{\frac{\pi}{2}} \sin x \cos x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \frac{1}{2}$, if m and n are both odd

EXAMPLE 1

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x\,dx = \frac{6-1}{5+6}\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x\,dx$$

$$= \frac{5}{11}\cdot\frac{4}{9}\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x\,dx$$

$$= \frac{5}{11}\cdot\frac{4}{9}\cdot\frac{3}{7}\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x\,dx$$

$$= \frac{5}{11}\cdot\frac{4}{9}\cdot\frac{3}{7}\cdot\frac{2}{5}\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$$

$$= \frac{5}{11}\cdot\frac{4}{9}\cdot\frac{3}{7}\cdot\frac{2}{5}\cdot\frac{1}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{5}{11}\cdot\frac{4}{9}\cdot\frac{3}{7}\cdot\frac{2}{5}\cdot\frac{1}{3}\cdot 1 = \frac{8}{693}$$

By putting $m = 0$ in the result (III 40) above, we have

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} I_{n-6}$$

and so on Ultimately we arrive at $I_1 = \int_0^{\frac{\pi}{2}} \cos x\,dx = 1$, if n is odd, and at $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$, if n is even

Putting $n = 0$ in the result (III 41), we have

$$I_m = \int_0^{\frac{\pi}{2}} \sin^m x\,dx = \frac{m-1}{m} I_{m-2} = \frac{m-1}{m}\cdot\frac{m-3}{m-2} I_{m-4}$$

and so on In this case we arrive ultimately at $I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1$, if m is odd and at $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$ if m is even

EXAMPLE 2

Prove that $12\int \sin^{12}\theta\,d\theta = -\sin^{11}\theta\,.\cos\theta + 11\int \sin^{10}\theta\,d\theta$, and deduce the value of $\int_0^{\frac{\pi}{2}} \sin^{12}\theta\,d\theta$ (U L)

$$
\begin{aligned}
\int \sin^{12}\theta\, d\theta &= -\int \sin^{11}\theta\, d(\cos\theta) \\
&= -[\sin^{11}\theta\cos\theta - \int \cos\theta \cdot 11\sin^{10}\theta \cdot \cos\theta\, d\theta] \\
&= -\sin^{11}\theta \cos\theta + 11\int \sin^{10}\theta(1 - \sin^2\theta)d\theta \\
&= -\sin^{11}\theta\cos\theta + 11\int \sin^{10}\theta\, d\theta - 11\int \sin^{12}\theta\, d\theta \\
12\int \sin^{12}\theta\, d\theta &= -\sin^{11}\theta \cos\theta + 11\int \sin^{10}\theta\, d\theta
\end{aligned}
$$

Again $\left[-\sin^{11}\theta \cos\theta\right]_0^{\frac{\pi}{2}} = 0$ hence

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^{12}\theta\, d\theta &= \frac{11}{12}\int_0^{\frac{\pi}{2}} \sin^{10}\theta\, d\theta \\
&= \frac{11}{12}\cdot\frac{9}{10}\int_0^{\frac{\pi}{2}} \sin^8\theta\, d\theta \\
&= \frac{11}{12}\cdot\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{231\pi}{2\,048}
\end{aligned}
$$

EXAMPLE 3

Evaluate $\int_0^{\frac{\pi}{2}} \cos^8 x\, dx$

Integral $= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \frac{35\pi}{256}$

EXAMPLES III

Find the values of the following integrals—

(1) $\int\left(4\sqrt{x} - \frac{3}{x} + 5x^2 - 3\right) dx$

(2) $\int \frac{du}{\sqrt[3]{u^2}}$

(3) $\int\left(\frac{2}{s^{1.2}} - 5s^{4} + \frac{1}{\sqrt{s}}\right) ds$

(4) $\int(2x + 7)^3\, dx$

(5) $\int \frac{dx}{(3x - 10)}$

(6) $\int \frac{dt}{4t - 1}$

(7) $\int 10^x\, dx$

(8) $\int \sin(5t + 2)\, dt$

(9) $\int_0^{\pi} \cos \tfrac{1}{2}t\, dt$

(10) $\int_0^2 \frac{dx}{4 + x^2}$

(11) $\int \frac{dx}{4 - x^2}$

(12) $\int_7^{10} \frac{dx}{x^2 - 25}$

(13) $\int \frac{dx}{\sqrt{x^2 + 4}}$

(14) $\int_{2.5}^{5} \frac{dx}{\sqrt{25 - x^2}}$

(15) $\int \frac{dx}{\sqrt{x^2 - 25}}$

(16) $\int \frac{6t-5}{3t^2-5t+4}\,dt$

(17) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$

(18) $\int_{2}^{4} \frac{6x^2-1}{2x^3-x-2}\,dx$

(19) $\int \frac{3x^2-8}{(x-2)(2x-3)}\,dx$

(20) $\int \frac{dy}{12y^2-y-35}$

(21) $\int_{5}^{6} \frac{2-x^2}{(x-3)(x-4)^2}\,dx$

(22) $\int \frac{dp}{18p^2-9p-20}$

(23) $\int \frac{dx}{x^2-6x-14}$

(24) $\int \frac{dy}{4y^2+28y-3}$

(25) $\int \frac{5u-8}{2u^2-u-9}\,du$

(26) $\int \frac{11-9x}{3x^2+4x+15}\,dx$

(27) $\int \frac{x}{\sqrt{x^2-3x+5}}\,dx$

(28) $\int \frac{1-8v}{\sqrt{16v^2+40v-47}}\,dv$

(29) $\int \frac{dx}{\sqrt{3-2x-5x^2}}$

(30) $\int \frac{12x-11}{\sqrt{4+7x-x^2}}\,dx$

(31) $\int \frac{dy}{(y-9)(y^2-9y+1)}$

(32) $\int \frac{(x^3-x+4)}{(x+1)^2(x^2-3)}\,dx$

(33) $\int \sqrt{v^2+6v+11}\,dv$

(34) $\int \sqrt{2x^2-2x-1}\,dx$

(35) $\int \sqrt{5+4x-6x^2}\,dx$

(36) $\int_{2}^{3} e^{2.5t}\,dt$

(37) $\int_{0}^{2\pi} \sin 6x\,dx$

(38) $\int_{0}^{2\pi} \cos \tfrac{1}{2}x\,dx$

(39) $\int \frac{dt}{e^{3t}+e^{-3t}}$

(40) $\int \frac{dx}{5\sin x+12\cos x}$

(41) $\int \frac{dt}{(3\sin t-4\cos t)^2}$

(42) $\int_{0}^{\frac{\pi}{2}} \frac{dx}{2+3\cos x}$

(43) $\int_{0}^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x+5\cos^2 x}$

(44) $\int \frac{x\,dx}{\sqrt{1-x^2}}$

(45) $\int_{2}^{7} \frac{5x}{1+x^2}\,dx$

(46) $\int \frac{y^2}{y^4-y^2-12}\,dy$

(47) $\int_{0}^{1} x\sqrt{1-x^2}\,dx$

(48) $\int_{0}^{4} \frac{x}{\sqrt{9+x^2}}\,dx$

(49) $\int_{0}^{\frac{2\pi}{3}} \sin^2\theta\,d\theta$

(50) $\int_{0}^{\frac{3\pi}{4}} \sin^2\theta \,.\, \cos\theta\,d\theta$

(51) $\int \sin^2\theta \,.\, \cos^3\theta\,d\theta$

(52) $\int_{0}^{\frac{\pi}{2k}} \cos 5kt \,.\, \cos 3kt\,dt$

(53) $\int_{0}^{\frac{2\pi}{p}} \sin apt \,.\, \cos bpt\,dt$ (a and b integers)

(54) $\int \tan^3 x\,dx$

(55) $\int_a^{2a} \frac{dx}{\sqrt{2ax - x^2}}$

(56) $\int \frac{dy}{y - \sqrt{y^2 - 1}}$

(57) $\int (x^2 + a^2)\, dx$

(58) $\int \frac{dx}{(x^2 + a^2)}$

(59) $\int \frac{x dx}{(a^2 - x^2)}$

(60) $\int_0^a (a^2 - x^2)\, dx$

(61) $\int \frac{dt}{(t - 1)\sqrt{t^2 - t + 1}}$

(62) $\int \frac{z + 4}{z\sqrt{4 - z}}\, dz$

(63) $\int_0^\infty \frac{dy}{(1 + y^2)^2}$

(64) $\int x^5 \log_e x\, dx$

(65) $\int \tan^{-1} x\, dx$

(66) $\int e^{3x} \sin 2x\, dx$

(67) $\int e^{x} \cos^2 x\, dx$

(68) $\int (a \sin pt + b \cos pt) e^{st}\, dt$

(69) $\int_0^{\frac{\pi}{2}} \sin^5 \theta\, d\theta$

(70) $\int_0^{\frac{\pi}{2}} \cos^6 \theta\, d\theta$

(71) $\int_0^{\frac{\pi}{2}} \sin^3 \theta \,.\, \cos^5 \theta\, d\theta$

(72) $\int_0^{\frac{\pi}{2}} \sin^4 \theta \,.\, \cos^2 \theta\, d\theta$

(73) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta \,.\, \cos^6 \theta\, d\theta$

(74) $\int_0^{\frac{\pi}{4}} \tan^4 x\, dx$

(75) $\int_0^{\pi} (3 \sin \theta + 4 \cos \theta)^3\, d\theta$

(76) By substituting z^2 for $9 - x^2$, prove that $\int_{2\sqrt{2}}^{3} \frac{12dx}{x(x^2 - 12)\sqrt{9 - x^2}} = -0.418$

(77) Define $\cosh x$ and $\sinh x$, and prove the relation between them. Integrate $\sqrt{a^2 + x^2}\, dx$ and $x\sqrt{a^2 + x^2}\, dx$.

The force on a body distant x from a fixed point is $\frac{k}{(x^2 + a^2)}$. Find the work done in moving the body from $x = a$ to $x = 2a$. (U.L.)

(78) Work out the integrals $\int_0^\infty \frac{dx}{x^2 + 4}$, $\int_0^\infty \frac{dx}{x^4 + 4}$ (U.L.)

(79) If $y^2 = a^2x^2 + c$, differentiate with respect to x the functions $\log_e (ax + y)$, xy, and express the results in terms of y. Hence, or otherwise, evaluate the integrals $\int \frac{dx}{y}$, $\int y dx$. Prove that $\int_0^{\cdot} \sqrt{x^2 - 4}\, dx = \frac{15}{2} - 2 \log_e 2$

(U.L.)

(80) By substituting $3 \sin^2 \theta$ for x, evaluate $\int_0^3 \sqrt{\frac{x^3}{3 - x}}\, dx$

(81) By writing $\sin \frac{\theta}{2} = \sin \frac{\alpha}{2} \cdot \sin \phi$, show that

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} = 2 \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} . \sin^2 \phi}}$$

Expand the integrand of the last expression in ascending powers of $\sin\phi$ as far as $\sin^6\phi$ and evaluate the integral to this degree of accuracy. Estimate the percentage error resulting from the neglect of the term in $\sin^6\phi$ if $\alpha = \frac{\pi}{3}$.

(U.L.)

(82) Prove that $\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$, hence evaluate

$$\int_0^{\frac{\pi}{4}} \cos^4\theta\, d\theta$$

(83) Evaluate (i) $\int \log(1 + x)\, dx$, (ii) $\int_0^{\pi} x \sin x\, dx$ (U.L.)

(84) State the formula for integration by parts. Find the value of

(i) $\int e^{x} \sin 2x\, dx$ (ii) $\int_0^{\pi} \theta \sin^3\theta\, d\theta$ (U.L.)

(85) Evaluate the integrals

(i) $\int_1^0 \sqrt{\frac{x}{1-x}}\, dx$ (ii) $\int_1^{\sqrt{3}} \frac{2x^2 - 3}{x^2(1 + x^2)}\, dx$ (iii) $\int_0^2 \sqrt{\frac{2+x}{2-x}}\, dx$

(86) Evaluate the integral $\int (x^2 - 4)^{\frac{1}{2}} dx$.

By aid of the substitution $x\sqrt{2}\sin\theta = \sin\phi$ or otherwise, show that

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{\cos 2\theta}}{\cos\theta}\, d\theta = \pi(\sqrt{2} - 1)/2$$

(U.L.)

(87) Prove that, if p and q are integers and $p \neq q$,

(i) $\int_0^{\pi} \cos px \cos qx\, dx = \int_0^{\pi} \sin px \sin qx\, dx = 0$

(ii) $\int_0^{\pi} \cos px \sin qx\, dx = 0$, if $q - p$ is even, and $= \frac{2q}{q^2 - p^2}$, if $q - p$ is odd

(88) Find the values of $\int \frac{x\, dx}{x^2 + 2x - 4}$, $\int x^2 \sin^2 x\, dx$, and prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab}$ (U.L.)

(89) Work out the integrals $\int_2^5 \frac{2x^2 - 3x + 4}{(x - 1)^2}\, dx$, $\int_0^{\infty} \frac{dx}{x^3 + 8}$ (U.L.)

(90) Prove that $\int_0^n x^2(n - x)^p\, dx = \int_0^n x^p(n - x)^2 dx$, and find the common value of the two integrals (U.L.)

(91) Evaluate (i) $\int x^2 e^{-x}\, dx$, (ii) $\int_0^{\frac{1}{4}\pi} \tan^6 x\, dx$, (iii) $\int x^3 (1 - x)^3 dx$, (iv) $\int \frac{1 + 2\cos x}{(2 + \cos x)^2}\, dx$ (U.L.)

(92) Evaluate the integrals

$$\int_0^1 \tan^{-1} x\, dx \qquad \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} \qquad \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x \quad 2\cos x} \qquad \int_0 \frac{x-1}{(x \quad 1)(x \quad 2x+2)}\, dx$$

(U L)

(93) Evaluate the following integrals

$$\int_0^1 x^3 e \quad dx, \quad \int_0^{\pi} e \quad \sin 2x\, dx \quad \int (x \quad 4x + 13)\, dx$$

(U L)

(94) Evaluate the integrals

(i) $\int_0^1 x \tan^{-1} x\, dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos x\, dx$, (iii) $\int_0^{\infty} \frac{x+1}{(x \quad 2)(x^2+1)}\, dx$

iv $\int x \, 8^{-1} \ldots$

CHAPTER IV

EXPANSION OF FUNCTIONS IN POWER SERIES—MACLAURIN'S THEOREM AND TAYLOR'S THEOREM—GRAPHICAL SOLUTION OF EQUATIONS—LIMITS

55. **Expansion in Power Series.** The infinite series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

is known as a *power series*. Let $f(x)$ be any finite, continuous, single-valued function of x. By suitably choosing the values of a_0, a_1, a_2, etc., we can ensure that the relation

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \quad \text{(IV.1)}$$

is satisfied for $(n + 1)$ values of x. Thus, consider the values of x; x_0, x_1, $x_2 \ldots x_{n-1}$, x_n, where $x_n > x_0$ and $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1}$. Substituting these values in turn in (IV.I), we have the $(n + 1)$ relations

$$\left.\begin{aligned}
f(x_0) &= a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + \ldots + a_nx_0^n \\
f(x_1) &= a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + \ldots + a_nx_1^n \\
f(x_2) &= a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + \ldots + a_nx_2^n \\
\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f(x_n) &= a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + \ldots + a_nx_n^n
\end{aligned}\right\} \quad \text{(IV.2)}$$

From these equations we can find the values of the $n + 1$ constants a_0, a_1, etc., $\ldots a_n$. On substituting these values in (IV.1) we obtain a finite series, which has the same values as $f(x)$ for the $n + 1$ given values of x. If the graphs of $f(x)$ and of this series are drawn in the same figure, from $x = x_0$ to $x = x_n$, they will intersect at $n + 1$ points whose projections on the axis of x will be equally spaced along the axis. If, now, we assume n to be increased so that n tends to infinity, the number of points of intersection will tend to infinity also, and the graph of the series will approach more and more closely to that of $f(x)$. In the limit the graphs will coincide and the function and the power series will become equivalent for values of x inside the given range. Thus we have the relation

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \quad \text{(IV.3)}$$

which holds for values of x within the given range $x = x_0$ to $x = x_n$.

We have assumed that $f(x)$ can be expanded in a convergent power series. For this to be possible $f(x)$ and its derivatives must be finite and continuous throughout the given range as well as at the end points. We shall assume that (IV.3) is differentiable, i.e. that

$$\frac{d}{dx}f(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

For a discussion of the validity of this and other assumptions made above, readers are referred to treatises on pure mathematics. We showed how to expand $(1 + x)^n$ in Art. 7, and gave a power series for e^{ax} in (1.15). We shall now deal with a general method of expanding functions of x in the form of power series.

56. **Maclaurin's Theorem.** We begin with the relation (IV.3) and by differentiating both sides obtain a power series for $f'(x)$ $\left(\text{or } \frac{dy}{dx}\right)$. Then by differentiating again and again we obtain in succession series for $f''(x), f'''(x), \ldots f^{(n)}(x) \ldots$ Thus

$$\left.\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \\
f'(x) &= 1.a_1 + 2.a_2x + 3.a_3x^2 + 4.a_4x^3 + 5.a_5x^4 + \ldots \\
f''(x) &= 2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + 5.4.a_5x^3 \\
&\qquad + 6.5.a_6x^4 + \ldots \\
f'''(x) &= 3.2.1.a_3 + 4.3.2.a_4x + 5.4.3.a_5x^2 \\
&\qquad + 6.5.4.a_6x^3 + 7.6.5.a_7x^4 + \ldots \\
\ldots &= \ldots\ldots\ldots\ldots\ldots \\
\ldots &= \ldots\ldots\ldots\ldots\ldots \\
\ldots &= \ldots\ldots\ldots\ldots\ldots \\
\ldots &= \ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \text{(IV.4)}$$

On putting $x = 0$ in each of the above, we obtain $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = \frac{1}{\lfloor 2}f''(0)$, $a_3 = \frac{1}{\lfloor 3}f'''(0)$, etc., and from the method of obtaining these we see that $a_n = \frac{1}{\lfloor n}f^{(n)}(0)$.

We have, then, on substitution in (IV.3)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \ldots + \frac{x^n}{\lfloor n}f^{(n)}(0) + \ldots \quad \text{(IV.5)}$$

which is known as *Maclaurin's Series.*

Assuming that $\frac{d}{dx}(e^{cx}) = ce^{cx}$, find a power series for e^{cx}.

Here

$$f(x) = e^{cx}, f'(x) = ce^{cx}, f''(x) = c^2e^{cx}, \ldots f^{(n)}(x) = c^n e^{cx}, \ldots$$

Hence,

$$f(0) = e^0 = 1, f'(0) = c, f''(0) = c^2, \ldots f^{(n)}(0) = c^n e^0 = c^n, \ldots$$

and

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \frac{c^4x^4}{\underline{|4}} + \ldots + \frac{c^nx^n}{\underline{|n}} + \ldots$$

as given in (I.15).

To expand $(a + x)^n$ in a power series.

Here

$$f(x) = (a + x)^n, f'(x) = n(a + x)^{n-1}, f''(x) = n\,.\,\overline{n-1}\,(a + x)^{n-2},$$
$$\ldots f^{(r)}(x) = n\,.\,\overline{n-1}\,.\,\overline{n-2} \ldots \overline{n-r+1}\,(a + x)^{n-r},$$

and

$$f(0) = a^n, f'(0) = na^{n-1}, f''(0) = n(n-1)a^{n-2}, \ldots$$
$$f^{(r)}(0) = n\,.\,\overline{n-1} \ldots \overline{n-r+1}\,a^{n-r}.$$

Hence,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n\,.\,\overline{n-1}}{\underline{|2}}a^{n-2}x^2 + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}a^{n-3}x^3$$
$$+ \ldots + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2} \ldots \overline{n-r+1}}{\underline{|r}}a^{n-r}x^r + \ldots \quad \text{(IV.6)}$$

which we gave in (I.6). If n is a positive integer this series terminates after $n + 1$ terms; otherwise it is an infinite series which as we have seen in Art. 7 is convergent if $|x| < |a|$.

To obtain a series for $\log_e(1 + x)$.

This series is easily obtained by Maclaurin's theorem, but we shall obtain it by a different method. By continued division

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \ldots$$

Hence, integrating both sides between the limits 0 and x, we have

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \ldots \quad \text{(IV.7)}$$

Both the above series are convergent for values of x between $x = -1$ and $x = +1$.

Putting $-x$ for x, we get

$$\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} \quad . \quad (\text{IV.8})$$

and subtracting (IV.8) from (IV.7), we have, since $\log \dfrac{1 + x}{1 - x}$

$$= \log_e (1 + x) - \log (1 - x)$$

$$\log_e \frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right) \quad (\text{IV.9})$$

Since (IV.7) and (IV.8) are convergent for $|x| < 1$, this last series holds for all positive values of $\dfrac{1 + x}{1 - x}$, since this varies from zero to infinity as x varies from -1 to $+1$.

Putting $\dfrac{n + 1}{n} = \dfrac{1 + x}{1 - x}$ in (IV.9), we have, since on solving this $x = \dfrac{1}{2n + 1}$,

$$\log_e \overline{n + 1} = \log_e n + 2\left(\frac{1}{2n + 1} + \frac{1}{3(2n + 1)^3} + \frac{1}{5(2n + 1)^5} + \ldots\right) \quad . \quad . \quad . \quad (\text{IV.10})$$

a rapidly converging series for positive values of n.

By giving to n the values 1, 2, 3, etc., in succession we find the values of $\log_e 2$, $\log_e 3$, $\log_e 4$, $\log_e 5$, etc. Since $\log 4 = 2 \log 2$ and $\log 6 = \log 2 + \log 3$, etc., it is only necessary to use the series for calculating the logarithms of prime numbers.

57. The Series for Sin x and Cos x. Put $f(x) = \sin x$, then $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = +\sin x$, etc. Also $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(4)}(0) = 0$, etc.

Substituting in (IV.5), we have

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots \quad . \quad . \quad . \quad (\text{IV.11})$$

By differentiation of this result, or by an application of Maclaurin's theorem as in the case of $\sin x$, we have

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} - \dots \qquad \text{(IV.12)}$$

58. **Taylor's Theorem.** Now consider the function $f(x+h)$. $f'(x+h)$ means $\frac{d}{d(x+h)}f(x+h)$, but since $\frac{d}{dx}(x+h) = 1$, this is the same as $\frac{d}{dx}f(x+h)$. Hence, in (IV.4) we may substitute $f(x+h)$ for $f(x)$, $f'(x+h)$ for $f'(x)$, $f''(x+h)$ for $f''(x)$, etc. On putting $x = 0$ in these we have $f(h)$ instead of $f(0)$, $f'(h)$ instead of $f'(0)$, and generally $f^{(n)}(h)$ instead of $f^{(n)}(0)$. Thus, instead of the relation (IV.5) we have

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{\underline{|2}}f''(h) + \frac{x^3}{\underline{|3}}f'''(h) + \dots$$
$$+ \frac{x^n}{|n}f^{(n)}(h) + \dots \qquad \text{(IV.13)}$$

which is known as *Taylor's Series.*

Expand $\tan(a+x)$ as far as the fifth power of x.

$f(a+x) = \tan(a+x)$ and putting $x = 0$

$f(a) = \tan a$

$f'(a+x) = \sec^2(a+x) = 1 + \tan^2(a+x)$ and

$f'(a) = 1 + \tan^2 a.$

$f''(a+x) = 2\tan(a+x)\sec^2(a+x)$

$= 2\tan(a+x)\{1 + \tan^2(a+x)\}$

$= 2\tan(a+x) + 2\tan^3(a+x)$, and

$f''(a) = 2\tan a + 2\tan^3 a$

$f'''(a+x) = 2\sec^2(a+x) + 6\tan^2(a+x)\sec^2(a+x)$

$= 2 + 8\tan^2(a+x) + 6\tan^4(a+x)$, and

$f'''(a) = 2 + 8\tan^2 a + 6\tan^4 a$

$f^{(4)}(a+x) = 16\tan(a+x)\sec^2(a+x) + 24\tan^3(a+x)\sec^2(a+x)$

$= 16\tan(a+x) + 40\tan^3(a+x) + 24\tan^5(a+x)$, and

$f^{(4)}(a) = 16\tan a + 40\tan^3 a + 24\tan^5 a$

$$
\begin{aligned}
f^{(5)}(a+x) &= 16 \sec^2 (a+x) + 120 \tan^2 (a+x) \sec^2 (a+x) \\
&\qquad + 120 \tan^4 (a+x) \sec^2 (a+x) \\
&= 16 + 136 \tan^2 (a+x) + 240 \tan^4 (a+x) \\
&\qquad + 120 \tan^6 (a+x), \text{ and} \\
f^{(5)}(a) &= 16 + 136 \tan^2 a + 240 \tan^4 a + 120 \tan^6 a.
\end{aligned}
$$

Hence, writing t for $\tan a$, and substituting in (IV.13), we have

$$
\begin{aligned}
\tan (a+x) &= t + (1+t^2)x + t(1+t^2)x^2 + \frac{1+4t^2+3t^4}{3} x^3 \\
&\quad + \frac{t(2+5t^2+3t^4)}{3} x^4 + \frac{2+17t^2+30t^4+15t^6}{15} x^5 + \ldots
\end{aligned}
$$

If $a = 0$, $t = \tan a = 0$, and

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

If $a = \frac{\pi}{4}$, $t = \tan a = 1$, and

$$\tan\left(\frac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \frac{64}{15}x^5 + \ldots$$

It is sometimes necessary to have an expansion for $f(x+h)$ in a series of powers of h. Interchanging x and h in (IV.13) we obtain Taylor's series in the form

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \ldots \\
&\quad + \frac{h^n}{\underline{|n}} f^{(n)}(x) + \ldots \qquad \text{(IV.14)}
\end{aligned}
$$

In Arts. 57 and 58 we have assumed that it is possible to expand $f(x)$ or $f(x+h)$ in a convergent series of ascending positive integral powers of x (or h). To be rigorous we should investigate the conditions under which our assumption is justifiable. We shall, however, content ourselves here with stating the conditions. The reader will find a proof in any standard textbook on the differential calculus. Provided that $f(x)$, $f'(x)$, $f''(x)$, . . ., $f^{(n)}(x)$ are all finite and continuous functions between the limits x to $x+h$ of x, then we can write the relation (IV.14) as

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \ldots \\
&\quad + \frac{h^n}{\underline{|n}} f^{(n)}(x + \theta h),
\end{aligned}
$$

where $1 > \theta > 0$.

Similarly we can write the relation (IV.5) as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0) + \frac{x^3}{\underline{|3}}f'''(0) + \ldots + \frac{x^n}{\underline{|n}}f^{(n)}(\theta x),$$

the limits of x being now 0 to x.

If $\underset{n \to \infty}{Lt.} \frac{h^n}{\underline{|n}}f^{(n)}(x + \theta h) = \varepsilon$, where ε is less than any assignable quantity, however small, then we are justified in assuming the expansion of $f(x + h)$ to be that given in (IV.14). Similarly, the expansion (IV.5) holds if $\underset{n \to \infty}{Lt.} f^{(n)}(\theta x)$ is less than any assignable quantity, however small.

The terms $\frac{h^n}{\underline{|n}}f^{(n)}(x + \theta h)$ and $\frac{x^n}{\underline{|n}}f^{(n)}(\theta x)$ are known as Lagrange's forms of the remainder after n terms in Taylor's and Maclaurin's series respectively.

59. **Approximations.** If h is sufficiently small we may write as successive approximations to the values of $f(x + h)$ the quantities

$$\left.\begin{aligned} f(x + h) &= f(x) + hf'(x) \quad . \quad . \quad . \quad . \\ f(x + h) &= f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) \quad . \quad . \\ f(x + h) &= f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) \\ \text{etc.} &= \text{etc.} \end{aligned}\right\} \text{(IV.15)}$$

The first of these is usually sufficient to enable us to find the value of $f(x + h)$ when the value of $f(x)$ is known and h is small. The quantity $\frac{h^2}{\underline{|2}}f''(x)$, which is added to give the next approximation, serves as a measure of the degree of accuracy of the first approximation. Similarly, if we proceed to a second approximation, the term $\frac{h^3}{\underline{|3}}f'''(x)$ is used as a measure of accuracy.

EXAMPLE 1

Tan 30° $= \frac{1}{\sqrt{3}}$. Find tan 31°.

From the above $\tan(x + h) = \tan x + h\frac{d}{dx}(\tan x)$ approx.

$$= \tan x + h \sec^2 x$$
$$= \tan x + h(1 + \tan^2 x)$$

Now, $31° = 0{\cdot}5410521$ radian, and $30° = 0{\cdot}5235988$ radian

Hence, $x = 0{\cdot}5235988$ and $h = 0{\cdot}0174533$.

$$\therefore \quad \tan 31° = \tan (x + h) = \tan 30° + 0{\cdot}0174533\,(1 + \tfrac{1}{3})$$
$$= 0{\cdot}5773503 + 0{\cdot}0232711$$
$$= 0{\cdot}6006214 \text{ approx.}$$

To find a second approximation, we take the next term in the series, which is $\frac{h^2}{\underline{|2}}f''(x)$. Now $f''(x) = \frac{d}{dx}(1 + \tan^2 x) = 2 \tan x \sec^2 x$.

Hence, the third term is

$$\frac{0{\cdot}0174533^2}{2} \times 2 \tan 30° \sec^2 30° = 0{\cdot}0174533^2 \times 0{\cdot}5773503 \times [illegible]$$
$$= 0{\cdot}0002345$$

and $$\tan 31° = 0{\cdot}6006214 + 0{\cdot}0002345$$
$$= 0{\cdot}6008559 \text{ approx.}$$

The next term in the series is $\frac{h^3}{\underline{|3}} f'''(x)$ and $f'''(x) = \frac{d}{dx}(2 \tan x \sec^2 x)$ $= \frac{d}{dx}(2 \tan x + 2 \tan^3 x) = 2 \sec^2 x + 6 \tan^2 x \,.\, \sec^2 x$, and the fourth term is

$$\frac{0{\cdot}0174533^3}{\underline{|3}} \times (2 \times [illegible] + 6 \times [illegible] \times [illegible]) = 0{\cdot}0000047$$

$$\therefore \quad \tan 31° = 0{\cdot}6008559 + 0{\cdot}0000047$$
$$= 0{\cdot}6008606$$

or $$\tan 31° = 0{\cdot}600861 \text{ correct to six significant figures.}$$

EXAMPLE 2

Given $\sin 30° = 0{\cdot}500000$, $\cos 30° = 0{\cdot}8660254$, find $\sin 31°$ correct to five significant figures.

$30°$ is equivalent to $\frac{\pi}{6}$ radians, therefore $x = \frac{\pi}{6}$ and $h = 0{\cdot}0174533$ radian.

$f(x + h) = f(x) + hf'(x)$, and $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x$. hence, substituting,

$$\sin (x + h) = \sin x + h \cos x$$

and $$\sin 31° = 0{\cdot}500000 + 0{\cdot}0174533 \times 0{\cdot}8660254$$
$$= 0{\cdot}515115 \text{ approx.}$$

For the second approximation we add the term $\frac{h^2}{\underline{|2}} f''(x)$

$$\frac{h^2}{\underline{|2}} f''(x) = \frac{0{\cdot}174533^2}{\underline{|2}}(-0{\cdot}500000) = -0{\cdot}0000761$$

and $$\sin 31° = 0{\cdot}515115 - 0{\cdot}0000761$$
$$= 0{\cdot}515039 \text{ approx.}$$

For the third approximation we add the term

$$\frac{h^3}{3!}f'''(x) = \frac{h^3}{6}(-\cos x)$$

$$= \frac{0{\cdot}0174533^3}{6}(-0{\cdot}8660254)$$

$$= -0{\cdot}0000007 = -0{\cdot}000001 \text{ approx.}$$

$$\sin 31^\circ = 0{\cdot}515038 \text{ approx.}$$

$$= 0{\cdot}51504 \text{ correct to five significant figures.}$$

60. Expansion in a Power Series by Differentiation or Integration of a Known Series. In Art. 57 we obtained the series for $\cos x$ by differentiating that for $\sin x$, and we found a series for $\log_e(1 + x)$ by integrating the series for $\frac{1}{1+x}$. These methods are often convenient, and we give further examples of each.

The series for $\tan x$ is given up to the term in x^5 by

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

Hence, differentiating both sides

$$\sec^2 x = 1 + x^2 + \tfrac{2}{3}x^4 + \ldots$$

a result which is easily verified, for $\tan^2 x + 1 = \sec^2 x$, and therefore

$$\sec^2 x = 1 + \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\right)^2 = 1 + x^2 + \tfrac{2}{3}x^4 + \ldots$$

Again, by continued division

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

Integrating both sides between the limits 0 and x

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots \qquad \text{(IV.16)}$$

Similarly, expanding by the binomial theorem

$$\frac{1}{\sqrt{1-x^2}} = (1 - x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots$$

Hence, integrating between the limits 0 and x

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots \qquad \text{(IV.17)}$$

We can obtain a series for finding the value of π from each of the relations (IV.16) and (IV.17). Putting $x = 1$ in the former and $x = \frac{1}{2}$ in the latter, we obtain

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \dots \qquad \text{(IV.18)}$$

and

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3 \,.\, 2^3} + \frac{1 \,.\, 3}{2 \,.\, 4} \cdot \frac{1}{5 \,.\, 2^5} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \cdot \frac{1}{7 \,.\, 2^7} + \dots \qquad \text{(IV.19)}$$

The series (IV.19) is much the better series for purposes of calculation, as it converges more rapidly than the other.

It is important to notice that in obtaining the series for $\tan^{-1} x$ and $\sin^{-1} x$ we have assumed that $|x|$ is less than 1.

EXAMPLE

From the equation $y = x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots$ prove that $\frac{dy}{dx} = 1 + y$.

From this prove that $x = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots$ (U.L.)

$$\frac{dy}{dx} = 1 + \frac{2x}{\underline{|2}} + \frac{3x^2}{\underline{|3}} + \frac{4x^3}{\underline{|4}} + \dots$$

$$= 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots$$

$$= 1 + y$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{1 + y} = 1 - y + y^2 - y^3 + y^4 - \dots$$

and $$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \dots$$

61. Solution of Equations by Graphs. The reader is familiar with the methods of solving by algebraic methods simple equations and quadratic equations. Methods are known of solving algebraic equations of the third and fourth degrees, but these are tedious and complicated, and graphical methods of solution are usually preferred. Equations of degree higher than the fourth cannot generally be solved by algebraic methods. There are also equations involving quantities which are not algebraic, such as e^{cx}, $\sin(ax + b)$, $\log_e x$, $\sinh x$, etc. Such equations (transcendental equations) cannot in general be solved without the use of tables or graphs. We shall describe a method of solving equations by graphs, and shall illustrate the method by solving a few equations.

Let $f_1(x) = f_2(x)$ represent any equation in one unknown. The method of solution is—

(1) Bring both quantities on to the left-hand side, thus

$$f_1(x) = f_2(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.20)}$$

$$\therefore \quad f_1(x) - f_2(x) = 0, \text{ or writing } \phi(x) \text{ for } f_1(x) - f_2(x)$$

$$\phi(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.21)}$$

(2) Draw the graph of $y = \phi(x)$ and measure the values of x for which the values of y are zero. Let these values be a, b, c, etc., respectively. These are approximate roots of (IV.21) or of (IV.20).

(3) In order to obtain any particular root, say, $x = a$, with a greater degree of accuracy, calculate values of $\phi(x)$ for values of x between $a - h$ and $a + k$, where h and k are as small as possible consistent with the condition that $f(a - h)$ and $f(a + k)$ must be of opposite signs. Plot the graph over this range of values of x, choosing your scale for values of x so that the length representing $h + k$ is as large as possible. The value of x for which the value of y is zero, is a more accurate value of the required root than a. This operation can be repeated until the root is found with the required degree of accuracy.

EXAMPLE 1

Find, correct to four significant figures, a root of $x^3 - 1{\cdot}8x^2 - 10x + 17 = 0$, and find approximate values of the other roots.

Let $y = x^3 - 1{\cdot}8x^2 - 10x + 17 = 0$.

We calculate the values in Table I—

TABLE I

x	-4	-3	-2	-1	0	1	2	3	4
$x^3 + 17$	-47	-10	9	16	17	18	25	44	81
$1{\cdot}8x^2$	28·8	16·2	7·2	1·8	0	1·8	7·2	16·2	28·8
$1{\cdot}8x^2 + 10x$	$-11{\cdot}2$	$-13{\cdot}8$	$-12{\cdot}8$	$-8{\cdot}2$	0	11·8	27·2	46·2	68·8
y	$-35{\cdot}8$	3·8	21·8	24·2	17	6·2	$-2{\cdot}2$	$-2{\cdot}2$	12·2

The quantities in the second, third, and fourth lines are arranged for convenience in calculation. Drawing a graph of these results (Fig. 22), we see that

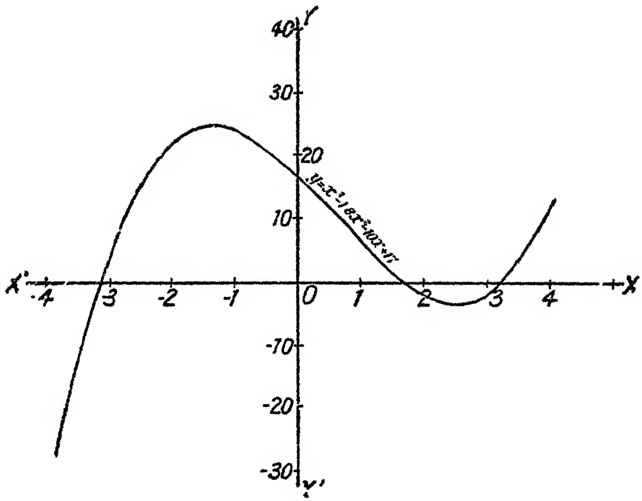


FIG. 22

the roots are approximately $x = -3{\cdot}1$, $x = 1{\cdot}6$, and $x = 3{\cdot}23$. To determine more accurately the root which is nearly $x = 1{\cdot}6$, we draw up Table II.

TABLE II

x	1·6	1·65	1·70
$x^3 + 17$	21·096	21·4921	21·9130
$1{\cdot}8x^2$	4·608	4·9005	5·2020
$1{\cdot}8x^2 + 10x$	20·608	21·4005	22·2020
y	0·488	0·0916	$-$ 0·2890

Drawing the graph of these in Fig. 23, we see that the root is $x = 1{\cdot}662$ correct to four significant figures.

The reader is warned against attempting to obtain more accuracy than the data permit of. We have no information about the accuracy of the coefficients in the given equation, and the above answer is only correct to four significant figures if the coefficients are correct to five significant figures. In many cases, when solving equations by graphs, it is only necessary to draw one graph. In the above case, after filling in Table I, and thus locating the root between $x = 1$ and $x = 2$, we could have drawn up a table of values between $x = 1$ and $x = 2$, in which way increasing x by 0·2 at a step we should have located the root between $x = 1{\cdot}6$ and $x = 1{\cdot}8$. A further tabulation would then have suggested

itself with the values given in Table II. Actually, we could locate the root to any required degree of accuracy by tabulation alone, and we have recourse to a graph only when such a procedure will shorten the operation of finding the root. The reader should notice that the graph in Fig. 23 is practically a straight line, and that we could have left out the point $x = 1{\cdot}6$, $y = 0{\cdot}488$ from the graph, drawing the graph as a straight line between the other two points. Again, it should be noticed that this graph need not be drawn, as the root can be calculated from the figures in Table II. Let the root be $x = 1{\cdot}65 + a$. Then, assuming that

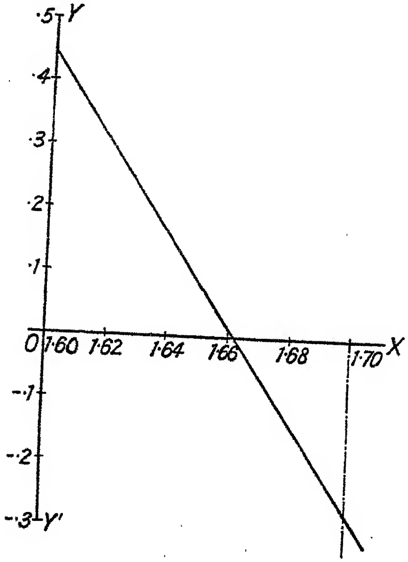


FIG. 23

changes in the value of the function are proportional to changes in the value of x, we have

$$\frac{a}{1{\cdot}70 - 1{\cdot}65} = \frac{0{\cdot}0916}{0{\cdot}3806}, \text{ from which we find } a = 0{\cdot}012$$

and $$x = 1{\cdot}662 \text{ as before.}$$

EXAMPLE 2

Solve the equation $x + \log_{10} x = 3{\cdot}375$

Here we find it convenient to vary our rule (1) above, and to express the equation in the form below, plotting the graphs of two functions and finding the abscissa of their point of intersection. This is often done when the graphs of $f_1(x)$ and $f_2(x)$ are well known, as in the present example.

Transposing the term x, we have

$$\log_{10} x = 3{\cdot}375 - x$$

We find values of the functions $y = \log_{10} x$ and $y = 3{\cdot}375 - x$ below.

x	0·5	1	2	3	4
$\log_{10} x$	− 0·3010	0	0·3010	0·4771	0·6021
$3{\cdot}375 - x$	2·875	—	—	—	− 0·625

The graphs of the two functions are drawn in Fig. 24, and their point of intersection has $x = 2{\cdot}9$ as abscissa. This is a first approximation to the root.

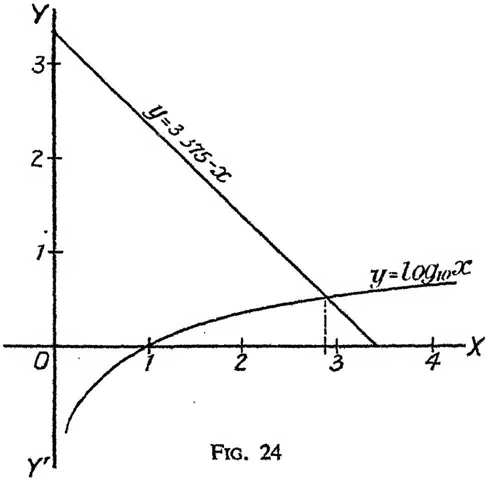


FIG. 24

Now, writing the equation in the form $y = x + \log_{10} x - 3{\cdot}375 = 0$, we tabulate values of y for values of x between $x = 2{\cdot}9$ and $x = 2{\cdot}92$.

x	2·9	2·91	2·92
$\log_{10} x$	0·4624	0·4639	0·4654
$x + \log_{10} x$	3·3624	3·3739	3·3854
$y = x + \log_{10} x - 3{\cdot}375$	− 0·0126	− 0·0011	0·0104

A portion of the graph of $y = x + \log_{10} x - 3{\cdot}375$ is drawn in Fig. 25, and the value of x is found to be $x = 2{\cdot}911$. The reader should note that as the number 3·375 and the logarithms in the table are correct only to four significant figures, the root $x = 2{\cdot}911$ is probably correct to three significant figures only.

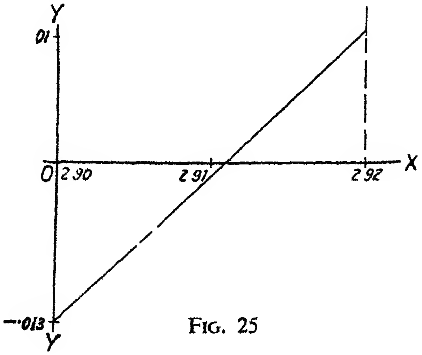


FIG. 25

EXAMPLE 3

Show that there are two roots of the equation

$$2 \log_{10} x = x^2 - 5x + 6$$

and find the root which is between $x = 3$ and $x = 4$.

Tabulating the values of $y = 2 \log_{10} x$ and $y = x^2 - 5x + 6$, we obtain

x	0	1	2	3	4	2·5
$y = 2 \log_{10} x$	$-\infty$	0	0·6021	0·9542	1·2041	0·7959
$y = x^2 - 5x + 6$	6	2	0	0	2	− 0·25

The graphs are shown in Fig. 26, and they intersect in the points for which $x = 1{\cdot}7$ and $x = 3{\cdot}6$ approximately. To find a more accurate value of the second root, we tabulate again after writing the equation in the form

$$y = 2 \log_{10} x - x^2 + 5x - 6 = 0$$

x	3·5	3·6	3·7	3·68	3·67
$2 \log_{10} x$	1·0882	1·1126	1·1364	1·1316	1·1294
$5x$	17·5000	18·0000	18·5000	18·4000	18·3500
$2 \log_{10} x + 5x$	18·5882	19·1126	19·6364	19·5316	19·4794
$x^2 + 6$	18·2500	18·9600	19·6900	19·5424	19·4689
y	0·3382	0·1526	− 0·0536	− 0·0108	0·0105

The reader should notice that we calculated the values of the functions for the values of x in the order of their occurrence in the top line of the above table, first locating the root between $x = 3{\cdot}6$ and $x = 3{\cdot}7$, and then between $x = 3{\cdot}67$ and $x = 3{\cdot}68$. By proportion or by the graph (Fig. 27), we find that $x = 3{\cdot}675$ is the root correct to four significant figures.

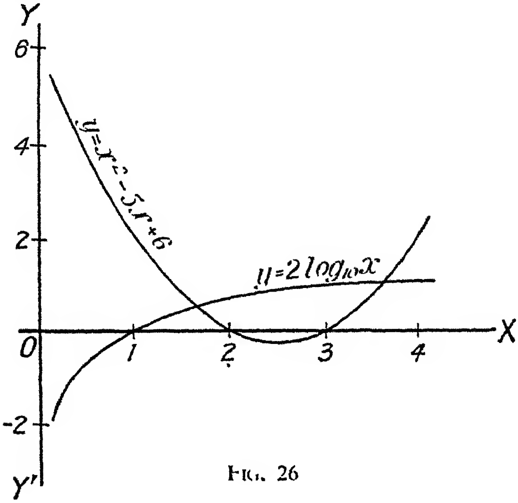


FIG. 26

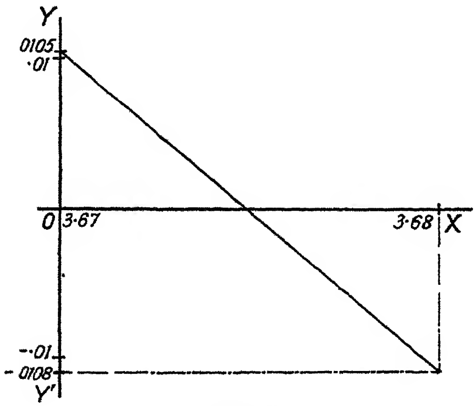


FIG. 27

EXAMPLE 4

Show that $3 \sin x = \frac{1}{x}$ has an infinite number of roots and that some of these are given approximately for large values of n by $x = n\pi$ where n is a positive or negative integer. Find the root which is nearly $x = 0{\cdot}6$.

Fig. 28 shows the graphs of $y = \frac{1}{x}$ and $y = 3 \sin x$. For large values of x, $y = \frac{1}{x}$ is practically coincident with the axis of x. The roots are given approximately by $x = 0{\cdot}6, 3{\cdot}1, 2\pi, 3\pi$, etc., and $x = -0{\cdot}6, -3{\cdot}1, -2\pi, -3\pi$, etc. Thus, the roots with large numerical values are given approximately by $x = n\pi$ where n is any large integer positive or negative. To find more accurately the value of

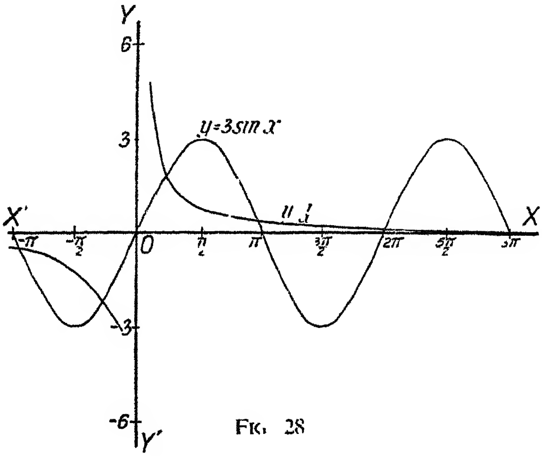


Fig. 28

the root, which is nearly $x = 0{\cdot}6$, we tabulate after putting the equation in the form $y = 3 \sin x - \frac{1}{x} = 0$.

x	0·6981	0·5760	0·6109	0·5934
$3 \sin x$	1·9284	1·6338	1·7208	1·6776
$\frac{1}{x}$	1·433	1·736	1·637	1·685
$y = 3 \sin x - \frac{1}{x}$	0·495	− 0·102	0·084	− 0·007

Fig. 29 shows the graph of $y = 3 \sin x - \frac{1}{x}$. The graph is assumed to be straight over the given range. The root is $x = 0{\cdot}595$ correct to three significant figures. A more accurate result would be obtained by using tables giving more than four figure accuracy, and values of sines of angles differing by less than one degree. Calculating the root by proportion, we have, if $x = 0{\cdot}5934 + a$,

$$a = \frac{0{\cdot}007}{0{\cdot}007 + 0{\cdot}084} \times (0{\cdot}6109 - 0{\cdot}5934) = 0{\cdot}0013 \text{ and } x = 0{\cdot}595 \text{ as above.}$$

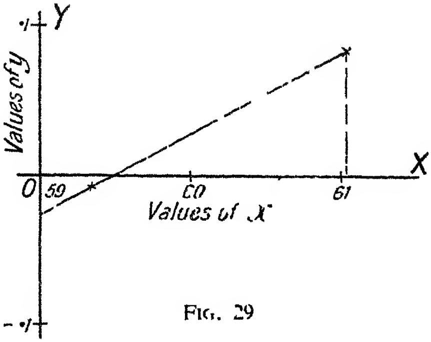


FIG. 29

62. **Approximating more Closely to Roots of Equations. Newton's Method. Method of Iteration.** In Examples 1 and 4 above we showed how to find by proportion a closer approximation to a root. We shall find a formula based on this method.

Let the equation be $f(x) = 0$, and suppose we have found that $f(a) = \xi$ and $f(a+h) = -\eta$, where ξ and η are small quantities. Suppose PQ (Fig. 30) to be the graph of $y = f(x)$ between $x = a$ and $x = a + h$. Then we have, if PM and QN are ordinates,

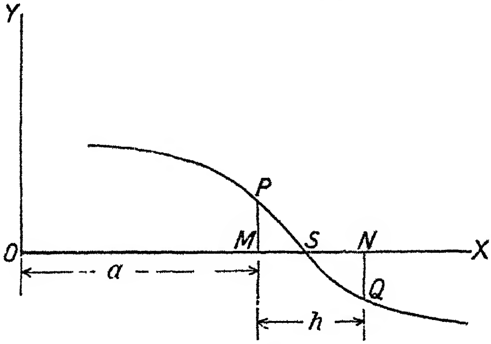


FIG. 30

$PM = \xi$, $NQ = \eta$, $OM = a$, and $MN = h$. If PQ cuts OX at S, and we assume PQ to be such a short arc of the graph that we can take it to be a straight line, we have by similar triangles $\dfrac{MS}{SN} = \dfrac{PM}{NQ}$ $= \dfrac{\xi}{\eta}$. But $\dfrac{MS}{MN} = \dfrac{MS}{MS + SN} = \dfrac{\xi}{\xi + \eta}$. $\therefore MS = \dfrac{h\xi}{\xi + \eta}$, and the

root of the equation is

$$x = a + MS = a + \frac{h\xi}{\xi + \eta}$$

i e
$$x = a + \frac{h\xi}{\xi + \eta} \qquad \text{(IV 22)}$$

When we have found a first approximation to a root (IV 22) may be applied repeatedly so as to produce closer and closer approximations to the actual root The method is rather tedious as it involves the calculation of two values of the function (for values of x one greater than and the other less than the actual root) in each approximation If, however, h, ξ, and η are sufficiently small, substitution of their values in (IV 22) will give a close approximation to the root of the equation We have made use of (IV 22) in solving Examples (1) and (4) above We shall now explain a method of approximation due to Newton and known as *Newton's Method*

Taking the first of the approximations (IV 15) we have, if h is sufficiently small, $f(x + h) = f(x) + hf'(x)$ If a is a first approximation to a root of $f(x) = 0$, and $a + h$ is a second approximation, then since

$$f(x) = 0 \text{ is approximately satisfied}$$

by $x = a + h$,
$$f(a + h) = 0 \text{ approximately}$$

$$f(a) + hf'(a) = 0$$

or
$$h = -\frac{f(a)}{f'(a)} \qquad \text{(IV 23)}$$

Hence,
$$x = a + h$$

i e
$$x = a - \frac{f(a)}{f'(a)} \qquad \text{(IV 24)}$$

is a second approximation to the root

By taking this value of x as the value of a in (IV 24) we get a third approximation to the root By successive applications of this method we may obtain a fourth, fifth, etc, approximation to the root, and in this way we can usually find a root of $f(x) = 0$ to any required possible degree of accuracy.

The reader is again warned that the greatest degree of accuracy possible depends upon the degree of accuracy of the numbers involved in the equation and of the tables of values used, if any

EXAMPLE 1

Solve the equation $x^3 - 1.8x^2 - 10x + 17 = 0$ of Art 61 to find the root, which is nearly 1.6

Looking upon $x = 1.6$ as a first approximation we assume as a second approximation that $x = a + h$ where h is given by (IV 23) and the root is then $1.6 + h$

Second Approximation

Let $f(x) = x^3 - 1.8x^2 - 10x + 17 = 0$ then $f'(x) = 3x^2 - 3.6x - 10$

Hence $f(1.6) = 1.6^3 - 1.8 \times 1.6^2 - 16 + 17 = 0.488$

and $f'(1.6) = 3 \times 1.6^2 - 3.6 \times 1.6 - 10 = -8.08$

Hence $$h = -\frac{f(1.6)}{f'(1.6)} = \frac{0.488}{8.08} = 0.06 \text{ nearly}$$

and the root is $x = 1.6 + h = 1.66$ nearly

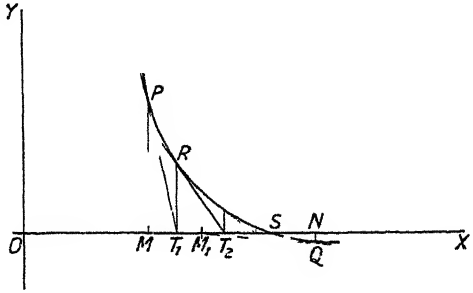


Fig 31

Third Approximation

$f(1.66) = 1.66^3 - 1.8 \times 1.66^2 - 16.6 + 17 = 0.01422$

and $f'(1.66) = 3 \times 1.66^2 - 3.6 \times 1.66 - 10 = -7.7092$

Hence $$h = -\frac{f(1.66)}{f'(1.66)} = \frac{0.01422}{7.7092} = 0.0019$$

and the root is $x = 1.66 + h = 1.6619$ nearly

Fourth Approximation

$f(1.6619) = 1.6619^3 - 1.8 \times 1.6619^2 - 16.619 + 17 = -0.00042$

$f'(1.6619) = 3 \times 1.6619^2 - 3.6 \times 1.6619 - 10 = -7.697$

and $$h = -\frac{f(1.6619)}{f'(1.6619)} = -\frac{0.00042}{7.697} = -0.00006$$

The root is therefore $x = 1.6619 + h = 1.66184$ or correct to four significant figures, $x = 1.662$

Fig 31 shows how the successive approximations approach the actual value of the root. $PRSQ$ is the graph. PT_1 is the tangent at P and RT_2 that at R

PM, RT_1, NQ are ordinates to the curve. If $OM = 1{\cdot}6$ and $f(1{\cdot}6) = \overline{PM}$, $f'(1{\cdot}6) = $ slope of $PT_1 = \frac{\overline{PM}}{\overline{MT_1}}$, and therefor

$$\frac{f(1{\cdot}6)}{f'(1{\cdot}6)} = -\frac{\overline{PM}}{\frac{\overline{PM}}{\overline{MT_1}}} = \overline{MT_1}$$

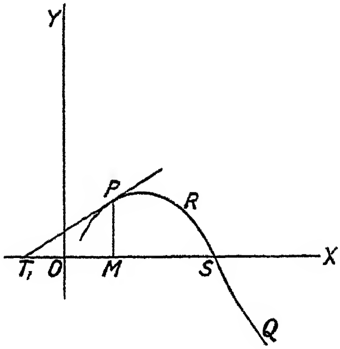


FIG. 32

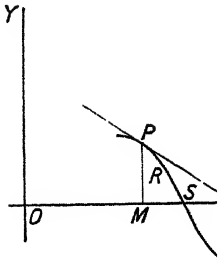


FIG.

The first approximation is $x = \overline{OM}$

The second approximation is $x = \overline{OM} - \frac{f(1{\cdot}6)}{f'(1{\cdot}6)}$

$$= \overline{OM} + \overline{MT_1}$$

$$= \overline{OT_1}$$

We thus obtain the second approximation, $x = \overline{OT_1}$, from th by erecting the ordinate MP and drawing the tangent PT_1.

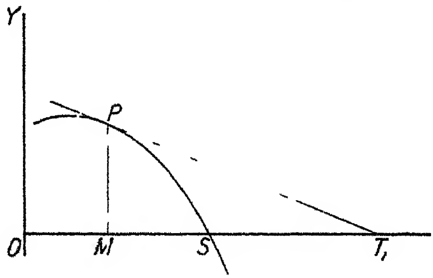


FIG. 34

by erecting the ordinate T_1R and drawing the tangent RT_2 w approximation $x = \overline{OT_2}$. By repeating this operation as dotted lines, we obtain closer and closer approximations t

$x = OS$. If the first approximation is $x = OA$ instead of $x = OM$, the first tangent QM will cut OY on the left of S, and the successive approximations will then approach the actual root from the left, as before.

The second approximation obtained by Newton's method may be less accurate than the first. If, however, after the second, the higher approximations approach closer and closer to the actual root, the method will still work. It is possible, however, that instead of approaching more closely to the actual root, the successive "approximations" will actually recede from it, in which case the method fails. The method fails if there is a maximum or minimum value of $f(x)$ between the points $x = \overline{OM}$ and $x = \overline{OS}$, as in Fig. 32. It will probably fail if there is a point of inflexion near to S, as in Fig. 33, or if the slope of PT_1 is small, as in Fig. 34. The reader should draw graphs of different kinds and examine for himself each case to see if the method applies.

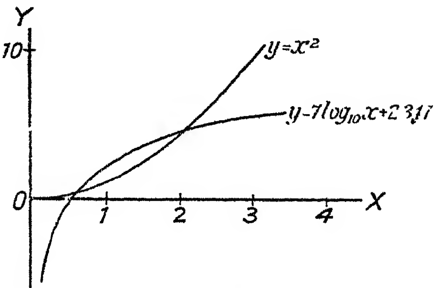


FIG. 35

Method of Iteration. If the equation to be solved, i.e. $f(x) = 0$, can be written in the form $x = F(x)$, and x_1 is a first approximation to the root, we can obtain a second approximation by substituting x_1 for x in $F(x)$. Successive approximations to the root are therefore x_1, $x_2 = F(x_1)$, $x_3 = F(x_2)$, $x_4 = F(x_3)$, etc. Sometimes these successive approximations converge rapidly, but often they either converge slowly or diverge, so that the method, known as the method of *iteration*, is not of general application. By carrying out the above operations on a graph the reader will see that if the gradient of the graph of $y = F(x)$ is numerically greater than unity in the neighbourhood of its intersection with the graph of $y = x$, the series x_1, x_2, x_3, x_4, etc., is divergent. If the gradient of the former graph is numerically less than unity, the series is convergent. If the gradient is nearly equal to unity, the iteration method becomes very lengthy.

EXAMPLE 2

Show graphically that the equation $x^2 - 7\log_{10}x = 2{\cdot}347$ has two real positive roots, and find the larger correct to three places of decimals. (U.L.)

The roots of the equation are the abscissae of the points of intersection of the graphs of $y = x^2$ and $y = 7\log_{10}x + 2{\cdot}347$. These are drawn in Fig. 35, and the approximate roots are $x = 0{\cdot}5$, and $x = 2{\cdot}2$.

Write the equation in the form

$$x = \frac{1}{x}\left(7\log_{10}x + 2{\cdot}347\right)$$

$$\text{Then } x_1 = 2.2 \text{ and } x_2 = \frac{1}{2.2}(7 \log_{10} 2.2 + 2.347) = 2.1561$$

$$x_3 = \frac{1}{2.1561}(7 \log_{10} 2.1561 + 2.347) = 2.1719$$

$$x_4 = \frac{1}{2.1719}(7 \log_{10} 2.1719 + 2.347) = 2.1662$$

$$x_5 = \frac{1}{2.1662}(7 \log_{10} 2.1662 + 2.347) = 2.1682$$

$$x_6 = \frac{1}{2.1682}(7 \log_{10} 2.1682 + 2.347) = 2.1676$$

The root is, therefore, 2.168 correct to four significant figures. Four-figure accuracy is impossible in this case, unless the number on the right of the equation is 2·3470 correct to five significant figures.

63. **Limiting Values.** In Chapter I we found certain limiting values. By means of Taylor's series we can obtain a method of finding the limiting value of a quotient of two functions which both vanish for a given value of the independent variable. Thus, suppose $F(a)$ and $f(a)$ are both zero and we require to find the value of $\underset{x \to a}{Lt.} \dfrac{F(x)}{f(x)}$, we have

$$\underset{x \to a}{Lt.} \frac{F(x)}{f(x)} = \underset{h \to 0}{Lt.} \frac{F(a+h)}{f(a+h)}$$

$$= \underset{h \to 0}{Lt.} \frac{F(a) + h F'(a) + \dfrac{h^2}{|\underline{2}} F''(a) + \ldots}{f(a) + h f'(a) + \dfrac{h^2}{|\underline{2}} f''(a) + \ldots}$$

or, since $F(a) = f(a) = 0$,

$$\underset{x \to a}{Lt.} \frac{F(x)}{f(x)} = \underset{h \to 0}{Lt.} \frac{F'(a) + \dfrac{h}{|\underline{2}} F''(a) + \dfrac{h^2}{|\underline{3}} F'''(a) + \ldots}{f'(a) + \dfrac{h}{|\underline{2}} f''(a) + \dfrac{h^2}{|\underline{3}} f'''(a) + \ldots}$$

$$\therefore \qquad \underset{x \to a}{Lt.} \frac{F(x)}{f(x)} = \frac{F'(a)}{f'(a)} \qquad . \quad . \quad . \quad . \qquad \text{(IV.25)}$$

unless $F'(a)$ and $f'(a)$ are both zero, in which case the limiting value is evidently $\dfrac{F''(a)}{f''(a)}$.

If this last fraction has both numerator and denominator zero the limiting value is $\dfrac{F'''(a)}{f'''(a)}$. If the first n derivatives of both $F(x)$ and $f(x)$ are zero for the value $x = a$ the above limit is easily seen to be $\dfrac{F^{(n+1)}(a)}{f^{(n+1)}(a)}$.

In a similar way we can find the limiting value of $\dfrac{F(x)}{f(x)}$ as x approaches a when $F(a) = f(a) = \infty$, for in this case we write

$$\underset{x\to a}{Lt.}\frac{F(x)}{f(x)} = \underset{x\to a}{Lt.}\frac{\dfrac{1}{f(x)}}{\dfrac{1}{F(x)}} = \underset{x\to a}{Lt.}\frac{\phi(x)}{\theta(x)}$$

where $\phi(a) = \theta(a) = 0$.*

EXAMPLE 1

Find $\underset{x\to 0}{Lt.}\dfrac{\sin x}{x}$. (Compare Art. 4.)

Here $F(x) = \sin x$, $F'(x) = \cos x$, $f(x) = x$, $f'(x) = 1$

Hence, $$\underset{x\to 0}{Lt.}\frac{\sin x}{x} = \left[\frac{\cos x}{1}\right]_{x=0} = \frac{1}{1} = 1$$

EXAMPLE 2

Find $\underset{x\to a}{Lt.}\dfrac{x^n - a^n}{x - a}$ (Art. 8, Ex. 5.)

Here $F(x) = x^n - a^n$, $F'(x) = nx^{n-1}$, $f(x) = x - a$, $f'(x) = 1$

Hence, $$\underset{x\to a}{Lt.}\frac{F(x)}{f(x)} = \left[\frac{F'(x)}{f'(x)}\right]_{x=a} = \frac{na^{n-1}}{1} = na^{n-1}$$

EXAMPLE 3

Find $\underset{x\to a}{Lt.}\dfrac{x^2 \sin a - a^2 \sin x}{x - a}$

$F(x) = x^2 \sin a - a^2 \sin x$, $F'(x) = 2x \sin a - a^2 \cos x$, $f(x) = x - a$, $f'(x) = 1$

Hence, $$\underset{x\to a}{Lt.}\frac{F(x)}{f(x)} = \left[\frac{F'(x)}{f'(x)}\right]_{x=a} = \frac{2a \sin a - a^2 \cos a}{1}$$

* Examples 1 and 2 are simple applications of the method of this article and must not be taken as formal proofs of the results. Formal proofs are given in Articles 4 and 8.

$$\therefore \quad \underset{x \to a}{Lt.} \frac{x^2 \sin a - a^2 \sin x}{x - a} = a(2 \sin a - a \cos a)$$

EXAMPLE 4

Find $\underset{x \to 1}{Lt.} \dfrac{\log_e x}{x - 1}$

Here $F(x) = \log_e x$, $F'(x) = \dfrac{1}{x}$, $f(x) = x - 1$, $f'(x) = 1$

Hence,
$$\underset{x \to 1}{Lt.} \frac{\log_e x}{x - 1} = \left[\frac{\frac{1}{x}}{1}\right]_{x = 1} = 1$$

EXAMPLE 5

Find $\underset{x \to 0}{Lt.} \dfrac{\sin x - x}{x^3}$

$F(x) = \sin x - x$, $F'(x) = \cos x - 1$, $f(x) = x^3$, $f'(x) = 3x^2$

Hence,
$$\underset{x \to 0}{Lt.} \frac{\sin x - x}{x^3} = \left[\frac{\cos x - 1}{3x^2}\right]_{x = 0} = \frac{0}{0}$$

In this case we put

$$\underset{x \to 0}{Lt.} \frac{F(x)}{f(x)} = \frac{F''(0)}{f''(0)} = \left[\frac{-\sin x}{6x}\right]_{x = 0} = \frac{0}{0}$$

We have next

$$\underset{x \to 0}{Lt.} \frac{F(x)}{f(x)} = \frac{F'''(0)}{f'''(0)} = \left[\frac{-\cos x}{6}\right]_{x = 0} = -\frac{1}{6}$$

or
$$\underset{x \to 0}{Lt.} \frac{\sin x - x}{x^3} = -\frac{1}{6}\ .$$

EXAMPLE 6

Find $\underset{x \to \frac{\pi}{2}}{Lt.} \dfrac{\tan 3x}{\tan x}$

Here $F(a) = f(a) = \infty$, and we have

$$\underset{x \to \frac{\pi}{2}}{Lt.} \frac{\tan 3x}{\tan x} = \underset{x \to \frac{\pi}{2}}{Lt.} \frac{\cot x}{\cot 3x}$$

which takes the form $\dfrac{0}{0}$. Putting $\phi(x) = \cot x$ and $\theta(x) = \cot 3x$, we have $\phi'(x) = -\operatorname{cosec}^2 x$ and $\theta'(x) = -3 \operatorname{cosec}^2 3x$. From (IV.25) we have, therefore,

$$\underset{x \to \frac{\pi}{2}}{Lt.} \frac{\cot x}{\cot 3x} = \underset{x \to \frac{\pi}{2}}{Lt.} \frac{-\operatorname{cosec}^2 x}{-3 \operatorname{cosec}^2 3x} = \frac{1}{3}$$

which is the required limit.

EXAMPLES IV

(1) Use Maclaurin's theorem to obtain the expansion of a^x in a power series.

(2) Obtain the first three terms of the series for (*a*) $\sin^{-1} x$, (*b*) $\tan^{-1} x$.

(3) Show that if we write Δy for $\frac{dy}{dx}$, $\Delta^2 y$ for $\frac{d^2y}{dx^2}$, etc., and assume that we may write an expression such as $\Delta^n y + a\Delta^{n-1} y + b\Delta^{n-2} y + \ldots + my$ in the form $(\Delta^n + a\Delta^{n-1} + b\Delta^{n-2} + \ldots + m)y$, then Taylor's theorem may be stated briefly as $f(x + y) = e^{y\Delta} f(x)$.

(4) Show by means of a graph that $\frac{f(x+h) - f(x)}{h}$ is the slope of the chord joining the two points which lie on the graph of $y = f(x)$, and whose abscissae are x and $x + h$ respectively. Hence, show that if h is small, $f(x + h) = f(x) + hf'(x + \theta h)$ where $\theta < 1$ and $f(x)$ and its derivatives are continuous inside the range $x = x$ to $x = x + h$.

(5) Expand e^{y+h} in a series of powers of h, and also obtain your result directly from the series for e^y.

(6) Expand $\tan x$ and $\tan(x + \frac{1}{4}\pi)$ as far as the term in x^4. Calculate tan 3° and tan 44° correct to four significant figures.

(7) Use Maclaurin's theorem to find expansions for $\sinh x$ and $\cosh x$, and show that for small values of $\frac{x}{c}$ the quantity $y = \cosh \frac{x}{c}$ may without great error be replaced by $y = \frac{x^2 + 2c^2}{2c^2}$. Find the greatest percentage error in making this replacement if the greatest value of $x = \frac{1}{10}c$.

(8) Obtain a series for $\tanh^{-1} x$, and show by means of this series and those for $\log_e (1 + x)$ and $\log_e (1 - x)$ that $\tanh^{-1} x = \frac{1}{2} \log_e \frac{1 + x}{1 - x}$. Prove this also by making use of the relation $x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$.

(9) Write down the values of $\sin x$ and $\cos x$ in series of ascending powers of x.

Prove that the length of an arc of a circle is given approximately by the formula $\frac{1}{3}(8b - a)$, where a is the chord of the arc and b the chord of half the arc.

Show that the error made in the length of an arc which subtends an angle of 90° at the centre of the circle calculated by this formula is about $\frac{1}{800}$ of the radius. (U.L.)

(10) Write down the series for $\sin \theta$ and $\cos \theta$ in powers of θ, and verify that the series satisfy the relation $\frac{d}{d\theta}(\sin \theta) = \cos \theta$. Using the series, prove that

$$\cot \theta + \theta \operatorname{cosec}^2 \theta - \frac{2}{\theta} = \frac{2}{45}(\theta^3)\left[1 + \frac{4}{21}\theta^2 + \ldots\right] \qquad \text{(U.L.)}$$

(11) Write down the expansion of $\log (1 + x)$ as a series in ascending powers of x, stating for what values of x the series is convergent.

Show that for a certain range of real values of x the two series

$$2\left(\frac{1-x}{1+x}\right) + \frac{2(1-x)^3}{3(1+x)^3} + \frac{2(1-x)^5}{5(1+x)^5} + \dots$$

$$(1-x) + \tfrac{1}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3 + \dots$$

represent the same function, and determine that range. (U.L.)

(12) State the series for the expansion of $\log_e(1+x)$ in ascending powers of x, and find the condition for the convergency of the series.

If a and b are small compared with x, show that

$$\log_e(x+a) - \log_e x = \frac{a}{b}\left(1 + \frac{b-a}{2x}\right)\{\log_e(x+b) - \log_e x\} \qquad \text{(U.L.)}$$

(13) Obtain the expansion of $\log_e(1+x)$ in ascending powers of x, stating the conditions under which the expansion is possible.

Obtain a series which will give the value of $\log_e 5$, and calculate its value to four places of decimals. (U.L.)

(14) Using the logarithmic expansion, show how the value of $\log_e(a+h)$ may be found, where h is less than a and the value of $\log_e a$ is given.

Prove that, if h and k are less than a,

$$\frac{h}{k}\cdot\frac{1}{1-\frac{k}{2a}} > \frac{\log_e(a+h) - \log_e a}{\log_e(a+k) - \log_e a} > \frac{h}{k}\left(1 - \frac{h}{2a}\right) \qquad \text{(U.L.)}$$

(15) Given $\log_{10} 73 = 1·863323$ correct to seven significant figures, use the method of Art. 59 to find correct to six significant figures $\log_{10} 73·55$.

$$(\log_e N = 2·302585 \log_{10} N)$$

(16) Sin $41° = 0·6560590$, cos $41° = 0·7547096$. Find sin 42° and cos 42° correct to six significant figures.

(17) Find the error in assuming that for large values of N, $\log_e(N+1) = \log_e N + \frac{2}{2N+1}$. What is the approximate error when $N = 50$?

(18) Show that a root of the equation $3x^3 - 5x^2 - 19x + 31 = 0$ lies between 1 and 2. Determine, graphically or otherwise, this root correct to four significant figures.

(19) Show by means of a rough graph that the least positive root of the equation $\tan x = \frac{1}{2}x$ lies between π and $\frac{3\pi}{2}$. Find this least root correct to four significant figures. (U.L.)

(20) If $f(x) = x^4 + 2x^3 + 6x^2 + 20x - 15$, show that $f''(x)$ is positive for all values of x. Thence show that $f(x) = 0$ cannot have more than two real roots.

Given that $x = -3$ is approximately a root of the last equation, find this root to three significant figures. (U.L.)

(21) Prove that if $x = a$ is an approximate value of a root of the equation $f(x) = 0$, then $a - \frac{f(a)}{f'(a)}$ is in general a closer approximation.

Illustrate this graphically, and indicate the circumstances in which this method of procedure will fail to give a closer approximation. (U.L.)

(22) Find the positive root of the equation $x^4 + x^3 - 2x^2 - 6x - 8 = 0$ accurately to four significant figures. (U.L.)

(23) Solve the equation $\tan\theta = \theta$ for the root which is nearly $\frac{3\pi}{2}$ (θ is in radians). Find the equivalent number of degrees. Show also that for large values of θ, the roots are approximately given by $(2n+1)\frac{\pi}{2}$, where n is any integer.

(24) Show that the equation $x^5 + 2x^3 - 4x^2 - 8x - 17 = 0$ has not more than one positive root, and calculate this root to four significant figures. (U.L.)

(25) If a root of the equation $f(x) = 0$ is known to lie between two values of x, a and $a + h$, where h is small, and if $f(a) = \xi$, and $f(a+h) = -\eta$, where ξ and η are small, show that a second approximation to the root is $a + \frac{h\xi}{\xi + \eta}$.

Prove that the equation $x^3 - 4x + 1 = 0$ has a root lying between 1 and 2, and find it correct to two decimal places. (U.L.)

(26) The equation $x^3(2 + x) = 11{\cdot}54$ has a root which does not differ much from $1{\cdot}5$; find this root correct to two decimal places. Use the iteration method.

(27) Using Maclaurin's theorem, expand θ in a series of ascending powers of x as far as the term in x^4, given that $\sin\theta = \sin\alpha \cdot \cos x$, where α is constant.

(28) Expand $\sqrt{1-x}$ by (1) the binomial theorem, and (2) Maclaurin's theorem.

(29) Prove the "rule of proportional parts" for logarithms, i.e. prove that if n is a large number (>1000 say) and $h < 1$, $\frac{\log(n+h) - \log n}{\log(n+1) - \log n} = h$, approx.

Given $\log_{10} 9779 = 3{\cdot}9902944$ and $\log_{10} 9780 = 3{\cdot}9903389$, find $\log_{10} 9779{\cdot}4$.

(30) Show, graphically or otherwise, that the equation $4^x - 5x = 0$ has two real roots, and state between which integers these roots lie. Find the smaller root correct to three decimal places.

(31) If $f(a) = 0$ and $F(a) = 0$ while $f'(a)$ and $F'(a)$ are not both zero, prove that $\lim_{x\to a} \frac{f(x)}{F(x)} = \frac{f'(a)}{F'(a)}$.

Work out the following limits—

$$\lim_{x\to a} \frac{x^2 \log_e x - a^2 \log_e a}{x^2 - a^2}, \quad \lim_{x\to a} \frac{x^2 \log_e a - a^2 \log_e x}{x^2 - a^2}, \quad \lim_{\theta\to a} \frac{1 - \cos(\theta - a)}{(\sin\theta - \sin a)^2}$$

(U.L.)

(32) The equation $x = \frac{3}{2 + x^3}$ has a root $x = 1$. Try to determine this root by the iteration method assuming as a first approximation $x = 0{\cdot}9$. Why does the method fail? Sketch the graphs of x and $\frac{3}{2 + x^3}$ and illustrate your answer.

(33) Find the values of $\underset{x\to 0}{Lt.} \frac{\cos x - 1}{x^2}$ and $\underset{x\to 0}{Lt.} \frac{2\cos x - 2 + x^2}{x^4}$

(34) Find the following limits—

$$\underset{x\to 0}{Lt.} \frac{\cosh x - 1}{x^2}, \quad \underset{x\to 0}{Lt.} \frac{\sinh x - x}{x^3}, \quad \underset{x\to 0}{Lt.} \frac{\sinh x}{x}$$

(35) Find $\underset{x\to a}{Lt.}\dfrac{x^n-a^n}{x^2-a^2}$ and $\underset{x\to 2}{Lt.}\dfrac{x^4-7x^3+8x^2+9x-26}{x^3-9x^2+7x+14}$.

(36) Show that $\underset{x\to 0}{Lt.}\dfrac{ax^2+bx}{dx^2+cx}=\dfrac{b}{c}$, $\underset{y\to \infty}{Lt.}\dfrac{ay^2+by}{dy^2+ey}=\dfrac{a}{d}$.

(37) Find $\underset{x\to 0}{Lt.}\dfrac{a^x-b^x}{x}$, (1) by using the relation (IV.25), (2) by using the expansions in power series of a^x and b^x.

(38) Prove that $\underset{x\to 0}{Lt.}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\infty$

and find $\underset{x\to 1}{Lt.}\dfrac{\sqrt{1+x}-\sqrt{1-x^2}}{\sqrt{1-x}-\sqrt{1-x^2}}$

CHAPTER V

AND MINIMA—PARTIAL DIFFERENTIATION

and Minima. In a great many practical problems he engineer has to deal it is necessary to find the greatest ue of some function and the particular value of the ich gives the function such a value. We proceed to es by means of which we may determine these turning hey are often called, and may be able to distinguish m.

N. A function $y = f(x)$ is said to have a *maximum* $= a$, if $f(a-h)$ and $f(a+h)$ are always less than $f(a)$, small but finite interval, and $f(x)$ is said to have a alue at $x = a$, if $f(a-h)$ and $f(a+h)$ are always $f(a)$.

er must note that the terms *maximum* and *minimum* as definition do not necessarily mean the *greatest possible ossible* values. There may, in fact, be more than one r *minimum* value of a function.

a point P to travel from left to right along the curve own in Fig. 36. As P approaches the point H the creases up to the value HK, which it assumes when , and as P recedes from H the ordinate decreases. Again, aches the point L the ordinate decreases down to the which it assumes when P reaches L, and as P recedes ordinate increases. According to the definition above, as a maximum value HK when $x = OK$, and a minimum hen $x = OM$. As P approaches I the ordinate increases alue IN which it assumes when P reaches I, and as P n I the ordinate continues to increase. At each of the , and I the tangent to the curve is parallel to the x-axis, dient $\frac{dy}{dx}$ is zero. A necessary condition, therefore, for or minimum value of $y = f(x)$ is $\frac{dy}{dx} = 0$; but that this not sufficient is evident from the existence of a point n the curve. I is called a "point of inflexion" on the consider such points more fully in Art. 65.

The rate of increase of the gradient $\frac{dy}{dx}$ with respect to x is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$, so that if $\frac{dy}{dx}$ is increasing as x increases, $\frac{d^2y}{dx^2}$ is positive, and if $\frac{dy}{dx}$ is decreasing, $\frac{d^2y}{dx^2}$ is negative. Confining our attention to positions of our travelling point P in the neighbourhood of H, L, and I, we note the behaviour of the quantities, y, $\frac{dy}{dx}$,

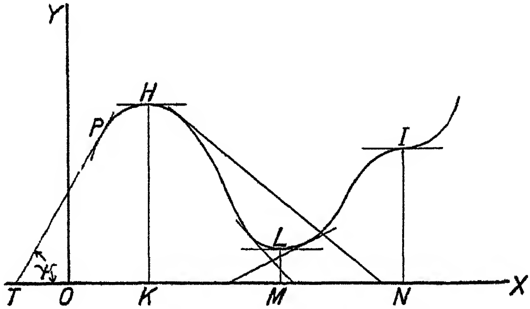


FIG. 36

$\frac{d^2y}{dx^2}$ in the various positions of P. The results are tabulated below.

Position of P	Ordinate y	Gradient $= \frac{dy}{dx} = \tan\psi$	Rate of Increase of Gradient $= \frac{d^2y}{dx^2}$
Approaching H.	Increasing	Positive and decreasing.	Negative.
At H.	HK (max. value).	Zero.	
Receding from H.	Decreasing.	Negative and decreasing.	
Approaching L.	Decreasing.	Negative and increasing.	Positive.
At L.	LM (min. value)	Zero.	
Receding from L.	Increasing.	Positive and increasing.	
Approaching I	Increasing	Positive and decreasing.	Negative.
At I.	IN.	Zero.	Zero.
Receding from I.	Increasing.	Positive and increasing.	Positive.

A consideration of this table shows us that a second test for maximum and minimum values of $y = f(x)$ is that $\frac{dy}{dx}$ changes from

positive to negative as x passes through a maximum value, and from negative to positive as x passes through a minimum value; or, alternatively, that $\frac{d^2y}{dx^2}$ is negative at a maximum point on the curve and positive at a minimum point. Sometimes it is convenient to use the former of these two alternative tests, at other times the latter. It may happen, however, that a value of x which makes $\frac{dy}{dx}$ zero, also makes $\frac{d^2y}{dx^2}$ zero. A further test is then necessary. By Taylor's theorem we have

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \ldots$$
$$+ \frac{h^r}{\lfloor r}f^{(r)}(x) + \ldots$$

Suppose that when $x = a$ all the differential coefficients from $f'(x)$ to $f^{(r-1)}(x)$ vanish; then

$$f(a + h) - f(a) = \frac{h^r}{\lfloor r}f^{(r)}(a) + \ldots \quad . \quad \text{(V.1)}$$

We can, in general, take h so small that the first term of the series on the right is numerically greater than the sum of all the terms which follow. The sign of $f(a + h) - f(a)$ will then depend on that of $\frac{h^r}{\lfloor r}f^{(r)}(a)$. Now, if r be odd, $f(a + h) - f(a)$ will change sign with h, and therefore $x = a$ will give neither a maximum nor a minimum value of $f(x)$; if r be even, then, irrespective of the sign of h, $f(a + h) > f(a)$ if $f^{(r)}(a)$ is positive, and $f(a)$ is then a minimum value; and $f(a + h) < f(a)$ if $f^{(r)}(a)$ is negative, and $f(a)$ is then a maximum value.

In many practical problems common sense tells us whether the value obtained for a given function is a maximum or minimum, and no further test is necessary.

EXAMPLE 1

If in testing for a maximum or a minimum value of $y = f(x)$, it is found that $\frac{d^2y}{dx^2} = 0$, what further tests must be tried?

A wire 3 ft long has to be bent into the form of a rectangle with an external circular loop at one corner, and the rectangle is to have one side double the other. Find the dimensions of the circle and rectangle so that the total area enclosed is a minimum. (U.L.)

The tests required are discussed in Art. 64 above. Let x and $2x$ be the sides of the rectangle and y the radius of the circle. Then by data,

$$6x + 2\pi y = 3 \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Also total area } A = 2x^2 + \pi y^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) $x = \frac{1}{6}(3 - 2\pi y)$, and substituting this value in (2) we obtain A in terms of y only.

Thus, $\qquad A = \dfrac{1}{18}(3 - 2\pi y)^2 + \pi y^2$

$$\therefore \qquad \frac{dA}{dy} = \frac{1}{9}(3 - 2\pi y)(-2\pi) + 2\pi y = \frac{2\pi}{9}[(9 + 2\pi)y - 3]$$

If A is a minimum, $\dfrac{dA}{dy} = 0$, whence $y = \dfrac{3}{9 + 2\pi}$

Again, $\dfrac{d^2A}{dy^2} = \dfrac{2\pi}{9}(9 + 2\pi)$, which is a positive quantity.

Therefore for the total area enclosed to be a minimum, the radius of the circle must equal $\dfrac{3}{9 + 2\pi}$ ft ($= 0{\cdot}196$ ft).

From above, $x = \dfrac{1}{6}\left(3 - 2\pi \cdot \dfrac{3}{9 + 2\pi}\right) = \dfrac{9}{2(9 + 2\pi)}$ ft.

Therefore sides of rectangle are $\dfrac{9}{2(9 + 2\pi)}$ ft and $\dfrac{9}{9 + 2\pi}$ ft, i.e. $0{\cdot}294$ ft and $0{\cdot}589$ ft.

EXAMPLE 2

The cross-section of a sheep trough is an arc of a circle of central angle 2θ and radius r; the length of the trough is a constant l. For a given internal trough volume find θ, so that the material used may be a maximum or a minimum, and discriminate between them. (U.L.)

The length of the arc $= 2r\theta$, and the area bounded by the arc and its chord $\frac{1}{2}r^2(2\theta - \sin 2\theta)$. Then volume of trough $= \frac{1}{2}r^2(2\theta - \sin 2\theta)l = V$, say; and surface area (inside) $= 2r\theta l = A$, say.

V is given, and $r^2 = \dfrac{2V}{(2\theta - \sin 2\theta)l}$, so that, in terms of the single variable θ,

$$A^2 = 4l^2\theta^2 \cdot \frac{2V}{(2\theta - \sin 2\theta)l} = 8lV\frac{\theta^2}{2\theta - \sin 2\theta}$$

A will be a maximum or minimum when A^2 is a maximum or minimum (since we are dealing with essentially positive quantities), i.e. when $\dfrac{\theta^2}{2\theta - \sin 2\theta}$ is a maximum or minimum, i.e. when $\dfrac{2\theta - \sin 2\theta}{\theta^2}$ is a minimum or maximum. The condition is

$$\frac{d}{d\theta}\left(\frac{2\theta - \sin 2\theta}{\theta^2}\right) = 0$$

i.e. $$\frac{\theta^2(2 - 2\cos 2\theta) - (2\theta - \sin 2\theta) \cdot 2\theta}{\theta^4} = 0$$

i.e. $$\frac{2}{\theta^3}[\sin 2\theta - \theta(1 + \cos 2\theta)] = 0$$

Hence, $2 \sin \theta \cos \theta = 2\theta \cos^2 \theta$, which gives $\cos \theta = 0$, or $\tan \theta = \theta$, i.e. $\theta = 90°$ or $257{\cdot}5°$ (nearly). (See Ex. IV, No. 23.)

Let $E = \frac{2}{\theta^3}[2 \sin \theta \cos \theta - 2\theta \cos^2 \theta] = \frac{4 \cos^2 \theta}{\theta^3}[\tan \theta - \theta]$. The sign of E will depend on that of $\tan \theta - \theta$, since $\cos^2\theta$ is positive and θ is positive. Now, when θ is slightly less than $90°$, $\tan \theta$ is positive and greater than θ. Therefore E is positive. When θ is slightly greater than $90°$, $\tan \theta$ is negative, and therefore E is negative. Hence, since E changes from positive to negative as θ passes through the value $90°$, $\frac{2\theta - \sin 2\theta}{\theta^2}$ is a maximum when $\theta = 90°$, and therefore A is a minimum when $\theta = 90°$.

The value $\theta = 257{\cdot}5°$ has no application here as it would make the central angle greater than $360°$.

EXAMPLE 3

Find the maximum and minimum values of the function $x^5 - 7{\cdot}5x^4 + 20x^3 - 20x^2 - 5$.

Let $y = x^5 - 7{\cdot}5x^4 + 20x^3 - 20x^2 - 5$.

Then $\frac{dy}{dx} = 5x^4 - 30x^3 + 60x^2 - 40x = 5x(x - 2)^3$; and $\frac{dy}{dx} = 0$ when $x = 0$ or 2.

Again, $\frac{d^2y}{dx^2} = 20x^3 - 90x^2 + 120x - 40 = 10(2x^3 - 9x^2 + 12x - 4)$.

When $x = 0$, $\frac{d^2y}{dx^2} = -40$, so that y is a maximum when $x = 0$ and $y_{max} = -5$.

When $x = 2$, $\frac{d^2y}{dx^2} = 10(16 - 36 + 24 - 4) = 0$, so that a further test is required.

Now, $\frac{d^3y}{dx^3} = 10(6x^2 - 18x + 12)$, and when $x = 2$, $\frac{d^3y}{dx^3} = 10(24 - 36 + 12) = 0$.

Again, $\frac{d^4y}{dx^4} = 10(12x - 18) = 60(2x - 3)$, and when $x = 2$, $\frac{d^4y}{dx^4} = +60$.

Hence, since this differential coefficient is of even order and positive, y is a minimum when $x = 2$; and $y_{min} = 32 - 120 + 160 - 80 - 5 = -13$.

NOTE. In this example it is easy to deduce that if x is slightly less than 2, $\frac{dy}{dx}$ is negative, and if x is slightly greater than 2, $\frac{dy}{dx}$ is positive; and hence $x = 2$ gives a minimum value of y. But we have adopted the alternative method in order to show the application of the further tests necessary when $\frac{d^2y}{dx^2} = 0$.

65. **Concavity and Convexity. Points of Inflexion.** Consider the curve shown in Fig. 37. As a point moves along the curve from left to right in the neighbourhood of the point A, the gradient $\frac{dy}{dx}$ $(= \tan \psi)$ increases, so that at A, $\frac{d^2y}{dx^2}$ is positive. By a similar

argument we deduce that at B, $\frac{d^2y}{dx^2}$ is negative. At A the curve is said to be concave upwards and at B convex upwards. Hence, a curve is concave upwards at any point (x, y) on it if $\frac{d^2y}{dx^2}$ is positive at that point, and convex upwards if $\frac{d^2y}{dx^2}$ is negative at that point.

We can regard the point C as the junction of two parts of the curve, the one concave upwards and the other convex upwards, so that the gradient $\frac{dy}{dx}$ increases up to C, and thereafter decreases.

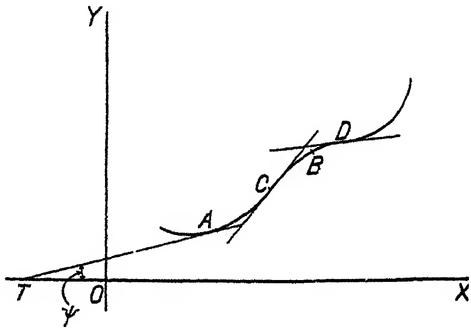


FIG. 37

It follows that at C, $\frac{dy}{dx}$ has a maximum value and therefore, by Art. 64, $\frac{d^2y}{dx^2} = 0$. At the point D which joins a part of the curve convex upwards with a part concave upwards, $\frac{dy}{dx}$ obviously has a minimum value, and again $\frac{d^2y}{dx^2} = 0$. C and D are called "points of inflexion" on the curve. The tests for these points of inflexion are really, as indicated above, the tests for maximum or minimum values of $\frac{dy}{dx}$.

Hence, the curve $y = f(x)$ has a point of inflexion at $x = a$ if (1) $\frac{d^2y}{dx^2} = 0$ when $x = a$; and (2) $\frac{d^2y}{dx^2}$ changes sign as x passes through the value a.

On referring to Fig. 36 and the table in Art. 64, the student will note that the above conditions are satisfied at the point I, which is,

therefore, a point of inflexion, but although $\frac{d^2y}{dx^2} = 0$ at I, this is not a necessary condition for a point of inflexion.

EXAMPLE 1

Find the maximum and minimum ordinates of the curve $y = (x - 2)^2(x - 7)$, and the position of the point of inflexion

Here $\frac{dy}{dx} = (x - 2)^2 (1) + (x - 7) . 2 . (x - 2)$

$= (x - 2)(x - 2 + 2x - 14) = (x - 2)(3x - 16)$

For turning values of y, $\frac{dy}{dx} = 0$, whence $(x - 2)(3x - 16) = 0$, $\therefore x = 2$ or $5\frac{1}{3}$

Again, $\frac{d^2y}{dx^2} = (x - 2)3 + (3x - 16)(1) = 6x - 22$.

When $x = 2$, $\frac{d^2y}{dx^2}$ is negative $(= -10)$, and when $x = 5\frac{1}{3}$, $\frac{d^2y}{dx^2}$ is positive $(= +10)$; so that y is a maximum when $x = 2$ and a minimum when $x = 5\frac{1}{3}$

$\therefore$ Maximum ordinate $= (2 - 2)^2(2 - 7) = 0$.

Minimum ordinate $= (5\frac{1}{3} - 2)^2(5\frac{1}{3} - 7) = -18\frac{14}{27}$.

When $\frac{d^2y}{dx^2} = 0$, $6x = 22$, or $x = \frac{11}{3}$; also when x is slightly less than $\frac{11}{3}$, $\frac{d^2y}{dx^2}$ is negative, and when x is slightly greater than $\frac{11}{3}$, $\frac{d^2y}{dx^2}$ is positive. There is, therefore, a point of inflexion at the point $\left(\frac{11}{3}, -\frac{250}{27}\right)$.

EXAMPLE 2

Give the tests for a point of inflexion (contra-flexure) in a plane. A beam of length $2l$ is supported at the same level at distances $\frac{l}{4}$ from each end, the load is $2W$ and distributed so that at any point the load per unit length $= kz$ where $z =$ distance from nearer end. The bending moment being approximately $EI\frac{d^2y}{dx^2}$, find its value at any point distant x from the centre and determine (a) the deflection at any point, (b) the slope at the supports, (c) the points of inflexion.

(U.L.)

The tests required are given above.

The load on half the beam $= \int_0^l kz\,dz = \frac{kl^2}{2}$; hence, $\frac{kl^2}{2} = W$, and therefore $k = \frac{2W}{l^2}$. By symmetry the reactions at the supports each $= W$ (Fig. 38). Let P be a point distant x from the centre of the beam; then the load on an element Δz of the portion $PB = kz\Delta z$ and its moment about $P = kz\Delta z(l - x - z)$.

Bending moment at $P = EI\dfrac{d^2y}{dx^2} = W\left(\dfrac{3l}{4} - x\right) - k\displaystyle\int_0^{l-x} z(l - x - z)\,dz$

$$= W\frac{3l - 4x}{4} - \frac{2W}{l^2}\left[(l-x)\frac{z^2}{2} - \frac{z^3}{3}\right]_0^{l-x}$$

$$= W\frac{3l - 4x}{4} - W\frac{(l-x)^3}{3l^2}$$

$$= \frac{W}{12l^2}[4x^3 - 12lx^2 + 5l^3] \quad (1)$$

The points of contra-flexure occur where $\dfrac{d^2y}{dx^2} = 0$, i.e. where $4x^3 - 12lx^2 + 5l^3 = 0$. On solving, we obtain $x = 0{\cdot}744l$ approximately.

Integrating (1) we have

$$EI\frac{dy}{dx} = \frac{W}{12l^2}[x^4 - 4lx^3 + 5l^3x] \quad . \quad . \quad (2)$$

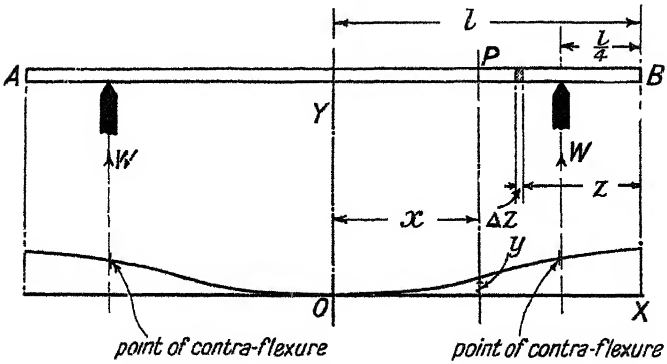


FIG. 38

the constant of integration being zero, since $\dfrac{dy}{dx} = 0$ when $x = 0$.

Now, the slope at each support = value of $\dfrac{dy}{dx}$ when $x = \dfrac{3l}{4}$

$$= \frac{W}{12EIl^2}\left[\frac{81l^4}{256} - \frac{27l^4}{16} + \frac{15l^4}{4}\right]$$

$$= \frac{203Wl^3}{1024EI}$$

Integrating (2) we have

$$EIy = \frac{W}{12l^2}\left[\frac{x^5}{5} - lx^4 + \frac{5l^3x^2}{2}\right] \quad . \quad . \quad . \quad . \quad (3)$$

the constant of integration being zero, since $y = 0$ when $x = 0$ by our choice of origin.

Hence, the deflection at P = value of y at point of support − value of y at P.

$$\text{Now at point of support, } y = \frac{W}{12EIl}\left[\frac{1}{5}\left(\frac{3l}{4}\right)^5 - l\left(\frac{3l}{4}\right)^4 + \frac{5l^3}{2}\cdot\left(\frac{3l}{4}\right)^2\right]$$

$$= \frac{W}{12EIl^2}\left[\frac{5823}{5120}l^6\right]$$

$$\therefore \qquad \text{Deflection at } P = \frac{W}{12EIl^2}\left[\frac{5823}{5120}l^6 - \frac{x^5}{5} + lx^4 - \frac{5l^3x^2}{2}\right]$$

66. Forces in the Members of Over-rigid Frames. Fig. 39 shows a framework of bars forming the quadrilateral $ABCD$. The bars are connected by frictionless pin joints at their ends, and additional bars join B to D and C to A. If one of the bars were removed, say BD, the framework would still remain rigid, and it would be possible to find the stress in each member due to the application of an external system of balanced forces to the points A, B, C, D. Frames or structures of this type are said to be redundant or over-rigid. Assuming that all the bars remain it is obvious that the structure may be self-strained. If, for instance, the bar AB is put in last and it is longer than the gap AB in the figure, the structure would need to be strained in order to get this bar into position. When such a structure is dealt with the assumption is usually made that, when there are no external forces acting, the stress in each member is zero. The stresses in the various parts due to any external system of loads cannot be determined by the use of the triangle or polygon of forces, or by ordinary analytical means as the number of unknown stresses is greater than the number of equations obtainable from statical principles. In order to solve the problem we make use of the "Principle of Least Action," which states that in any position of equilibrium of a material system the total energy of the system is a maximum or a minimum. The method of dealing with problems concerning redundant frames is shown in the following example—

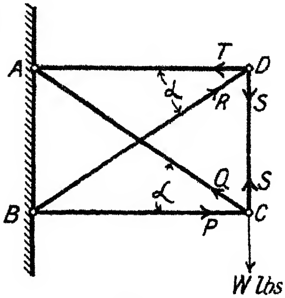


FIG. 39

EXAMPLE

A square $ABCD$ with its diagonals BD, AC, forms a braced structure. It is fixed in a vertical position with AB attached to a vertical wall. A load W lb is hung from C. Find the forces in the bars AD, DC, CB, BD, and AC.

Let P, Q, R, S, T be the forces in the bars BC, CA, BD, CD, DA, as shown in Fig. 39, and let α be the magnitude of the angle ACB. Resolving horizontally

and vertically for the equilibrium of the forces acting at the points C and D, we have

$$Q\cos\alpha = P \qquad \text{i.e. } Q = \sqrt{2}.P \text{ (since } \alpha = 45^\circ)$$
$$Q\sin\alpha = W - S \qquad \therefore S = W - P$$
$$R\sin\alpha = S \qquad \therefore R = \sqrt{2}(W - P)$$
$$R\cos\alpha = T \qquad \therefore T = W - P$$

These four equations contain the five unknowns, P, Q, R, S, T. To obtain a fifth equation we apply the above principle. It is proved in textbooks on strength of materials that the elastic energy of a straight bar of length L and cross-sectional area A under the action of an axial tension or thrust F is $\dfrac{F^2L}{2EA}$, where E is constant. If the side of the square is of length L the diagonal is of length $\sqrt{2}L$. Suppose A to be the same for all the bars and that the bars are all made of the same material. Then

$$U = \text{total elastic energy of the bars}$$
$$= \frac{L}{2EA}\{P^2 + \sqrt{2}Q^2 + \sqrt{2}R^2 + S^2 + T^2\}$$

Substitute for Q, R, S, and T in terms of P so as to obtain U as a function of P only. Then

$$U = \frac{L}{2EA}\{P^2 + 2\sqrt{2}P^2 + 2\sqrt{2}(W - P)^2 + (W - P)^2 + (W - P)^2\}$$
$$= \frac{L}{2EA}\{P^2(3 + 4\sqrt{2}) - 2WP(2\sqrt{2} + 2) + 2W^2(\sqrt{2} + 1)\}$$

By the principle of least action U is a maximum or a minimum. Hence $\dfrac{dU}{dP} = 0$, i.e. $(3 + 4\sqrt{2})P = (2\sqrt{2} + 2)W$, or $P = 0{\cdot}558W$.

By substitution, we find that $Q = 0{\cdot}789W$, $R = 0{\cdot}625W$, $S = T = 0{\cdot}442W$. In this solution we have assumed the bar AB fixed to the wall to be infinitely rigid. If the bar is free to stretch or contract its strain energy must be included in the expression for U.

67. **Partial Differentiation.** Let $z = f(x, y)$ be a continuous function of x and y. If y is assumed constant and z is differentiated with respect to x, the result so obtained is called the *partial differential coefficient* of z with respect to x, and is denoted by the symbol $\dfrac{\partial z}{\partial x}$. Similarly $\dfrac{\partial z}{\partial y}$ is the partial differential coefficient of z with respect to y, x being assumed constant.

The geometrical interpretation of these partial differential coefficients is evident from a study of Fig. 40, which shows part of the surface $z = f(x, y)$. Let the planes $x = a$, $y = b$ cut the surface

along the curves APB, CPD respectively, and let the tangents to these curves at P meet the x, y plane at T and R respectively. The equation of the curve APB in the plane $x = a$ is $z = f(a, y)$, so that in this plane $\frac{\partial z}{\partial y}$ has the same meaning as $\frac{dz}{dy}$.

i.e. $$\frac{\partial z}{\partial y} = -\tan PTM = \text{gradient of curve } APB \text{ at } P \qquad \text{(V.2)}$$

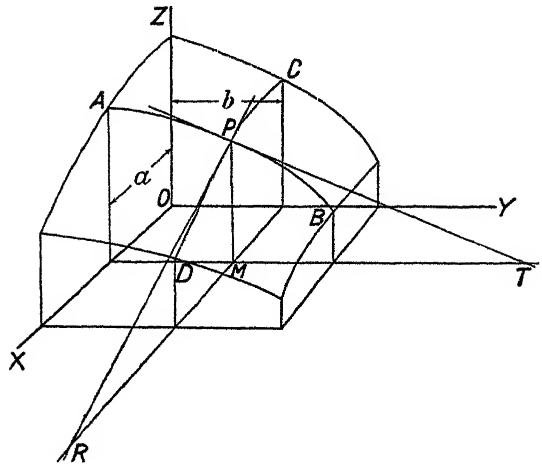


Fig. 40

By similar reasoning we find that

$$\frac{\partial z}{\partial x} = -\tan PRM = \text{gradient of curve } CPD \text{ at } P \qquad \text{(V.3)}$$

In general, if $u = f(x, y, z, \ldots)$, $\frac{\partial u}{\partial x}$ is found by differentiating with respect to x, assuming y, z, etc., constant; $\frac{\partial u}{\partial y}$ by differentiating with respect to y, assuming x, z, etc., constant, and so on.

By differentiating $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ with respect to x and y in turn, we obtain the second partial differential coefficients as given below

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) : \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \quad . \quad . \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) : \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \quad . \quad . \end{aligned}\right\} \quad \text{(V.4)}$$

In general, $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$, so that the order in which the differentiations are performed does not affect the final result.

EXAMPLE 1

Differentiate $\tan^{-1}\left(\dfrac{y}{x}\right)$ partially with respect to x and also with respect to y.

Verify that, if $z = 3xy - y^3 + (y^2 - 2x)^{\frac{3}{2}}$, then

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x} \text{ and } \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 \qquad \text{(U.L.)}$$

$$\frac{\partial}{\partial x} \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1 + \dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial}{\partial y} \tan^{-1}\left(\frac{y}{x}\right) = \frac{x^2}{x^2 + y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$z = 3xy - y^3 + (y^2 - 2x)^{\frac{3}{2}}$$

$$\therefore \qquad \frac{\partial z}{\partial x} = 3y + \tfrac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(-2) = 3[y - (y^2 - 2x)^{\frac{1}{2}}]$$

$$\frac{\partial^2 z}{\partial x^2} = -\tfrac{3}{2}(y^2 - 2x)^{-\frac{1}{2}} \cdot (-2) = 3(y^2 - 2x)^{-\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = 3[1 - \tfrac{1}{2}(y^2 - 2x)^{-\frac{1}{2}}(2y)] = 3[1 - y(y^2 - 2x)^{-\frac{1}{2}}]$$

$$\frac{\partial z}{\partial y} = 3x - 3y^2 + \tfrac{3}{2}(y^2 - 2x)^{\frac{1}{2}} \cdot (2y) = 3[x - y^2 + y(y^2 - 2x)^{\frac{1}{2}}]$$

$$\frac{\partial^2 z}{\partial y^2} = 3[-2y + (y^2 - 2x)^{\frac{1}{2}} + \frac{y}{2}(y^2 - 2x)^{-\frac{1}{2}} \cdot (2y)]$$

$$= 3[-2y + (y^2 - 2x)^{\frac{1}{2}} + y^2(y^2 - 2x)^{-\frac{1}{2}}]$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = 3\left[1 + \frac{y}{2}(y^2 - 2x)^{-\frac{1}{2}} \cdot (-2)\right] = 3\,[1 - y(y^2 - 2x)^{-\frac{1}{2}}]$$

We see that

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$$

Also $$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = 3(y^2 - 2x)^{-\frac{1}{2}} \times 3[-2y + (y^2 - 2x)^{\frac{1}{2}} + y^2(y^2 - 2x)^{-\frac{1}{2}}]$$

$$= 9[-2y(y^2 - 2x)^{-\frac{1}{2}} + 1 + y^2(y^2 - 2x)^{-1}]$$

$$= 9[1 - y(y^2 - 2x)^{-\frac{1}{2}}]^2$$

$$= \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2$$

EXAMPLE 2

The ellipsoid $\dfrac{x^2}{16} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$ is cut by the plane $x = 3$; find the gradient of the curve of section at the point where $y = 1{\cdot}5$ and z is positive.

EXAMPLE 3

If z be a function of x and y, and u and v be two other variables, such that

$$u = lx + my$$
$$v = ly - mx$$

show that
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

assuming that, on the right-hand side of this last equation, z is expressed as a function of u and v only. (U.L.)

We have
$$\frac{\partial u}{\partial x} = l$$
$$\frac{\partial v}{\partial x} = -m$$
$$\frac{\partial u}{\partial y} = m$$
$$\frac{\partial v}{\partial y} = l$$

Since z may be regarded as a function of u and v, where u and v are each functions of x and y, then by (V.10)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}$$

and
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \left(l\frac{\partial^2 z}{\partial u^2} - m\frac{\partial^2 z}{\partial u\partial v}\right)(l) + \left(l\frac{\partial^2 z}{\partial v\partial u} - m\frac{\partial^2 z}{\partial v^2}\right)(-m)$$

i.e.
$$\frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u\partial v} + m^2\frac{\partial^2 z}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 z}{\partial y^2} = m^2\frac{\partial^2 z}{\partial u^2} + 2lm\frac{\partial^2 z}{\partial u\partial v} + l^2\frac{\partial^2 z}{\partial v^2}$$

By addition,
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

71. **Small Corrections.** We have shown in Art. 40 that if y is a function of a single variable x, then the change Δy in y due to a small change Δx in x is given approximately by the relation $\Delta y = \frac{dy}{dx}\Delta x$. We have now to find a corresponding relation in the case of a function of two or more variables. This is supplied by the

result obtained in (V.6), namely, that if $z = f(x, y, u, \ldots)$, then the total variation Δz in z due to changes Δx, Δy, Δu, . . . in x, y, u, . . . is given by

$$\Delta z = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y + \frac{\partial z}{\partial u}\,\Delta u + \ldots$$

The following examples illustrate the application of this relation to the calculation of small corrections.

EXAMPLE 1

The angles of a triangle are calculated from the sides a, b, c; if small changes $\delta a, \delta b, \delta c$ are made in the sides, show that approximately

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cdot \cos C - \delta c \cdot \cos B)$$

where Δ is the area of the triangle.

Apply the formula to calculate in degrees and minutes the angle A of a triangle in which $a = 99{\cdot}5$, $b = 100{\cdot}2$, $c = 100{\cdot}7$. [1 degree $= 0{\cdot}0175$ radian.] (U.L.)

[*Note.* The symbol δ is frequently used as in this example to denote "a small increment of." Thus, $\delta a, \delta b$, . . . mean the same as $\Delta a, \Delta b$, . . .]

We have $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Let $z = \log \cos A = \log (b^2 + c^2 - a^2) - \log 2 - \log b - \log c.$

Now $$\frac{\partial z}{\partial a} = \frac{-2a}{b^2 + c^2 - a^2} = \frac{-2a}{2bc \cos A} = -\frac{a}{bc \cos A}$$

$$\frac{\partial z}{\partial b} = \frac{2b}{b^2 + c^2 - a^2} - \frac{1}{b} = \frac{2b}{2bc \cos A} - \frac{1}{b} = \frac{b - c \cos A}{bc \cos A} = \frac{a \cos C}{bc \cos A}$$

so $$\frac{\partial z}{\partial c} = \frac{a \cos B}{bc \cos A}$$

By (V.6) $$\delta z = \delta(\log \cos A) = \frac{\partial z}{\partial a}\cdot \delta a + \frac{\partial z}{\partial b}\cdot \delta b + \frac{\partial z}{\partial c}\cdot \delta c$$

i.e. $$-\frac{\sin A}{\cos A}\cdot \delta A = -\frac{a}{bc \cos A}\,\delta a + \frac{a \cos C}{bc \cos A}\,\delta b + \frac{a \cos B}{bc \cos A}\,\delta c$$

$$\therefore \quad \delta A = \frac{a}{bc \sin A}[\delta a - \delta b \cdot \cos C - \delta c \cdot \cos B]$$

$$= \frac{a}{2\Delta}[\delta a - \delta b \cdot \cos C - \delta c \cos B]$$

Assume $a = b = c = 100$, then $A = \frac{\pi}{3}$ radians; also $\delta a = -0{\cdot}5$, $\delta b = +0{\cdot}2$, $\delta c = +0{\cdot}7$; and $\Delta =$ area of triangle $= 2500\sqrt{3}.$

Hence, $\quad \delta A = \dfrac{100}{5000\sqrt{3}}\left[\,0.5 + 0.2(0.5) + 0.7(0.5)\right]$

$$= \frac{1}{50\sqrt{3}}[0.5 + 0.1 + 0.35]$$

$$= \frac{\sqrt{3}}{150} \cdot 0.95 = 0.011 \text{ radian}$$

i.e. $\quad \delta A = -38$ minutes

$\therefore \quad A = 60° - 38' = 59° 22'$

EXAMPLE 2

The shape of a hanging rod of uniform strength is given by $y = Ae^{\frac{w}{f}x}$ where y is the radius at any height x above a fixed point and A, w, and f are constants. Find the change in y produced by small changes δw in w and δf in f. Show that the percentage error in y is $\dfrac{wx}{f}$ times the difference in the percentage errors in w and f.

By Art. 69, $\quad \delta y = \dfrac{\partial y}{\partial w} \cdot \delta w + \dfrac{\partial y}{\partial f} \cdot \delta f$

$$= A\frac{x}{f} \cdot e^{\frac{w}{f}x} \cdot \delta w + A\left(-\frac{w}{f^2} \cdot x\right) e^{\frac{w}{f}x} \cdot \delta f$$

$$= Ae^{\frac{w}{f}x} \cdot \frac{wx}{f}\left[\frac{\delta w}{w} - \frac{\delta f}{f}\right]$$

i.e. $$\frac{\delta y}{y} = \frac{wx}{f}\left[\frac{\delta w}{w} - \frac{\delta f}{f}\right]$$

whence it follows that the percentage error in y

$$= 100\frac{\delta y}{y} = \frac{wx}{f}\left[100 \cdot \frac{\delta w}{w} - 100 \cdot \frac{\delta f}{f}\right]$$

$$= \frac{wx}{f}[\text{difference of percentage errors in } w \text{ and } f].$$

72. Extension of Taylor's Theorem. In Art. 58 we have

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \ . \ . \ . \qquad \text{(V.11)}$$

Treating $\phi(x + h, y + k)$ as a function of a single variable x, we have by the above series

$$\phi(x + h, y + k) = \phi(x, y + k) + h\frac{\partial}{\partial x}\phi(x, y + k) + \frac{h^2}{\underline{|2}}\frac{\partial^2}{\partial x^2}\phi(x, y + k) + \ . \ . \ . \qquad \text{(V.12)}$$

Now, looking upon $\phi(x, y+k)$ as a function of a single variable y

$$\phi(x, y+k) = \phi(x, y) + k\frac{\partial}{\partial y}\phi(x, y) + \frac{k^2}{\underline{|2}}\frac{\partial^2}{\partial y^2}\phi(x, y) + \ldots \quad \text{(V.13)}$$

In the same way

$$\frac{\partial}{\partial x}\phi(x, y+k) = \frac{\partial}{\partial x}\phi(x, y) + k\frac{\partial^2}{\partial y\,\partial x}\phi(x, y) + \ldots \quad \text{(V.14)}$$

and $$\frac{\partial^2}{\partial x^2}\phi(x, y+k) = \frac{\partial^2}{\partial x^2}\phi(x, y) + k\frac{\partial^3}{\partial y\,\partial x^2}\phi(x, y) + \ldots \quad \text{(V.15)}$$

Substituting from (V.13), (V.14), and (V.15) in (V.12) and writing $\frac{\partial^2}{\partial x\,\partial y}\phi(x, y)$ for $\frac{\partial^2}{\partial y\,\partial x}\phi(x, y)$, we have as far as terms of the second degree in h and k

$$\phi(x+h, y+k) = \phi(x, y) + \left(h\frac{\partial}{\partial x}\phi(x, y) + k\frac{\partial}{\partial y}\phi(x, y)\right)$$
$$+ \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2}{\partial x^2}\phi(x, y) + 2hk\frac{\partial^2}{\partial x\,\partial y}\phi(x, y) + k^2\frac{\partial^2}{\partial y^2}\phi(x, y) + \ldots \right. \quad \text{(V.16)}$$

We make use of the result (V.16) in the next article to discriminate between maximum and minimum values of a function of two variables.

73. **Maxima and Minima of a Function of Two Variables.** Let $u = \phi(x, y)$ be a continuous function of the variables x and y. Then u has a maximum or minimum value when $x = a$, $y = b$ if $\phi(a, b)$ is always greater or always less than $\phi(a+h, b+k)$, where h and k are small but finite intervals. In other words, if u has a maximum or minimum value, the function $\phi(x+h, y+k) - \phi(x, y)$ must preserve the same sign as h and k vary over their small but finite ranges. Now, by (V.16)

$$\phi(x+h, y+k) - \phi(x, y) = \left(h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y}\right) + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 u}{\partial x^2}\right.$$
$$\left. + 2hk\frac{\partial^2 u}{\partial x\,\partial y} + k^2\frac{\partial^2 u}{\partial y^2}\right) + \ldots \quad \text{(V.17)}$$

By taking h, k sufficiently small we can make the sign of the expression on the left depend on that of $h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y}$. Now this latter expression changes sign with h, k so that a necessary condition

for a maximum or minimum value of u is $h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} = 0$, and, as h and k are quite independent of each other, this implies that $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$ simultaneously. With this condition satisfied (V.17) becomes

$$\phi(x + h, y + k) - \phi(x, y) = \frac{1}{|2}\left(h^2\frac{\partial^2 u}{\partial x^2} + 2hk\frac{\partial^2 u}{\partial x\,\partial y} + k^2\frac{\partial^2 u}{\partial y^2}\right)$$

$+$ terms containing higher powers of h and k (V.18)

For small values of h and k the algebraic sign of the left-hand side of (V.18) is the same as that of the term in brackets on the right-hand side. Thus u is a maximum or minimum according as $Ah^2 + 2Hhk + Bk^2$ is negative or positive, where

$$A = \frac{\partial^2 u}{\partial x^2},\ H = \frac{\partial^2 u}{\partial x\,\partial y},\ B = \frac{\partial^2 u}{\partial y^2}$$

Now $Ah^2 + 2Hhk + Bk^2$

$$= A\left(h^2 + 2\frac{H}{A}hk + \frac{B}{A}k^2\right)$$

$$= A\left[\left(h^2 + 2\frac{H}{A}hk + \frac{H^2}{A^2}k^2\right) + \frac{AB - H^2}{A^2}k^2\right]$$

$$= A\left[\left(h + \frac{H}{A}k\right)^2 + \frac{AB - H^2}{A^2}k^2\right]$$

which has the same sign for all values of h and k if, and only if, $AB > H^2$. If this condition is satisfied, u is either a maximum or minimum, and A and B are of the same sign. The sign of the expression $Ah^2 + 2Hhk + Bk^2$ is then that of A.

We see, then, that the conditions that $u = \phi(x, y)$ should have a maximum or a minimum value, are

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 0,\ \text{and}\ \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} > \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 \quad . \tag{V.19}$$

These conditions being satisfied u is a maximum or minimum according as the sign of $\frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 u}{\partial y^2}$ is negative or positive.

EXAMPLE 1

Find the values of x and y for which $u = y^2 + x^2 + 6x + 12$ has a minimum value.

$$\frac{\partial u}{\partial x} = 2x + 6, \frac{\partial u}{\partial y} = 2y$$

Hence, for stationary values, $\left.\begin{aligned} 2x + 6 &= 0 \\ y &= 0 \end{aligned}\right\}$ i.e. $x = -3, y = 0$

Again, $$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 2, \text{ and } \frac{\partial^2 u}{\partial x \, \partial y} = 0$$

Hence, the conditions (V.19) are satisfied and u has a stationary value for $x = -3, y = 0$. Also, since $\frac{\partial^2 u}{\partial x^2}$ is positive, the value is a *minimum*.

EXAMPLE 2

If $z^2 = 6xy + 9$, find the values of x and y for which z has a stationary value.

$z = \pm \sqrt{6xy + 9}$. Hence, $\frac{\partial z}{\partial x} = \pm \frac{3y}{\sqrt{6xy + 9}}$ and $\frac{\partial z}{\partial y} = \pm \frac{3x}{\sqrt{6xy + 9}}$

For stationary values, $\frac{\partial z}{\partial x} = 0$, and $\frac{\partial z}{\partial y} = 0$

i.e. $$\frac{3y}{\sqrt{6xy + 9}} = 0 \text{ or } y = 0$$

and $$\frac{3x}{\sqrt{6xy + 9}} = 0 \text{ or } x = 0$$

There is a stationary value when $x = 0$ and $y = 0$.

$$\frac{\partial^2 z}{\partial x^2} = \pm \frac{9y^2}{(6xy + 9)}, \frac{\partial^2 z}{\partial y^2} = \pm \frac{9x^2}{(6xy + 9)}, \frac{\partial^2 z}{\partial x \partial y} = \pm \frac{9(xy + 3)}{(6xy + 9)}$$

Hence, the condition $\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ becomes

$$\frac{81x^2y^2}{(6xy + 9)^3} > \frac{81(xy + 3)^2}{(6xy + 9)^3}$$

which is not satisfied by $x = 0$, $y = 0$. Hence, the stationary value is not a maximum or a minimum.

EXAMPLE 3

Prove that the rectangular solid of maximum volume which can be inscribed in a given sphere is a cube.

Let x, y, z, be the length, breadth, and height of the solid, and let d be the diameter of the sphere. Then, a diagonal of the solid will be a diameter of the sphere; whence $x^2 + y^2 + z^2 = d^2$.

If $V =$ volume of solid, then $V = xyz = xy\sqrt{d^2 - x^2 - y^2}$.

Now, if V is a maximum, $V^2 = x^2y^2(d^2 - x^2 - y^2) = d^2x^2y^2 - x^4y^2 - x^2y^4$ is also a maximum.

The initial conditions are $\dfrac{\partial(V^2)}{\partial x} = 2d^2xy^2 - 4x^3y^2 - 2xy^4 = 0$

and $\dfrac{\partial(V^2)}{\partial y} = 2d^2x^2y - 2x^4y - 4x^2y^3 = 0$

Since $x = y = 0$ does not apply here, these equations reduce to $d^2 - 2x^2 - y^2 = 0$ and $d^2 - x^2 - 2y^2 = 0$, which give $x = y = z$.

$$\text{Again,}\quad \left.\begin{aligned} \frac{\partial^2(V^2)}{\partial x^2} &= 2d^2y^2 - 12x^2y^2 - 2y^4 = -8x^4 \\ \frac{\partial^2(V^2)}{\partial y^2} &= 2d^2x^2 - 2x^4 - 12x^2y^2 = -8x^4 \\ \frac{\partial^2(V^2)}{\partial x \partial y} &= 4d^2xy - 8x^3y - 8xy^3 = -4x^4 \end{aligned}\right\} \begin{aligned} &\text{when } x = y = z \text{ and} \\ &\text{therefore } d^2 = 3x^2 \end{aligned}$$

We see that $\dfrac{\partial^2(V^2)}{\partial x^2}\cdot\dfrac{\partial^2(V^2)}{\partial y^2} - \left(\dfrac{\partial^2(V^2)}{\partial x \partial y}\right)^2 = 64x^8 - 16x^8$ is positive, and that $\dfrac{\partial^2(V^2)}{\partial x^2}$ is negative.

Hence, V^2 (and therefore V) is a maximum when $x = y = z$; the solid is then a cube.

EXAMPLES V

(1) The horse-power generated by a Pelton wheel is proportional to $u(v - u)$, where u = velocity of wheel which is variable and v = velocity of jet which is fixed. Show that the maximum efficiency is given when $u = \frac{1}{2}v$.

(2) Prove that the curve given by the equation $y^3 = (x + 1)(2x^2 - 7x + 7)$ has turning points at $x = 0$ and $x = \frac{5}{3}$, and a point of inflexion at $x = 1$. Give a graph of the curve for values of $x \not> 3$. (U.L.)

(3) Find the maximum and minimum values of $\dfrac{(x + 2)(x + 5)}{x + 1}$ and distinguish between them.

(4) Find the values of x for which $y = \dfrac{1 + \frac{1}{2}x - \frac{1}{10}x^2}{\sqrt{1 + x}}$ has stationary values.

Sketch the curve representing y as a function of x, and show that y does not differ from unity by as much as 0·0008 for any value of x between 0 and 0·6. (U.L.)

(5) The stress p lb per sq in. on a section of a wheel tooth which makes an angle θ with the pitch surface is given by $p = \dfrac{6P \sin\theta \cos\theta}{bt^2}$, where P is the load acting on a corner of the tooth and the constants b and t are its breadth and thickness respectively in inches. For what value of θ is p a maximum, and what is the maximum stress?

(6) The greatest horse-power which a belt can transmit is given by $P = k\left(Tv - \dfrac{wv^3}{g}\right)$, where v = the velocity in feet per second, w = the weight in pounds of a piece of belt 1 ft in length, $g = 32{\cdot}2$, and T is the greatest tensile strength of the belt. Find v for maximum horse-power.

(7) Find the dimensions of (1) the heaviest, (2) the strongest, (3) the stiffest beam of rectangular section which can be cut from a round log of wood of length

L ft and diameter D ft. [If b = breadth and d = depth of beam, the strength varies as bd^2 and the stiffness as bd^3.]

(8) A rectangular tank, open at the top and with a square base, is to have a given inner surface area A; find the dimensions (inner) of the tank for maximum capacity. Find also the dimensions under the same conditions of a tank with a closed top.

(9) Work Ex. 8, assuming that the tank is in the form of a right circular cylinder.

(10) Find $\dfrac{dy}{dx}$ when $y = \log_e x$.

If in a submarine cable the range of signalling varies as $x^2 \log_e \dfrac{1}{x}$, where x is the ratio of the radius of the core to that of the cable, find the value of x for which the range of signalling is a maximum. (U.L.)

(11) Draw a rough graph of the expression $4x^3 + 3x^2 - 18x - 9$, calculating the positions of the turning points and the point of inflexion. Find from your graph the positive value of x for which the expression is zero, and, by the use of Taylor's theorem, calculate a better approximation to this value. (U.L.)

(12) The shape of a hole bored by a drill is a cone surmounting a cylinder. If the cylinder be of height h and radius r, and the semi-vertical angle of the cone be α where $\tan \alpha = \dfrac{h}{r}$, show that for a total fixed depth H of the hole the volume removed is a maximum if $h = \dfrac{H}{6}(\sqrt{7} - 1)$. (U.L.)

(13) Prove that the necessary and sufficient conditions for $f(x)$ to have a maximum or minimum value when $x = a$ are that $f'(a)$ should vanish and $f'(x)$ change sign as $f(x)$ passes through the value $f(a)$.

How much water must be put into a cylindrical vessel standing on a horizontal plane in order to bring the centre of mass as low as possible, the weight of the vessel being four-fifths of the weight of the water it can contain, and the centre of mass of the vessel being taken at the middle point of its height? (U.L.)

(14) Show that in general, $f(x)$ has a maximum or minimum value, when x is one of the roots of the equation $f'(x) = 0$. If the above method gives a maximum value of $f(x)$ in the interval from $x = a$ to $x = b$, is this necessarily the greatest value of $f(x)$ in this interval?

A uniform beam of weight W rests on supports at its extremities A and B. The beam carries a load W' at C, where $AC = \frac{1}{4}AB$. Prove that, if $W' = \frac{1}{2}W$, the bending moment is greatest at a point of the beam between C and the mid-point of AB; but that if $W' = 2W$, the bending moment is greatest at C. (U.L.)

(15) A right circular cone, including a flat circular base, is constructed of sheet metal of uniform small thickness. Express the total area of surface in terms of the volume and vertical semi-angle of the cone, and show that for a given volume the area of surface is a minimum if the vertical semi-angle $= \sin^{-1} \frac{1}{3}$. (U.L.)

(16) The weight W lb of gas flowing through the throat of a nozzle of area A is given by $W = A\left[\dfrac{2g\gamma}{\gamma - 1} w_0 p_0 \left(\alpha^{\frac{2}{\gamma}} - \alpha^{1 + \frac{1}{\gamma}}\right)\right]^{\frac{1}{2}}$, where g, γ, w_0, p_0 are constants. Find the value of α for which W is a maximum and find the numerical value of α in this case given that $\gamma = 1{\cdot}41$.

(17) If $p \sin\theta \cos\theta = \mu(s + p\cos^2\theta)$, where s and μ are constants, show that p is a minimum when $\cot 2\theta = -\mu$.

(18) The efficiency e of a screw jack is given by $e = \dfrac{\tan\alpha}{\tan(\alpha + \phi)}$ where ϕ = constant. Prove that for maximum efficiency $\alpha = 45° - \dfrac{\phi}{2}$ and that the maximum efficiency is $\dfrac{1 - \sin\phi}{1 + \sin\phi}$.

(19) The discharge Q gallons per hour of water through a circular channel of radius r is given by $Q = kr\sqrt{\dfrac{(\theta - \sin\theta)^3}{\theta}}$, where θ is the angle subtended by the wetted perimeter at the centre of the circle. Show that for maximum discharge $\sin\theta = \theta(3\cos\theta - 2)$ and show by plotting or otherwise that this is approximately satisfied by $\theta = 308°$.

(20) In $E = \dfrac{a\omega I}{1 + kI}$, E is the E.M.F. of a dynamo in volts, I the current in amperes, ω the angular velocity of the armature, and a and k are constants. If r and R are the internal and external resistances respectively, then $I = \dfrac{E}{R + r}$ and the power P given out is proportional to I^2R. Find the value of R which makes P a maximum, having given $a\omega = 1{\cdot}2$, $k = 0{\cdot}03$, and $r = 0{\cdot}10$.

(21) The total cost C of a ship per hour (including interest, wages, depreciation, coal, etc.) is given (in pounds) by $C = 3{\cdot}2 + \dfrac{v^3}{1\,800}$, where v = speed of ship in knots. For what value of v will the total cost of a passage of 1 500 nautical miles be a minimum?

(22) If $y = f(x)$ discuss the conditions that y may have a maximum or minimum value.

A uniform thin rod of length l and mass m is suspended by one end so that it can oscillate as a pendulum; a particle of mass $\frac{4}{3}m$ is attached to the rod at a distance x from the point of support. Determine x, so that the period for small oscillations may be a minimum, and find the period. (U.L.)

(23) Show that there are three points of inflexion on the graph of $y = 3x^5 + 10x^4 - 10x^3 - 60x^2 + 5$, and hence find the ranges of x over which the graph is convex upwards.

(24) Find the ranges over which the following graphs are convex upwards or downwards: (1) $y = \log_a x$; (2) the catenary $y = c\cosh\dfrac{x}{c}$; (3) $y = \dfrac{1}{x^n}$ (where $n > 1$); (4) $y = x^n$ where n is (*a*) an even positive integer, (*b*) an odd positive integer.

(25) A rectangular table of length L and breadth B is supported on four legs. A load W lb is placed on the table at a distance l from one of the edges L, and a distance b from one of the edges B. Find the forces in each of the four legs. (Use the principle of least action, neglecting the weight of the table.)

(26) A braced structure is in the form of a rectangle $ABCD$ with its diagonals BD, AC. It is fixed in a vertical position with AB attached to a vertical wall. A load of 24 lb is hung from C. Find the forces in the bars AD, DC, CB, BD, and AC, given that $AB = 3$ ft, $AD = 4$ ft. All bars of the same cross-section.

(27) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in the following cases: (1) $u = 3x^3 - 2xy^2 - 4y^3$;

(2) $u = \sin^{-1}\frac{y}{x}$; (3) $u = \log_e \tan(xy)$

(28) If $F(x, y, z) = \frac{x^2}{a} + \frac{y^2}{b} - 2z$, where a and b are constants, show that the equation $l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0$ reduces to $\frac{lx}{a} + \frac{my}{b} - n = 0$

(29) The equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point (x_1, y_1, z_1) is $(x - x_1)\frac{\partial F}{\partial x_1} + (y - y_1)\frac{\partial F}{\partial y_1} + (z - z_1)\frac{\partial F}{\partial z_1} = 0$, where $\frac{\partial F}{\partial x_1}$, $\frac{\partial F}{\partial y_1}$, $\frac{\partial F}{\partial z_1}$ are the values of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ respectively when $x = x_1$, $y = y_1$, $z = z_1$. For example, if the equation of the surface be $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$, then $\frac{\partial F}{\partial x} = \frac{2x}{a^2}$, $\frac{\partial F}{\partial y} = \frac{2y}{b^2}$, $\frac{\partial F}{\partial z} = \frac{2z}{c^2}$, and the equation of the tangent plane at the point (x_1, y_1, z_1) on the surface is $(x - x_1)\frac{2x_1}{a^2} + (y - y_1)\frac{2y_1}{b^2} + (z - z_1)\frac{2z_1}{c^2} = 0$, which reduces to $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1$.

Find the equations of the tangent planes to the surfaces—

(1) $x^2 + y^2 + z^2 = 14$ at the point (1, 2, 3);

(2) $2z = 4x^2 - 5y^2$ at the point (0·5, 2, − 9·5).

(30) Assuming $pv = R\theta$, where R is constant, show that

$$\frac{\partial p}{\partial \theta} = -\frac{\partial v}{\partial \theta}\Big/\frac{\partial v}{\partial p}; \quad \frac{\partial v}{\partial p} = -\frac{\partial \theta}{\partial p}\Big/\frac{\partial \theta}{\partial v}$$

(31) If x, y are the rectangular co-ordinates and r, θ the polar co-ordinates of a point on a plane, express r in terms of x and y, and show that $r\frac{\partial r}{\partial x} = x$, $r\frac{\partial r}{\partial y} = y$. (See Art. 77.)

(32) Find $\frac{dy}{dx}$ from the following implicit relations—

(1) $3x^2 + 2xy + y^2 + 4x - 5y + 1 = 0$ (2) $x^4 + y^4 = a^4$

(3) $xy = e^{x^2 + y^2}$ (4) $xy \log_e (x + y) = 1$

(33) Let x, y, z be the three edges of a rectangular homogeneous solid at temperature θ degrees, and let a, b, c be the values of x, y, z respectively when $\theta = 0$. Assuming the coefficient of linear expansion to be k, find the volume of the solid at temperature θ. Hence, show that the coefficient of expansion of volume is three times that of length.

(34) Two adjacent sides of a rectangle are u ft and v ft in length, and are increasing at rates of r_1 ft/sec and r_2 ft/sec respectively; prove that the area is increasing at the rate $ur_2 + vr_1$ ft²/sec. Use your result to illustrate the truth of the relation $\frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$

Illustrate also the relation $\frac{d}{dt}(uvw) = uv\frac{dw}{dt} + vu\frac{du}{dt} + wu\frac{dv}{dt}$

(35) Find the total differential dz in each of the following cases—

(1) $z = \frac{x + y}{x - y}$. (2) $z = 2x^3y - 3xy^3 + y^4$. (3) $z = \log_e\left(\cos\frac{y}{x}\right)$

(36) A captive balloon, 500 ft high, is moving horizontally at the constant rate of 20 ft/sec and is also rising at the rate of 6 ft/sec. At what rate is it receding from a point over which it passed 15 seconds ago?

(37) The height of a frustum of a right circular cone is increasing at the rate of 5 in./sec, and the radii of its ends are each decreasing at the rate of 2 in./sec. At what rate is the volume changing when the height is 50 in. and the radii of the ends are 24 in. and 18 in. respectively? [$V = \frac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2)$, where h = height and r_1 and r_2 are the radii of the ends.]

(38) If u is a homogeneous function of the nth degree in $x, y, z, \ldots$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots = nu$. This result is known as Euler's Theorem on Homogeneous Functions. Verify Euler's theorem in the following cases—

(1) $u = x^4yz + 2x^3y^3 + 5xyz^4$. (2) $u = x^3\sin\left(\frac{y}{x}\right)$

(3) $u = \frac{x^4 + y^4}{x + y}$ (4) $u = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{x + y}$

(39) Verify the relation $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$ in each of the following cases—

(1) $z = \frac{xy}{a + y}$. (2) $z = \log_e\frac{x^3 + y^3}{xy}$. (3) $z = (x - y)\sqrt{x^2 + y^2}$.

(40) Show that Laplace's equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ is satisfied by $V = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$; show also that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ is satisfied by $V = \tan^{-1}\left(\frac{y}{x}\right)$

(41) Show that the equation $\frac{\partial^2 z}{\partial y^2} = c^2\frac{\partial^2 z}{\partial x^2}$ is satisfied by $z = f(x + cy) + \phi(x - cy)$.

(42) If $f(x, y, z) = 0$, prove that $\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z} = -1$.

(43) If $u = f(x, y)$ and $y = \phi(x)$, prove that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

(44) Given $z = \phi(x, y)$ and $u = f_1(z)$, $v = f_2(z)$ prove that $\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$ and that $\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)$

(45) Define a *partial differential coefficient.*

Denoting by suffixes those variables which are taken as independent in each case, show that if x, y, z are connected by a relation $f(x, y, z) = 0$, it is not, in general, true that $\left(\frac{\partial x}{\partial z}\right)_{z,y} = 1\Big/\left(\frac{\partial z}{\partial x}\right)_{x,y}$

The pressure p, volume v, and absolute temperature T of a gas are connected by the equation $pv = RT$, where R is a constant. Prove that if Q is a function of the state of the gas such that $\left(\frac{\partial Q}{\partial v}\right)_{v,T} = p$, then $\left(\frac{\partial Q}{\partial T}\right)_{p,T} = \left(\frac{\partial Q}{\partial T}\right)_{v,T} + R$, and find the value of $\left(\frac{\partial Q}{\partial p}\right)_{p,T}$ (U.L.)

(46) If $u = \phi(x, y, z)$, show that to the first order of small quantities, $du = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz$.

The position of a point P with reference to a measured base line AB of length a is given by the angles PAB and PBA. Find the percentage error in the calculated distance of P from AB, when $a = 500$ yd, $PAB = 30°$, and $PBA = 60°$, due to underestimating a by 5 ft, and each angle by 5 minutes. (U.L.)

(47) If Δ = area of a triangle ABC, find the error in Δ due to—

(1) errors δa, δb, δc in the sides a, b, c.

(2) errors δa, δb, δC in the sides a, b, and the angle C.

(3) errors δA, δB, δc in the angles A, B, and the side c.

(4) errors δa, δb, δc in a, b, c, the sum remaining constant.

(48) Two sides of a triangle are 70 ft and 40 ft in length; the angle between these sides is 64° 42′, but this is measured by mistake as 65°. What is the error in the calculated length of the third side?

(49) A beam of span l in. is freely supported at its ends, and carries a load of w lb per inch run. The curvature at any point distant x from one end is given by $\frac{d^2y}{dx^2}\Big/\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$. The value of $\frac{dy}{dx}$ at the end is $\frac{wl^3}{24EI}$, assuming this to be small compared with unity. Prove that the fractional error in the value of the curvature at the end due to assuming the curvature to be $\frac{d^2y}{dx^2}$ is $-\frac{w^2l^6}{384E^2I^2}$ and calculate the value of this when $w = 300$, $l = 240$, $E = 30\,000\,000$, $I = 500$.

(50) The indicated horse-power (I) of an engine is calculated from the formula $I = \frac{PLAN}{33\,000}$, where $A = \frac{\pi d^2}{4}$. Assuming that errors of r per cent may have been made in measuring P, L, N, and d, find the greatest possible error in I. Show that the error in I cannot be less than r per cent.

(51) The angular momentum M of the earth is given by $M = I\omega$, where I is the moment of inertia of the earth's mass about its axis of rotation, and ω is the speed of rotation in radians per second. Prove that, if M remains constant, and the earth's radius R contracts by r per cent, where r is small, the day will shorten by $2r$ per cent $\left[I = \frac{2m}{5g}R^2\text{, where } m = \text{mass of earth.}\right]$

(52) The time of swing t of a pendulum of length l under certain conditions is given by $t = 2\pi\sqrt{\frac{l}{g'}}$ where $g' = g\left(\frac{r}{r+h}\right)^2$. Find the error in t due to errors of p per cent in h and q per cent in l.

(53) If $F = \dfrac{M}{(d-r)^2} - \dfrac{M}{(d+r)^2}$, and r is small compared with d, show that $F = \dfrac{4rM}{d^3}$ approximately, and that the percentage error is $200\dfrac{r^2}{d^2}$.

(54) Given that $y = A\sin(px + \alpha)\cos(qt + \beta)$, find the error in y due to small errors δx in x and δt in t.

(55) Prove that, if $C = M - a\sqrt{M^2 - T^2}$, where C = constant, and there are small changes δM in M and δT in T, then $T\delta T - M\delta M + \dfrac{\sqrt{M^2 - T^2}}{a}\delta M = 0$.

(56) The distance x of a point on the earth's surface from the meridian of Greenwich is given by $x = r\sin\theta\sin\lambda$, where θ = latitude, λ = longitude and r = radius at the given point. Assuming errors δr, $\delta\theta$, $\delta\lambda$, in r, θ, λ, find the error in x.

(57) If z is a function of x and y, explain and justify the equation

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Establish the tests for a maximum or minimum value of z. The plan of the part of a matchbox which contains the matches is as sketched; if z be the depth and the volume be 36 cm³, find x, y, and z for minimum wood used. (The two y sides near together are shown separated, actually they are in contact.) (Fig. 42.) (U.L.)

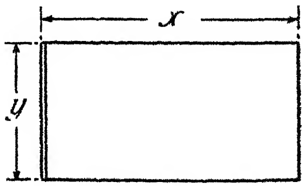


FIG. 42

(58) Prove that the rectangular solid of maximum volume for a given surface area is a cube.

(59) If $\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} = 6$, find the values of x, y, z which make $x + y + z$ a minimum.

(60) The sum of three numbers is constant; prove that their product is a maximum when they are equal.

(61) Show that if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.

[Use the formula: Area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.]

(62) Given $f(x, y, z) = \dfrac{5xyz}{x + 2y + 4z}$, find the values of x, y, z for which $f(x, y, z)$ is a maximum with the condition $xyz = 8$.

(63) Find the co-ordinates of a point $P(x, y)$ such that the sum of the squares of its distances from the rectangular axes of reference OX, OY and the line $x + y = 8$ is a minimum.

(64) Show that the expression $2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} + \cos(x+y)$ is a minimum when $x = y = \dfrac{3\pi}{2}$, and find for what values of x and y the expression is a maximum.

(65) A tent has its plan a square (side equal to x), its sides are vertical and of height z, and the top is a regular pyramid of height h. Find x and h in terms of z if the area of the material is a minimum, when x, z, and h all vary subject to the condition that the volume of the tent is given. (U.L.)

(66) Show that $y = \sin ax \sin bt$ (a and b constants) has turning values for any of the points $x = \dfrac{p\pi}{2a}$, $t = \dfrac{q\pi}{2b}$, where p and q are integers.

(67) The following values of x and y follow approximately the law $y = ax + b$, ($x = 1, y = 3$), ($x = 4, y = 4$), ($x = 7, y = 4{\cdot}5$). Find by the method of least squares the probable values of a and b. [The method of least squares assumes that the sum of the squares of the errors is a minimum.]

(68) If the base BC of a triangle ABC is kept fixed and the vertex A moved to A', where AA', of small length δx, is parallel to BC and in the direction B to C, find the corresponding changes in the lengths of the sides AB, AC, and prove that the change in the angle A, expressed in radians, is

$$h\,\delta x\left(\frac{1}{c^2}-\frac{1}{b^2}\right)$$

where h is the unaltered height of the triangle. (U.L.)

(69) A tent on a square base of side $2a$ consists of four vertical sides of height b surmounted by a regular pyramid of height h. If the volume enclosed is V, show that the area of canvas in the tent is

$$\frac{2V}{a}-\frac{8}{3}ah+4a\sqrt{h^2+a^2}$$

Hence, show that, if a and h can both vary, the least area of canvas corresponding to a given volume V will be given by

$$a = \frac{\sqrt{5}}{2}h,\quad b = \tfrac{1}{2}h$$

(U.L.)

(70) If $V = f(z)$ where $z^2 = t^2 - x^2$ and x and t are independent variables, show that $z\,\partial V/\partial t = t\,df/dz$. Show that if V satisfies the equation $\dfrac{\partial^2 V}{\partial x^2} - V = \dfrac{\partial^2 V}{\partial t^2}$, then $f(z)$ is a solution of the equation $\dfrac{d^2 f}{dz^2} + \dfrac{1}{z}\dfrac{df}{dz} + f = 0$

(U.L.)

CHAPTER VI

TANGENTS AND NORMALS TO PLANE CURVES—CURVATURE

74. **Summary.** It is convenient at this stage to summarize briefly certain important results in elementary co-ordinate geometry. The reader will already be acquainted with the results (a) to (h), and he should revise the proofs of these from a standard textbook on the subject.

(a) The equation of a straight line can be expressed in any one of the following forms—

(i) $ax + by + c = 0$ (a, b, c constants).

(ii) $\dfrac{x}{k} + \dfrac{y}{l} = 1$ (k, l intercepts on x and y axes).

(iii) $x \cos \alpha + y \sin \alpha = p$ (p = length of perpendicular from the origin on the line, and α = angle between this perpendicular and positive direction of x-axis).

If the equation (i) be written as $y = -\dfrac{a}{b}x - \dfrac{c}{b}$, i.e. $y = mx + n$, m gives the gradient of the line, so that $\tan \psi = m$, where ψ is the angle between the line and the positive direction of the x-axis.

(b) The length of the perpendicular from a point (x', y') to a straight line is $x' \cos \alpha + y' \sin \alpha - p$ or $\dfrac{ax' + by' + c}{\sqrt{a^2 + b^2}}$, according as the equation of the line is given in the form (iii) or (i) above.

(c) The angle θ between two straight lines $y = m_1x + n_1$ and $y = m_2x + n_2$ is given by the relation $\tan \theta = \dfrac{m_1 - m_2}{1 + m_1m_2}$. If the lines are parallel, $m_1 = m_2$; if the lines are at right angles $1 + m_1m_2 = 0$, or $m_2 = -\dfrac{1}{m_1}$.

(d) The equation $y - y' = m(x - x')$ represents a straight line of gradient m through the point (x', y').

(e The equation of the straight line through two given points (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.

(f) If the equation of a curve is $y = f(x)$ referred to rectangular axes OX, OY, then its equation referred to rectangular axes $O'X'$, $O'Y'$ through a given point $O'(p, q)$ and parallel respectively to OX, OY, is $y + q = f(x + p)$, the x and y now referring to the new axes.

(g) If the axes OX, OY in (f) be each turned through an angle θ in the same direction, then the equation $y = f(x)$, when referred to the new axes, becomes $x \sin \theta + y \cos \theta = f(x \cos \theta - y \sin \theta)$, the x and y now referring to these new axes. (Art. 119.)

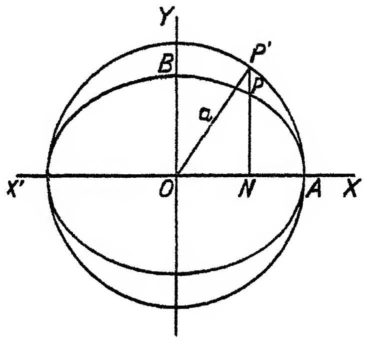


FIG. 43

(h) The equation of a circle of radius a, whose centre is at the point (p, q) is $(x - p)^2 + (y - q)^2 = a^2$; and if the centre is at the origin, the equation becomes $x^2 + y^2 = a^2$.

(i) In Fig. 43 P' is any point on the circle $x^2 + y^2 = a^2$, and the ordinate $P'N$ is divided at P such that $\dfrac{PN}{P'N} = \dfrac{b}{a}$ $(b < a)$. Now $ON^2 + P'N^2 = OP'^2 = a^2$, and $P'N = \dfrac{a}{b}PN$. Hence, $ON^2 + \dfrac{a^2}{b^2}PN^2 = a^2$ or $\dfrac{ON^2}{a^2} + \dfrac{PN^2}{b^2} = 1$. The locus of P is, then, the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. This curve is an *ellipse*, and $OA\ (= a)$ and $OB\ (= b)$ are called the *semi-major* and *semi-minor* axes respectively. The circle $x^2 + y^2 = a^2$ is called the *auxiliary* circle of the ellipse (see Art. 89).

75. **Equations of Tangents and Normals.** Let $P(x_1, y_1)$ be any point on a curve $y = f(x)$ (Fig. 44); then the equation of any straight line through P is $y - y_1 = m(x - x_1)$ (Art. 74 (d)). If this line is a tangent to the curve at P, its gradient m will be equal to the

gradient of the curve at P; whence m = value of $\frac{dy}{dx}$ at P (Art. 25). The equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is then

$$y - y_1 = \frac{dy}{dx}(x - x_1) \quad . \quad . \quad . \quad \text{(VI.1)}$$

where x_1, y_1 are substituted for x, y in the expression for $\frac{dy}{dx}$.

The normal to the curve at P is perpendicular to the tangent and its gradient is, therefore, equal to $-\frac{1}{\frac{dy}{dx}}$ (Art. 74 (c)). The equation of the normal at P is then $y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$, or

$$(y - y_1)\frac{dy}{dx} + (x - x_1) = 0 \quad . \quad . \quad \text{(VI.2)}$$

where, as before, x_1, y_1 are substituted for x, y in the expression for $\frac{dy}{dx}$.

If the equation of the curve is given in the implicit form $f(x, y) = 0$, then, since $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ (Art. 70), the equation of the tangent at the point (x_1, y_1) becomes

$$(y - y_1)\frac{\partial f}{\partial y} + (x - x_1)\frac{\partial f}{\partial x} = 0 \quad . \quad . \quad \text{(VI.3)}$$

and the equation of the normal becomes

$$(y - y_1)\Big/\frac{\partial f}{\partial y} = (x - x_1)\Big/\frac{\partial f}{\partial x} \quad . \quad . \quad \text{(VI.4)}$$

where x_1, y_1 are substituted for x, y in the partial differential coefficients $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ after differentiation.

EXAMPLE 1

(i) For the plane curve $y = b \sin \frac{\pi x}{a}$, find the equations to the tangent and normal at the point on the curve where $x = \frac{a}{4}$. (U.L.)

(ii) Find the equations of the tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1)

(i) Here $\frac{dy}{dx} = \frac{\pi b}{a} \cos \frac{\pi x}{a}$, and when $x = \frac{a}{4}, \frac{dy}{dx} = \frac{\pi b}{a} \cos \frac{\pi}{4} = \frac{\pi b}{\sqrt{2}a}$

Also, when $x = \frac{a}{4}$, $y = b \sin \frac{\pi}{4} = \frac{b}{\sqrt{2}}$

Equation of tangent is

$$y - \frac{b}{\sqrt{2}} = \frac{\pi b}{\sqrt{2}a}\left(x - \frac{a}{4}\right) \text{ or } 4(\sqrt{2}\,ay - \pi bx) = ab(4 - \pi),$$

and the equation of the normal is

$$\left(y - \frac{b}{\sqrt{2}}\right)\frac{\pi b}{\sqrt{2}a} + \left(x - \frac{a}{4}\right) = 0 \text{ or } 2(\sqrt{2}\,\pi by + 2ax) = a^2 + 2\pi^2 b^2$$

(ii) Here $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, $\frac{\partial f}{\partial x} = \frac{2x}{a^2}$ and $\frac{\partial f}{\partial y} = \frac{2y}{b^2}$ At point (x_1, y_1)

$$\frac{dy}{dx} = -\frac{2x_1}{a^2}\Big/\frac{2y_1}{b^2} = -\frac{b^2 x_1}{a^2 y_1}$$

Equation of tangent at point (x_1, y_1) is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \text{ or } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

Now $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, since (x_1, y_1) is a point on the ellipse, so that the equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ The equation of the normal is $(y - y_1)\Big/\frac{y_1}{b^2} = (x - x_1)\Big/\frac{x_1}{a^2}$

EXAMPLE 2

Find the equation of the tangent at the point (am^2, am^3) on the curve $ay^2 = x^3$ The tangent at a point P on this curve cuts the curve again at Q, and is normal to it there, find the value of m at P (U L)

Differentiating, we have $2ay \frac{dy}{dx} = 3x^2$ or $\frac{dy}{dx} = \frac{3x^2}{2ay}$

At the point (am^2, am^3) $\frac{dy}{dx} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}$

Equation of tangent at the point (am^2, am^3) is

$$y - am^3 = \frac{3m}{2}(x - am^2) \text{ or } 2y = m(3x - am^2) \tag{1}$$

Let Q be the point (am_1^2, am_1^3) on the curve The condition that Q lies on the tangent (1) is $2am_1^3 = m(3am_1^2 - am^2)$, which reduces to $(m_1 - m)^2(2m_1 + m) = 0$ The solution $m_1 = m$ obviously does not apply here, hence,

$$m = -2m_1 \tag{2}$$

Again, the gradient of the normal at $Q = -\frac{2}{3m_1}$, and if the tangent at P is normal to the curve at Q, $-\frac{2}{3m_1}$ will equal $\frac{3m}{2}$, whence $m_1 = -\frac{4}{9m}$.

Substituting in the relation (2) we have

$$m = -2\left(-\frac{4}{9m}\right) \text{ i.e. } m = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$$

76 **Subtangents and Subnormals.** If the tangent and normal at P (Fig 44) cut the x-axis at T and G respectively and if N is the foot

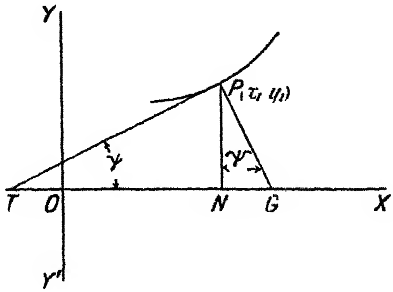


Fig 44

of the ordinate of P, the intercept TN is called the *subtangent*, and the intercept NG the *subnormal* of the point P

$$\text{Subtangent } TN = PN \cot\psi = y \Big/ \frac{dy}{dx} \qquad \text{(VI 5)}$$

$$\text{Subnormal } NG = PN \tan\psi \text{ (since } N\hat{P}G = \psi) = y\frac{dy}{dx} \qquad \text{(VI 6)}$$

where x_1, y_1 are substituted for x, y in these two expressions

EXAMPLE

Obtain the equations of the tangent and normal at any point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ on the parabola $y^2 = 4ax$

Show that the subtangent is double the abscissa, and that the subnormal is constant (U L)

Differentiating, we have $2y\frac{dy}{dx} = 4a$, or $\frac{dy}{dx} = \frac{2a}{y}$ and at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, $\frac{dy}{dx} = m$

The equation of the tangent is $y - \frac{2a}{m} = m\left(x - \frac{a}{m^2}\right)$, i.e. $y = mx + \frac{a}{m}$;

and the equation of the normal is $\left(y - \frac{2a}{m}\right)m + \left(x - \frac{a}{m^2}\right) = 0$, i.e. $my + x = a\left(2 + \frac{1}{m^2}\right)$

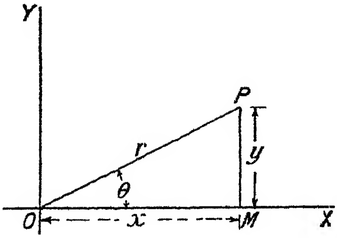


FIG. 45

The subtangent $= y\Big/\frac{dy}{dx} = \frac{2a/m}{m} = 2\cdot\frac{a}{m^2} =$ double the abscissa; and the subnormal $= y\frac{dy}{dx} = \frac{2a}{m}\cdot m = 2a =$ constant.

The latter of these two results is important in connection with the theory of the centrifugal governor for controlling the speed of the steam engine or internal combustion engine. The speed of the engine is a function of the "height" of the governor. If the governor balls move on a parabolic path the "height" is the subnormal, and as this is constant the speed is constant. In an isochronous governor,

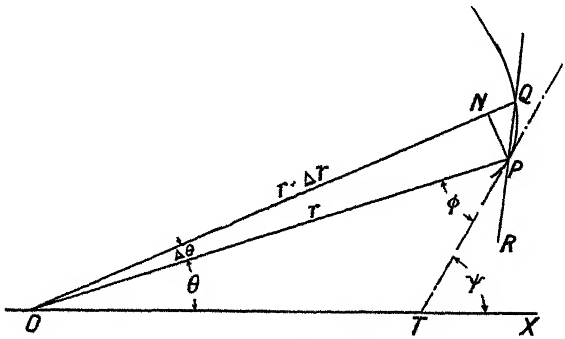


FIG. 46

therefore, the path of the balls in a vertical plane rotating with them is a parabola with its axis vertical.

77. **Polar Co-ordinates.** Let O (Fig. 45) be a fixed point and OX a fixed straight line in a plane; let P be a point in that plane such that $X\hat{O}P = \theta$ and $OP = r$. Then, if r and θ be known, the position of P in the plane is uniquely determined. The quantities r and θ are

called the polar co-ordinates of the point P. If the rectangular co-ordinates of P be (x, y) referred to axes OX, OY, then we see from the figure that $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$, $x = r\cos\theta$, and $y = r\sin\theta$.

Let P be the point (r, θ) on a given curve (Fig. 46) and Q the point $(r + \Delta r, \theta + \Delta\theta)$. Draw $PN \perp OQ$ and produce the line QP to any point R. Let arc $PQ = \Delta s$.

$$\text{Then } \bar{NQ} = \bar{OQ} - \bar{ON} = r + \Delta r - r\cos\Delta\theta$$

$$= \Delta r + r(1 - \cos\Delta\theta) = \Delta r + r\,.\,2\sin^2\frac{\Delta\theta}{2};$$

and $\qquad \bar{PN} = r\sin\Delta\theta$

Since $\underset{\Delta\theta \to 0}{Lt.}\ \frac{\sin\Delta\theta}{\Delta\theta} = 1$, we have to the first order of small quantities $NQ = \Delta r$ and $\bar{PN} = r\Delta\theta$.

Let ϕ be the angle between $\bar{OP}$ (the "radius vector," as it is termed) and the tangent to the curve at P drawn on the side of OP on which θ lies; then as $\Delta\theta$ approaches the limit zero, the line QPR tends to the position of the tangent PT at P, and the angles OQR and OPR both tend to the value ϕ. Hence,

$$\tan\phi = Lt.\ \tan OQR = Lt.\ \frac{PN}{NQ} = Lt.\ \frac{r\Delta\theta}{\Delta r} = r\frac{d\theta}{dr} \qquad \text{(VI.7)}$$

$$\text{and} \qquad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} \qquad . \quad . \quad . \quad . \qquad \text{(VI.8)}$$

$$\text{Also} \qquad \sin\phi = Lt.\ \sin OQR = Lt.\ \frac{PN}{PQ} = Lt.\ \frac{r\Delta\theta}{\Delta s} = r\frac{d\theta}{ds} \quad . \qquad \text{(VI.9)}$$

$$\text{and} \qquad \cos\phi = Lt.\ \cos OQR = Lt.\ \frac{NQ}{PQ} = Lt.\ \frac{\Delta r}{\Delta s} = \frac{dr}{ds} \qquad \text{(VI.10)}$$

EXAMPLE

Express in polar form the equation $x^2 - y^2 = a^2$ of the rectangular hyperbola and deduce that the lines which bisect the angles between the radius vector and the tangent at any point on the curve have a constant gradient.

By substituting $x = r\cos\theta$ and $y = r\sin\theta$, we obtain as the equation of the rectangular hyperbola $r^2(\cos^2\theta - \sin^2\theta) = a^2$ or $r^2\cos 2\theta = a^2$; hence, $2r\frac{dr}{d\theta} = a^2\frac{d}{d\theta}(\sec 2\theta) = 2a^2\sec 2\theta\tan 2\theta$.

With the notation above, $\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{a^2\sec 2\theta\,.\,\tan 2\theta}{r^2} = \tan 2\theta$ so that $\phi = \frac{\pi}{2} - 2\theta$. Now, the angle between the bisector of the angle ϕ and the positive direction of the x-axis is obviously equal to $\theta + \frac{\phi}{2}$, and with the value of ϕ just found this is $\theta + \frac{\pi}{4} - \theta = \frac{\pi}{4}$. Hence, the bisector is of constant gradient 1; and the other bisector is perpendicular to this line.

78. **Polar Subtangents and Subnormals.** In Fig. 47 P is any point (r, θ) on a given curve and KOH is drawn perpendicular to OP to meet the tangent and the normal to the curve at P in the points H and K respectively. OH is called the *polar subtangent* and OK

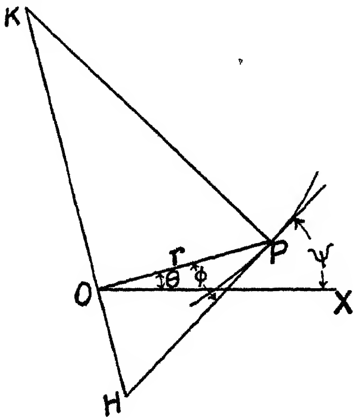


FIG. 47

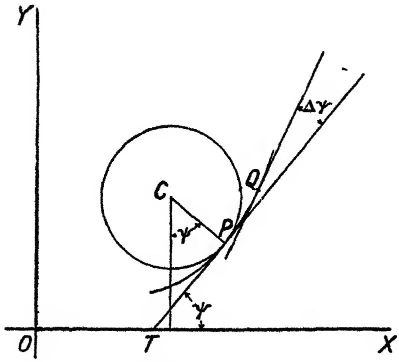


FIG. 48

the *polar subnormal* of the point P. Using the results of Art. 77, we have

Polar subtangent

$$= OH = OP\tan\phi = r\,.\,r\frac{d\theta}{dr} = r^2\frac{d\theta}{dr}\ . \qquad . \text{(VI.11)}$$

Polar subnormal

$$= OK = OP\cot\phi = r\,.\,\frac{1}{r}\frac{dr}{d\theta} = \frac{dr}{d\theta}\ . \qquad . \text{(VI.12)}$$

EXAMPLE

Find the lengths of the polar subtangent and the polar subnormal the curve $r = a\theta$ (spiral of Archimedes).

Here $$\frac{dr}{d\theta} = a \text{ and } \frac{d\theta}{dr} = \frac{1}{a}$$

∴ Polar subtangent $= r^2\dfrac{d\theta}{dr} = \dfrac{r^2}{a}$ or $a\theta^2$, and polar subnormal $= \dfrac{dr}{d\theta} = a$ (constant).

79. **Curvature.** The tangents at the extremities, P, Q of an arc PQ of length Δs of a given curve (Fig. 48) intersect at an angle $\Delta\psi$. The ratio $\dfrac{\Delta\psi}{\Delta s}$ gives the mean change of direction or "mean curvature" over the arc PQ; and the limit to which this ratio tends as Q tends to the position P is the measure of the "curvature" of the given curve at the point P, i.e.

$$\text{Curvature at } P = \frac{d\psi}{ds} \qquad . \qquad . \qquad (\text{VI}.13)$$

In the case of a circle of radius ρ the angle $\Delta\psi$ between the tangents is equal to the angle subtended at the centre by the arc Δs, and $\Delta s = \rho\Delta\psi$ or $\dfrac{\Delta\psi}{\Delta s} = \dfrac{1}{\rho} =$ constant. Hence the curvature $\dfrac{d\psi}{ds}$ of a circle is constant and equal to the reciprocal of the radius. In Fig. 48 let a circle be drawn with the same curvature as the given curve at P such that the tangent at P to the curve is also a tangent to the circle and the concavities of the curve and the circle are in the same sense. Then, if ρ be the radius of this circle and C its centre, $\rho = \dfrac{ds}{d\psi}$ is called the "radius of curvature," the circle the "circle of curvature," and C the "centre of curvature" of the given curve at P. In general ρ will vary as P moves along the curve; C will then trace out a locus, which is called the *evolute* of the given curve (see Art. 83).

80. **Radius of Curvature.** The formula

$$\rho = \frac{ds}{d\psi} \qquad . \qquad . \qquad . \qquad . \qquad (\text{VI}.14)$$

is immediately applicable when the equation of a curve is given in what is termed the "intrinsic" form, that is $s = F(\psi)$. For example, the intrinsic equation of the cycloid is $s = 4a \sin \psi$ (see Exs. X, No. 44), and hence $\rho = \dfrac{ds}{d\psi} = 4a \cos \psi$.

Other formulae for ρ can be deduced to suit various forms of the equation to a curve.

Since, with the usual notation,

$$\frac{dy}{dx} = \tan \psi \text{ or } \psi = \tan^{-1}\left(\frac{dy}{dx}\right)$$

then, differentiating, we have

$$\frac{d\psi}{ds} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{dy}{dx}\right)\right] \cdot \frac{dx}{ds}$$

$$= \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2} \cdot \cos\psi$$

$$= \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

$$\left(\text{since } \cos\psi = \frac{1}{\sqrt{1 + \tan^2\psi}}\right)$$

Hence, when the equation of a curve is given in the form $y = f(x)$ or $F(x, y) = 0$, the curvature at any point on the curve is given by

$$\text{Curvature} = \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \quad . \quad . \quad . \text{(VI.15)}$$

and the radius of curvature ρ is given by

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} \quad . \quad . \quad . \text{(VI.16)}$$

If we take the positive value of the radical $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ in the numerator, then ρ will have the same sign as $\dfrac{d^2y}{dx^2}$, and will therefore be positive or negative according as the curve is concave upwards or concave downwards in the neighbourhood of the point considered (see Art. 65). It is usually only the numerical value of ρ that we require.

EXAMPLE 1

Find an expression for the radius of curvature at any point of the catenary $y = c \cosh \dfrac{x}{c}$.

Prove also that the radius of curvature and the normal are both equal to $\frac{y^2}{c}$.

(U.L.)

Here $\quad \frac{dy}{dx} = \sinh\frac{x}{c}$ and $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \left[1 + \sinh^2\frac{x}{c}\right]^{\frac{3}{2}}$

$$= \left(\cosh^2\frac{x}{c}\right)^{\frac{3}{2}} = \cosh^3\frac{x}{c}$$

Also $\quad \frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}$

Hence, $\quad \rho = \dfrac{\cosh^3\frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c} = c\left(\frac{y^2}{c^2}\right) = \frac{y^2}{c}$

Again, at any point (x, y), the normal

$$= \sqrt{(\text{ordinate})^2 + (\text{subnormal})^2}$$

$$= \sqrt{y^2 + \left(y\frac{dy}{dx}\right)^2}$$

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y\cdot\cosh\frac{x}{c} = \frac{y^2}{c}$$

EXAMPLE 2

If the co-ordinates x, y of a curve are given as functions of a parameter θ, show that the radius of curvature at any point is

$$\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}} \Big/ \left\{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\cdot\frac{dy}{d\theta}\right\}$$

Prove that the radius of curvature at any point of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is $4a\sin\frac{\theta}{2}$. (U.L.)

Since x and y are both functions of θ, $\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$, and, hence,

$$\frac{d^2y}{dx^2} = \left(\frac{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^2}\right)\frac{d\theta}{dx}$$

$$\therefore \text{ Radius of curvature} = \frac{\left[1 + \left(\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\cdot\frac{dy}{d\theta}\right)\Big/\left(\frac{dx}{d\theta}\right)^3}$$

$$= \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\cdot\frac{dy}{d\theta}}$$

From the given equations, we have $\dfrac{dx}{d\theta} = a(1 - \cos\theta)$, $\dfrac{dy}{d\theta} = a \sin\theta$,

$$\frac{d^2x}{d\theta^2} = a\sin\theta \qquad \frac{d^2y}{d\theta^2} = a\cos\theta$$

Substituting in the formula just obtained we have

$$\rho = \frac{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}}{a\cos\theta(1 - \cos\theta) - a^2\sin^2\theta} = \frac{a^3\,[2(1 - \cos\theta)]^{\frac{3}{2}}}{a^2\,(\cos\theta - 1)}$$

$$= a\frac{\left(4\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}}}{-2\sin^2\frac{\theta}{2}} = -4a\sin\frac{\theta}{2} \quad \text{(See note above on sign of } \rho\text{.)}$$

EXAMPLE 3

Obtain the formula for the radius of curvature at any point of the curve $y = f(x)$. In the case of beams and girders, explain why the radius can be taken as approximately $1\Big/\dfrac{d^2y}{dx^2}$.

A uniform beam of length l and weight W is supported at the extremities. Assuming the bending moment to be $\dfrac{IE}{\rho}$, where ρ is the radius of curvature, show that the deflection at the centre is $\dfrac{5Wl^3}{384EI}$, where E is Young's modulus for the material of the beam, and I is the moment of inertia of a cross-section about the neutral axis. (U L)

For the first part of the question, see above.

If x and y are the co-ordinates of a point in a beam, the x-axis being horizontal and the y-axis vertical and drawn through some convenient point, $\dfrac{dy}{dx}$ is small compared with unity. Hence, to a sufficiently close degree of approximation, $\rho = 1\Big/\dfrac{d^2y}{dx^2}$.

The reaction at each of the supports (Fig. 49) is $\dfrac{W}{2}$. Taking O, the centre of the beam, as origin and the axes as shown in the figure, we have

Bending moment M at a section of the beam distant x from O

$$= \frac{W}{2}\left(\frac{l}{2} - x\right) - W\frac{\frac{l}{2} - x}{l}\cdot\frac{\frac{l}{2} - x}{2} = \frac{W}{2l}\left(\frac{l^2}{4} - x^2\right)$$

By hypothesis, $M = \dfrac{IE}{\rho}$, and by above, ρ can be taken here as $1\Big/\dfrac{d^2y}{dx^2}$.

Hence, $\qquad EI\dfrac{d^2y}{dx^2} = M = \dfrac{W}{2l}\left(\dfrac{l^2}{4} - x^2\right)$

Integrating, $\qquad EI\dfrac{dy}{dx} = \dfrac{W}{2l}\left(\dfrac{l^2x}{4} - \dfrac{x^3}{3}\right) + \text{constant}$

Now, when $x = 0$ (i.e. at O), $\frac{dy}{dx} = 0$ so that constant $= 0$

$$EI\frac{dy}{dx} = \frac{W}{2l}\left(\frac{l^2x}{4} - \frac{x^3}{3}\right)$$

Integrating again, $EIy = \frac{W}{2l}\left(\frac{l^2x^2}{8} - \frac{x^4}{12}\right) +$ constant

But when $x = 0$, $y = 0$ so that constant $= 0$

$$EIy = \frac{W}{2l}\left(\frac{l^2x^2}{8} - \frac{x^4}{12}\right)$$

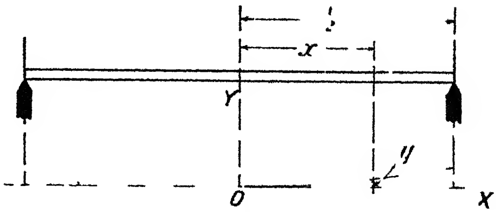


Fig. 49

The deflection y_0 at the centre is the value of y when $x = \frac{l}{2}$, hence,

$$EIy_0 = \frac{W}{2l}\left(\frac{l^4}{32} - \frac{l^4}{192}\right) = \frac{5Wl^3}{384}$$

$$y_0 = \frac{5Wl^3}{384EI}$$

The formula, $$EI\frac{d^2y}{dx^2} = \text{bending moment} \qquad \text{(VI 17)}$$

is very important and should be remembered.

81. **Polar Formula for ρ.** When the equation of a curve is given in the form $r = f(\theta)$, an expression for ρ is found as follows—

We see from Fig 46 that $\psi = \theta + \phi$

$$\therefore \qquad \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

Now $\tan\phi = r\frac{d\theta}{dr}$ (VI.7).

i.e. $$\phi = \tan^{-1}\left(\frac{r}{\frac{dr}{d\theta}}\right)$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{1}{1 + r^2\Big/\left(\frac{dr}{d\theta}\right)^2} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad \text{(VI.18)}$$

Substituting from (VI.18) in the expression for $\frac{d\psi}{d\theta}$ above,

$$\frac{d\psi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad . \quad . \quad \text{(VI.19)}$$

From (VI.8), $$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\therefore \quad \operatorname{cosec}\phi = \sqrt{1 + \cot^2\phi} = \sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}$$

and from (VI.9), $\sin\phi = r\frac{d\theta}{ds}$

$$\therefore \quad \operatorname{cosec}\phi = \frac{1}{\sin\phi} = \frac{1}{r}\frac{ds}{d\theta}$$

Hence, $$\frac{1}{r}\frac{ds}{d\theta} = \frac{1}{r}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

so that $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad . \quad . \quad . \quad \text{(VI.20)}$$

Dividing (VI.20) by (VI.19) we obtain

$$\rho = \frac{ds}{d\psi} = \frac{\frac{ds}{d\theta}}{\frac{d\psi}{d\theta}} = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad . \quad \text{(VI.21)}$$

EXAMPLE

Find the radius of curvature at any point on the cardioid $r = a(1 - \cos\theta)$.

Here $\dfrac{dr}{d\theta} = a\sin\theta$ and $\dfrac{d^2r}{d\theta^2} = a\cos\theta$, so that

$$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^3[2(1-\cos\theta)]^{3/2}$$

$$= 8a^3\sin^3\frac{\theta}{2}; \text{ and}$$

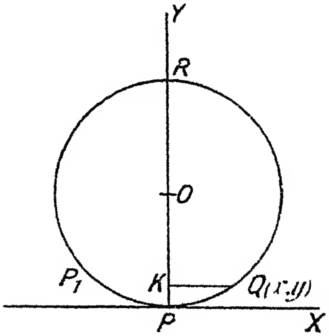


FIG. 50

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta\,(1-\cos\theta)$$

$$= a^2[3 - 3\cos\theta] = 6a^2\sin^2\frac{\theta}{2}$$

Hence, $$\rho = \frac{8a^3\sin^3\dfrac{\theta}{2}}{6a^2\sin^2\dfrac{\theta}{2}} = \frac{4}{3}a\sin\frac{\theta}{2} \text{ or } \frac{2}{3}\sqrt{2ar}$$

82. **Newton's Method.** The circle of curvature at any point on a given curve may be defined as follows: Let P_1, P, Q (Fig. 50) be three contiguous points on the curve. Then, when P_1 and Q move up to P and become ultimately coincident with P, the circle through P_1, P, Q becomes in the limit the circle of curvature at P. The chord P_1Q will in the limit be the tangent to the curve at P. Let this tangent be taken as x-axis and the normal at P as y-axis, and let Q be the point (x, y) on the curve.

By the above the circle passing through Q and touching the x-axis at P becomes the circle of curvature at P when Q ultimately coincides with P. The centre O of this circle lies on the y-axis.

Let QK be drawn perpendicular to PO and let R be the other end of the diameter through P. Then

$$QK^2 = PK \,.\, KR, \text{ or } KR = \frac{QK^2}{PK} = \frac{x^2}{y}$$

Now in the limit $KR = 2\rho$, where ρ = radius of curvature at P. Hence,

$$2\rho = Lt. \frac{x^2}{y}, \text{ or } \rho = \tfrac{1}{2}\, Lt. \frac{x^2}{y} \quad . \quad . \quad (\text{VI.22})$$

If the curve touches the y-axis at the origin, we have similarly

$$\rho = \tfrac{1}{2}\, Lt. \frac{y^2}{x}. \quad . \quad . \quad . \quad (\text{VI.23})$$

This method of finding the radius of curvature of a curve at the origin is due to Newton.

EXAMPLE

Find the radius of curvature of the curve $y^2(3 - x)(1 + x) = 8(x - 1)$ at the point (1, 0).

Transferring the origin to the point (1, 0) [see Art. 74 (f)], we have as the new equation

$$y^2(3 - \overline{x + 1})(1 + \overline{x + 1}) = 8(\overline{x + 1} - 1)$$

or

$$y^2(4 - x^2) = 8x$$

For small values of x this may be written $y^2 = 2x$, since $\underset{x \to 0}{Lt.}\,(4 - x^2) = 4$.
The graph of this touches OY at the origin, and, using (VI.23),

$$\rho = \tfrac{1}{2}\, Lt. \frac{y^2}{x} = 1$$

i.e. the radius of curvature is unity.

83. **Evolutes.** As stated in Art. 79, the *evolute* of a given curve is the locus of the centre of curvature. We see from Fig. 48 that if C be the centre of curvature for the point (x, y) on a curve, then if (ξ, η) are the co-ordinates of C

$$\left.\begin{aligned} \xi &= x - \rho \sin \psi \quad . \quad . \quad . \\ \text{and} \qquad \eta &= y + \rho \cos \psi \quad . \quad . \quad . \end{aligned}\right\} \quad (\text{VI.24})$$

Now
$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

$$\cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

and
$$\sin\psi = \sqrt{1-\cos^2\psi} = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Hence,
$$\left.\begin{aligned}\xi &= x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}\\ \text{and}\qquad \eta &= y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(VI.25)}$$

With ρ, ξ, η known, the equation of the circle of curvature at the point (x, y) on the curve is

$$(x-\xi)^2 + (y-\eta)^2 = \rho^2 \quad . \quad . \quad \text{(VI.26)}$$

EXAMPLE 1

Prove that the equation of a conic touching the axis of x at the origin is of the form $2y = ax^2 + 2hxy + by^2$; and find the equation of the circle of curvature at the origin.

A conic is drawn touching the axis of x at the origin and passing through the points (0, 2), (3, 1), (4, 2). Show that the equation of the circle of curvature at the origin is $6y = x^2 + y^2$, and that it meets the conic again in the point (3, 3).
(U.L.)

The general equation of a conic is $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$. Since (0, 0) is a point on the curve, $C = 0$.

Also, when $y = 0$, x must have two values each equal to zero.

Hence, the equation $Ax^2 + 2Gx = 0$ must have two zero roots, whence $G = 0$.

The equation of the conic can then be written

$$-2Fy = Ax^2 + 2Hxy + By^2$$

or
$$2y = ax^2 + 2hxy + by^2, \text{ where } a = -\frac{A}{F},\ h = -\frac{H}{F},\ b = -\frac{B}{F}$$

Let ρ_0 = radius of curvature at the origin; then, using Newton's method, we have

$$\rho_0 = \tfrac{1}{2}\,Lt.\frac{x^2}{y} = \tfrac{1}{2}\,Lt.\frac{\frac{1}{a}(2y - 2hxy - by^2)}{y}$$

$$= \tfrac{1}{2}\,Lt.\frac{2 - 2hx - by}{a}$$

$$= \tfrac{1}{2}\left(\frac{2}{a}\right) = \frac{1}{a}$$

The co-ordinates of the centre of curvature are then $\left(0,\frac{1}{a}\right)$, and the equation of the circle of curvature is $x^2 + \left(y - \frac{1}{a}\right)^2 = \frac{1}{a^2}$ or $\frac{2y}{a} = x^2 + y^2$.

The equation of the conic in the second part of the question must be of the form $2y = ax^2 + 2hxy + by^2$. Since it passes through the points (0, 2), (3, 1), (4, 2), we must have

$$4 = b(4), \quad \therefore b = 1$$

$$\left.\begin{aligned} 2 &= 9a + 6h + 1, \quad &\therefore 9a + 6h &= 1 \\ \text{and}\quad 4 &= 16a + 16h + 4, \quad &\therefore a + h &= 0 \end{aligned}\right\}\ \text{whence } a = \tfrac{1}{3},\ h = -\tfrac{1}{3}.$$

The equation of the conic is then $2y = \frac{1}{3}x^2 - \frac{2}{3}xy + y^2$.

By the above, $\rho_0 = \frac{1}{a} = \frac{1}{\frac{1}{3}} = 3$; and the equation of the circle of curvature at the origin is $\frac{2y}{\frac{1}{3}} = x^2 + y^2$ or $6y = x^2 + y^2$.

The two curves $2y = \frac{1}{3}x^2 - \frac{2}{3}xy + y^2$ and $6y = x^2 + y^2$ meet where $x^2 - 2xy + 3y^2 = x^2 + y^2$, i.e. where $2y^2 = 2xy$. The two solutions of this equation are $y = 0$ and $y = x$. The former solution refers to the origin; the latter implies that $6x = x^2 + x^2$ (from equation of circle); whence $x = 3$ and therefore $y = 3$. The circle of curvature therefore meets the conic again in the point (3, 3).

EXAMPLE 2

Prove that the equations of the evolutes of the parabola $y^2 = kx$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $27ky^2 = 2(2x - k)^3$ and $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ respectively.

For the parabola, $2y\frac{dy}{dx} = k$, i.e. $\frac{dy}{dx} = \frac{k}{2y}$ and $\frac{d^2y}{dx^2} = -\frac{k}{2y^2}\cdot\frac{dy}{dx} = -\frac{k^2}{4y^3}$

Also $$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{k^2}{4y^2}$$

Substituting in (VI.25), we have

$$\xi = x - \left(\frac{4y^2 + k^2}{4y^2}\cdot\frac{k}{2y}\right)\Big/\left(-\frac{k^2}{4y^3}\right) = x + \frac{4y^2 + k^2}{2k} = \frac{6x + k}{2}$$

and $$\eta = y + \left(\frac{4y^2 + k^2}{4y^2}\right)\Big/\left(-\frac{k^2}{4y^3}\right) = y - \frac{y(4y^2 + k^2)}{k^2} = -\frac{4y^3}{k^2}$$

Hence, $x = \frac{2\xi - k}{6}$ and $y = -\left(\frac{k^2\eta}{4}\right)^{\frac{1}{3}}$

Substituting in the equation $y^2 = kx$, we obtain

$$\left(\frac{k^2\eta}{4}\right)^{\frac{2}{3}} = k\left(\frac{2\xi - k}{6}\right)$$

Cubing both sides, $$\frac{k^4\eta^2}{16} = \frac{k^3}{216}(2\xi - k)^3$$

or $$27k\eta^2 = 2(2\xi - k)^3$$

The equation of the evolute is then $27ky^2 = 2(2x - k)^3$

For the ellipse, $\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$, whence $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$; and

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = -\frac{b^2}{a^2}\left[\frac{y + \frac{b^2x^2}{a^2y}}{y^2}\right] = -\frac{b^2}{a^4y^3}[a^2y^2 + b^2x^2]$$
$$= -\frac{b^2}{a^4y^3}\cdot(a^2b^2) = -\frac{b^4}{a^2y^3}$$

Also $$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^4x^2}{a^4y^2} = \frac{a^2b^2(a^2 - x^2) + b^4x^2}{a^4y^2} = \frac{b^2[a^4 - x^2(a^2 - b^2)]}{a^4y^2}$$

Substituting in (VI.25) as before, we have

$$\xi = x - \left[\frac{b^2[a^4 - x^2(a^2 - b^2)]}{a^4y^2}\left(-\frac{b^2x}{a^2y}\right)\right]\Big/\left(-\frac{b^4}{a^2y^3}\right)$$
$$= x - \frac{x[a^4 - x^2(a^2 - b^2)]}{a^4} = \frac{(a^2 - b^2)x^3}{a^4}$$

and $$\eta = y + \left(\frac{a^4y^2 + b^4x^2}{a^4y^2}\right)\Big/\left(-\frac{b^4}{a^2y^3}\right)$$
$$= y - \frac{y}{a^2b^4}[a^4y^2 + a^2b^2(b^2 - y^2)] = -\frac{(a^2 - b^2)y^3}{b^4}$$

Hence, $x = \left(\frac{a^4\xi}{a^2 - b^2}\right)^{\frac{1}{3}}$ and $y = -\left(\frac{b^4\eta}{a^2 - b^2}\right)^{\frac{1}{3}}$

Substituting in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and simplifying, we obtain $(a\xi)^{\frac{2}{3}} + (b\eta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

The equation of the evolute is then $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

84. **Properties of Evolutes.** From (VI.24) we have

$$\xi = x - \rho\sin\psi \text{ and } \eta = y + \rho\cos\psi$$

Differentiating with respect to s, the length of arc on the original curve, we obtain

$$\frac{d\xi}{ds} = \frac{dx}{ds} - \left[\rho \frac{d}{d\psi}(\sin \psi)\frac{d\psi}{ds} + \frac{d\rho}{ds} \cdot \sin \psi\right]$$

$$= \frac{dx}{ds} - \rho \cos \psi \cdot \frac{1}{\rho} - \frac{d\rho}{ds} \cdot \sin \psi$$

i.e. $$\frac{d\xi}{ds} = -\frac{d\rho}{ds} \cdot \sin \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.27)}$$

and $$\frac{d\eta}{ds} = \frac{dy}{ds} + \left[\rho \frac{d}{d\psi}(\cos \psi)\frac{d\psi}{ds} + \frac{d\rho}{ds} \cdot \cos \psi\right]$$

$$= \frac{dy}{ds} - \rho \sin \psi \cdot \frac{1}{\rho} + \frac{d\rho}{ds} \cdot \cos \psi$$

i.e. $$\frac{d\eta}{ds} = \frac{d\rho}{ds} \cdot \cos \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.28)}$$

$$\left[\text{For } \frac{d\psi}{ds} = \frac{1}{\rho};\ \frac{dx}{ds} = \cos \psi;\ \frac{dy}{ds} = \sin \psi\right]$$

By division, $$\frac{d\eta}{ds}\bigg/\frac{d\xi}{ds} = -\cot \psi = -1\bigg/\frac{dy}{dx}$$

i.e. $$\frac{d\eta}{d\xi} = -1\bigg/\frac{dy}{dx}$$

Now $\dfrac{d\eta}{d\xi}$ = gradient of tangent to the evolute at the point (ξ, η);

and $-1\Big/\dfrac{dy}{dx}$ = gradient of normal to the original curve at the point (x, y). Since the normal at the point (x, y) passes through the corresponding point (ξ, η) on the evolute, it follows that

The normal at any point on a given curve is a tangent to its evolute. (VI.29)

From (VI.27) and (VI.28) we obtain in differential notation $d\xi = -d\rho \,.\, \sin \psi$ and $d\eta = d\rho \,.\, \cos \psi$. Now, if σ be a length of arc of the evolute,

$$(d\sigma)^2 = (d\xi)^2 + (d\eta)^2 = (d\rho)^2 [\sin^2 \psi + \cos^2 \psi] = (d\rho)^2$$

Hence, $$d\sigma = \pm\, d\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.30)}$$

If the length of arc between two points C_1 and C_2 on the evolute be σ', and ρ_1 and ρ_2 be the radii of curvature at the corresponding points P_1 and P_2 on the original curve, then on integrating (VI.30), we have

$$\sigma' = \rho_2 - \rho_1$$

Hence, *the length of an arc of the evolute is equal to the difference between the radii of curvature at the points on the original curve corresponding to the extremities of the arc.* (VI.31)

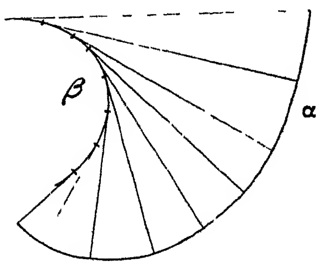


FIG. 51

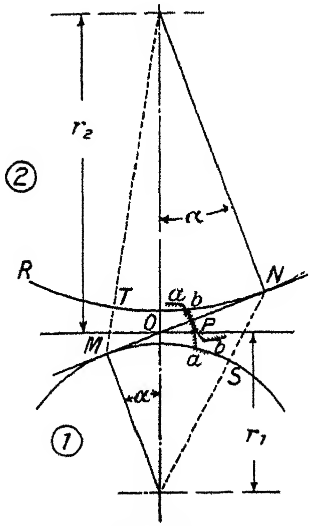


FIG. 52

Again, the radius of curvature ρ' of the evolute $= \dfrac{d\sigma}{d\psi'}$ and ψ' = slope of tangent to evolute = slope of normal to original curve = $\dfrac{\pi}{2} + \psi$; so that $d\psi' = d\psi$.

Hence, $d\sigma = \rho' d\psi' = \rho' d\psi$; and from (VI.30) $d\sigma = d\rho$

$$\therefore \qquad d\rho = \rho' d\psi \text{ or } \rho' = \frac{d\rho}{d\psi} = \frac{d}{d\psi}\left(\frac{ds}{d\psi}\right) = \frac{d^2 s}{d\psi^2}$$

Hence, *the radius of curvature of the evolute is* $\dfrac{d^2s}{d\psi^2}$ *where s and* ψ *refer to the original curve.* (VI.32)

85. **Involutes.** If a curve β is an evolute of a curve α, then α is called *an* involute of β.

From the properties (VI.29) and (VI.31) of an evolute, it follows that if a string be supposed fixed at one end to a point on the curve β (Fig. 51) and held tight while it is wound on the curve, the free end will describe a curve α of which β is the evolute. In other words, α is an involute of β. Since any point on the string describes an involute of β, a given curve has an infinite number of involutes. The involute curve is of importance to engineers, as the profiles of the teeth of gear wheels are usually involutes of circles.

Fig. 52 shows how the involute teeth are developed for two toothed wheels (1) and (2) of radii r_1 and r_2 respectively whose pitch circles touch at O. MaS and RTN are known as the base circles of (1) and (2) respectively. The tooth surface aa is an involute of the circle MaS, and the tooth surface bb is an involute of the circle RTN. Involute teeth have practical advantages over other shapes, and their use is almost universal (see *Theory of Machines*, Pitman's).

EXAMPLES VI

(1) Find the equation of the tangent to the curve $y^3 = 2ax^2$ at the point $(2am^3, 2am^2)$.

If O is the origin, and P, Q are two points on the curve such that the angle POQ is a right angle, show that the locus of the intersection of the tangents at P and Q is the parabola $2x^2 + 2a^2 = 3ay$. (U.L.)

(2) Show how to find $\frac{dy}{dx}$ when x and y are connected by the implicit relation $\phi(x, y) = 0$.

Prove that the tangent to the curve $x^3 + y^3 = 3axy$ at the point x', y' is $x(x'^2 - ay') + y(y'^2 - ax') = ax'y'$. Write down the equation of the tangent at the point $\frac{6a}{7}, -\frac{12a}{7}$, and verify that it meets the curve again in the point $-\frac{16a}{21}, \frac{4a}{21}$. (U.L.)

(3) Find the co-ordinates of the points on the curve $y = 5 \log_e (3 + x^2)$ at which the slope is 2, and find the equations to the tangents at these points. Find the maximum slope of the given curve. (U.L.)

(4) Find the equation of the tangent to the curve $y = x^3$ at the point (t, t^3), and the condition that it should pass through the point $(0, -r)$.

By consideration of the graphs of $y = x^3$ and $y + qx + r = 0$, or otherwise, show that the equation $x^3 + qx + r = 0$ has three real roots if $27r^2 + 4q^3$ is less than, or equal to, zero. (U.L.)

(5) The co-ordinates of any point on a certain curve are given by the relations $x = pt^2$ and $y = qt^3$, where p and q are constants. Find the equations of the tangent and normal to the curve at the point t, and show that the tangent cuts the curve again at the point $-\frac{t}{2}$.

(6) Find the equations of the tangent and normal and the lengths of the subtangent and subnormal to each of the following curves at the points given—

(i) $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1)$.

(ii) $y = \dfrac{c^3}{c^2 + x^2}$ at the point $\left(c, \dfrac{c}{2}\right)$.

(iii) $3x^2 - 2xy + y^2 = 1$ at the point $(0, -1)$.

(iv) Ellipse $x = a\cos\phi$, $y = b\sin\phi$, at the point where $\phi = \dfrac{\pi}{3}$.

(7) Show that for the curve $y = ae^{kx}$ the subtangent is constant, and the subnormal is ky^2.

(8) A curve is given by the equations $x = k\cos^3\phi$, $y = k\sin\phi$. Show that the lengths of the tangent and normal at any point are $\dfrac{k}{\sin\phi}$ and $\dfrac{k}{\cos\phi}$ respectively.

(9) Show that in the logarithmic spiral $r = ae^{k\theta}$, the tangent is inclined to the radius vector at a constant angle.

(10) Show that in the curve $r = a\theta$, the polar subnormal is constant and the polar subtangent is equal to $a\theta^2$.

(11) Find the angle at which the curves $r^2 = a^2 \operatorname{cosec} 2\theta$ and $r^2 = b^2 \sec 2\theta$ intersect.

(12) Find the lengths of the polar subtangent and the polar subnormal to the curve $r^2 = a^2 \sin 2\theta$.

(13) Show that the angle α between the tangents to the curves $r = f(\theta)$ and $r = \phi(\theta)$ at a point of intersection (r', θ') is given by

$$\tan\alpha = \frac{\phi(\theta').f'(\theta') - \phi'(\theta').f(\theta')}{\phi'(\theta').f'(\theta') - \phi(\theta').f(\theta')}$$

$f'(\theta')$ and $\phi'(\theta')$ being the values of $f'(\theta)$ and $\phi'(\theta)$ when $\theta = \theta'$.

(14) A chord AB drawn through the pole of the cardioid $r = a(1 + \cos\theta)$ cuts the curve at A and B. Prove that the tangents at A and B intersect at right angles.

(15) Find the radius of curvature for each of the following curves—

(i) $y^2 = kx^3$.

(ii) $2x^2 + 5y^2 = 4$ at the point $(\sqrt{2}, 0)$.

(iii) $y = \sin x$ at the point $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$.

(iv) $\left(\dfrac{x}{a}\right) + \left(\dfrac{y}{b}\right) = 1$ at the point $(a, 0)$.

(v) $y = e^x$ at the point $(0, 1)$.

(vi) $x^2 - y^2 = a^2$ at the point $(a, 0)$.

(16) The rectangular co-ordinates of a point on a curve are $x = a\sin pt$, $y = a\cos 2pt$, where p is constant and t variable. Find the direction of the tangent at the point where $y = 0$ and the radius of curvature at the point where $x = 0$. (U.L.)

(17) Find the condition that must hold in order that the hyperbolae $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $xy = c^2$ may intersect orthogonally, and in this case determine

the radii of curvature at the points of intersection. What further condition must be fulfilled if these are equal in length? (U.L.)

(18) A curve is given by the equations $x = a \sin \theta$, $y = b \cos 2\theta$, where a and b are constants. Find the radius of curvature at the point where $\theta = \frac{\pi}{3}$.

(19) Using Newton's method, prove that the radius of curvature at the lowest point (i.e. where $\theta = 0$) of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ is equal to $4a$.

(20) Show that the radius of curvature at any point on the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $3\sqrt[3]{axy}$.

(21) Apply Newton's method to prove that the radius of curvature at the lowest point of the catenary $y = c \cosh \frac{x}{c}$ is equal to c.

(22) Find the radius of curvature at any point on each of the following curves—

(i) The rectangular hyperbola $r^2 = a^2 \sec 2\theta$.

(ii) The conic $\frac{l}{r} = 1 + e \cos \theta$.

(iii) The parabola $r \cos^2 \frac{1}{2}\theta = a$.

(iv) The spiral $r = a\theta$.

(23) In the equiangular spiral $r = ke^{\theta \cot \alpha}$, α is the constant angle which the curve makes with the radius vector. Show that at any point on the curve the radius of curvature is r cosec α, and subtends a right angle at the pole.

(24) Find the co-ordinates of the centre of curvature and the equation of the evolute for the cycloid in Question 19.

(25) Show that the equation of the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.

(26) If (ξ, η) are the co-ordinates of the centre of curvature in the rectangular hyperbola $xy = c^2$, show that $2(\xi + \eta) = \frac{(y + x)^3}{c^2}$ and $2(\xi - \eta) = \frac{(y - x)^3}{c^2}$.
Hence, deduce the equation of the evolute $(x + y)^{\frac{2}{3}} - (x - y)^{\frac{2}{3}} = (16c^2)^{\frac{1}{3}}$.

(27) Find the co-ordinates of the centre of curvature at any point on the ellipse $x = a \cos \phi$, $y = b \sin \phi$, and deduce the equation of the evolute $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

(28) If the involutes of two given circles touch at a point P, show that P lies on a common tangent to the circles. Explain the use of involutes of circles in connection with wheel gearing.

(29) The rectangular co-ordinates of a point on a curve are given by

$$x = a \sin t - b \sin \frac{at}{b}, \; y = a \cos t - b \cos \frac{at}{b}$$

Find the maximum distance of a point on the curve from the origin, and show that the curvature at such a point of maximum distance is $\frac{a + b}{4ab}$. (U.L.)

(30) Prove from first principles that the radius of curvature at the point θ on the curve

$$x = 3a \cos \theta - a \cos 3\theta, \; y = 3a \sin \theta - a \sin 3\theta, \text{ is } 3a \sin \theta.$$

(U.L.)

CONIC SECTIONS—CATENARIES

86. **The Conic Sections.** Let S (Fig. 53) be a fixed point and ZZ' a fixed straight line, and let a point P move in the plane of S and ZZ' in such a way that its distance SP from S always bears a constant ratio e to its distance PM from ZZ'. The curve which P traces out is called a parabola, ellipse, or hyperbola, according as e is equal to, less than, or greater than unity. The quantity e is called the "eccentricity" of the curve, the point S is the "focus," and the line ZZ' the "directrix." The focal chord LSL' parallel to ZZ' is called the "latus-rectum," and its length is usually denoted by $2l$. If a double right circular cone, generated by the rotation of two intersecting straight lines about the bisector of the angle between them is cut by a plane, the curve of section is a parabola, ellipse, or hyperbola, according as the plane is parallel to a generating line of the cone, cuts all the generating lines on one side of the vertex, or cuts some generating lines on one side of the vertex and some on the other. The ellipse becomes a circle when the cutting plane is perpendicular to the axis. To the connection of these curves with the cone is due the term "conic sections" or "conics."

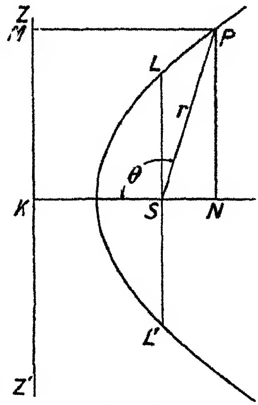


FIG. 53

87. **Polar Equation of Conic.** In Fig. 53 let SK be drawn perpendicular to ZZ' and PN perpendicular to SK. Let P be the point (r, θ) referred to S as pole and SK as initial line; then $SP = r$ and $K\hat{S}P = \theta$. We have

$$r = SP = ePM = e\,(NS + SK)$$

i.e. $\quad r = e \times r \cos N\dot{S}P + e \times$ (perpendicular distance of L from ZZ')

$$= - er \cos \theta + l.$$

The polar equation of the conic is, then,

$$\frac{l}{r} = 1 + e \cos \theta \quad . \quad . \quad . \quad \text{(VII.1)}$$

88. **The Parabola.** In Fig. 54 S is the focus and ZZ' the directrix of a parabola. A, the mid-point of SK, which is obviously a point on the curve, is called the vertex of the parabola. Choose AS as x-axis and AY perpendicular to AS as y-axis, and let P be a point (x_1, y_1) on the curve. Then, since $e = 1$ for a parabola, $SP = PM = NK$, where N is the foot of the ordinate of P. Let $KA = AS = a$;

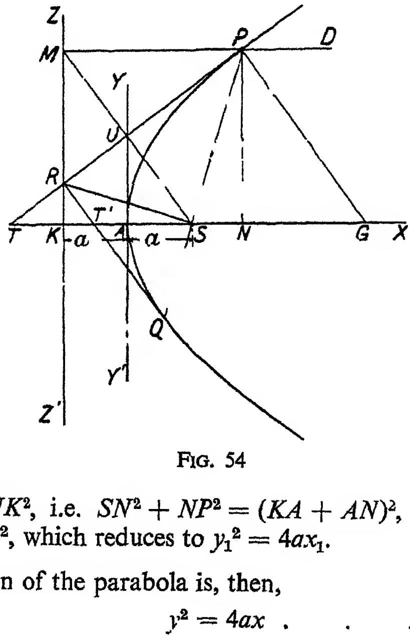


FIG. 54

then $SP^2 = NK^2$, i.e. $SN^2 + NP^2 = (KA + AN)^2$, or $(x_1 - a)^2 + y_1^2 = (a + x_1)^2$, which reduces to $y_1^2 = 4ax_1$.

The equation of the parabola is, then,

$$y^2 = 4ax \quad . \quad . \quad . \quad . \quad \text{(VII.2)}$$

Sometimes the co-ordinates of a point on the parabola are expressed in terms of a single variable. For example, the point given by $x = at^2$, $y = 2at$ is obviously on the parabola $y^2 = 4ax$; we can call this point the point t. Compare Art. 76, Example.

Differentiating (VII.2), we have: $2y\frac{dy}{dx} = 4a$, so that at the point (x_1, y_1), $\frac{dy}{dx} = \frac{2a}{y_1}$.

The equation of the tangent PT at P is, therefore

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

which reduces to

$$yy_1 = 2a(x + x_1) \qquad \text{(VII.3)}$$

and the equation of the normal PG at P is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

or

$$(y - y_1)2a + y_1(x - x_1) = 0 \qquad \text{(VII.4)}$$

In Fig. 54 PT and PG are respectively the tangent and normal to the curve at P, T and G being on the x-axis; let PT cut ZZ' at R. Join SR.

Putting $y = 0$ in (VII.3) we obtain $x = -AT = -x_1$, whence $TA = AN$, $\therefore TS = KN = PM = SP$ and $S\hat{P}T = S\hat{T}P$, but $S\hat{T}P = M\hat{P}T$ $(TS \parallel MP)$, so that $S\hat{P}T = M\hat{P}T$.

Hence, *the tangent at any point on a parabola bisects the angle between the line joining the point to the focus and the line through the point perpendicular to the directrix.* (VII.5)

Since $SP = PM$, $PR = PR$, and $S\hat{P}R = M\hat{P}R$, the triangles SPR and PMR are congruent. $\therefore R\hat{S}P = R\hat{M}P = 90°$.

Hence, *the part of any tangent to a parabola intercepted by the point of contact and the directrix subtends a right angle at the focus.* (VII.6)

The equation of any straight line through $S(a, 0)$ is $y = m(x - a)$ (see Art. 74 (d)), and if this line is perpendicular to the tangent at P, $m = -\frac{y_1}{2a}$. The lines $y = -\frac{y_1}{2a}(x - a)$ and $yy_1 = 2a(x + x_1)$ intersect where $-\frac{y_1}{2a}(x - a) = \frac{2a}{y_1}(x + x_1)$, which reduces to $x = 0$. It follows that if PT cuts the y-axis (i.e. the tangent at the vertex) at U, then SU is perpendicular to PT. Also, if MU be joined, the triangles MPU and SPU are congruent and SUM is, therefore, a straight line.

Hence, *the straight line through the focus perpendicular to any tangent to a parabola meets the tangent on the tangent at the vertex and passes through the foot of the perpendicular drawn from the point of contact to the directrix.* (VII.7)

We have already proved (Art. 76) that in the parabola the subtangent TN is double the abscissa and the subnormal NG is constant.

The straight line $y = mx + c$ meets the parabola $y^2 = 4ax$ where $(mx + c)^2 = 4ax$, and if the line is a tangent to the parabola the two roots of this equation, $m^2x^2 + 2(mc - 2a)x + c^2 = 0$, are equal. The required condition is $4(mc - 2a)^2 = 4m^2c^2$, which reduces to $c = \dfrac{a}{m}$

Hence, *the line* $y = mx + \dfrac{a}{m}$ *is a tangent to the parabola* $y^2 = 4ax$ *for all values of* m. (VII.8)

In the equation $y^2 = 4ax$, y^2 is essentially positive so that x cannot be negative; also as x increases, $|y|$ increases, and for any given value of x there are two values of y differing only in sign. The parabola, therefore, consists of one open branch on the focal side of the directrix, symmetrical about the x-axis, and receding continually without limit from both the axes of reference.

Let RQ (Fig. 54) be the other tangent from R to the parabola. Then by (VII.6) $R\hat{S}Q = 90°$; also $R\hat{S}P = 90°$, so that PSQ is a straight line.

Let RQ cut the x-axis at T'. We have already proved that $S\hat{P}T = S\hat{T}P$; similarly $S\hat{Q}T' = S\hat{T'}Q = R\hat{T'}T$.

$$\therefore \qquad S\hat{P}T + S\hat{Q}T' = S\hat{T}P + R\hat{T'}T$$

$$\text{i.e.} \qquad 180° - P\hat{R}Q = 180° - T\hat{R}T'$$

$$\therefore \qquad P\hat{R}Q = T\hat{R}T'; \text{ whence } PRQ = 90°$$

Hence, *the tangents at the extremities of any focal chord intersect at right angles on the directrix.* (VII.9)

In Fig. 55 $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a parabola such that the chord PQ is of constant gradient m'. The equation of the line PQ is $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ (Art. 74 (*e*)); also $y_1^2 = 4ax_1$

and $y_2^2 = 4ax_2$, so that $y_2^2 - y_1^2 = 4a(x_2 - x_1)$, or $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4a}{y_2 + y_1}$.

The equation of the line PQ becomes, then, $\dfrac{y - y_1}{x - x_1} = \dfrac{4a}{y_2 + y_1}$; its gradient $= \dfrac{4a}{y_2 + y_1} = m'$ (m' = constant).

If (x', y') be the co-ordinates of the mid-point of PQ, then

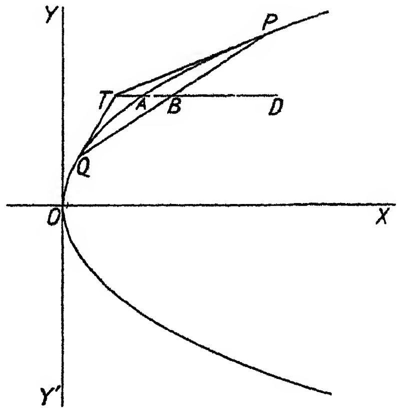


FIG. 55

$x' = \dfrac{x_1 + x_2}{2}$, $y' = \dfrac{y_1 + y_2}{2}$; hence, $\dfrac{4a}{2y'} = m'$ or $\dfrac{2a}{m'} = y'$. The locus of B, the mid-point of PQ, is therefore the line $y = \dfrac{2a}{m'}$.

Hence, *the locus of the mid-points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.* (VII.10)

The equations of the tangents at P and Q are $yy_1 = 2a(x + x_1)$ and $yy_2 = 2a(x + x_2)$ respectively; these lines meet at T where

$$yy_1 - 2ax_1 = yy_2 - 2ax_2$$

or $$y = \frac{2a(x_2 - x_1)}{y_2 - y_1} = 2a\left(\frac{y_2 + y_1}{4a}\right), \text{ (from above)}$$

i.e. $y = \dfrac{2a}{m'}$, so that T lies on the line through B parallel to OX

The locus of the mid-points of a system of parallel chords of a conic is called a *diameter* of the conic.

Hence, *the tangents at the extremities of any chord of a parabola intersect on the diameter bisecting the chord.* (VII.11)

Referring again to Fig. 54, let us suppose S to be a source of light and the curve a section of the reflecting surface of a parabolic mirror made by a plane through the vertex. A ray of light emanating from S in the direction SP will be reflected along the diameter PD by virtue of the property (VII.5).

If the axis of such a mirror is directed towards the sun, the parallel rays of light proceeding from the sun will after reflection all converge towards S, and will be intensified there.

EXAMPLE 1

TP, *TQ* are two tangents to a parabola (Fig. 55), and a line through T parallel to the axis of the parabola meets the curve in A and PQ in B; show that $TA = AB$ and $PB = QB$.

A telegraph wire may be assumed to hang in the form of a parabola between two posts a known distance apart. Show how to estimate the sag in the centre of the wire by observing the angle that the wire makes with the horizontal chord through its ends. (U.L.)

From (VII.11) we deduce that $PB = BQ$. The equation of the diameter TB is $y = \dfrac{2a}{m}$ where m = gradient of chord PQ; this diameter meets the parabola $y^2 = 4ax$ where $\left(\dfrac{2a}{m}\right)^2 = 4ax$, i.e. where $x = \dfrac{a}{m^2}$. Again, the tangents at P and Q meet at T, where $\dfrac{2a(x + x_1)}{y_1} = \dfrac{2a(x + x_2)}{y_2}$,

$$\text{i.e. where} \qquad x = \frac{x_1y_2 - x_2y_1}{y_1 - y_2} = \frac{\dfrac{y_1^2}{4a} \cdot y_2 - \dfrac{y_2^2}{4a} \cdot y_1}{y_1 - y_2} = \frac{y_1y_2}{4a};$$

also the x co-ordinate of B is

$$\frac{x_1 + x_2}{2} = \frac{y_1^2 + y_2^2}{8a}$$

Therefore sum of x co-ordinates of T and B

$$\begin{aligned} &= \frac{y_1^2 + y_2^2}{8a} + \frac{y_1y_2}{4a} = \frac{(y_1 + y_2)^2}{8a} \\ &= \left(\frac{4a}{m}\right)^2 \Big/ 8a \\ &= \frac{2a}{m^2} \end{aligned}$$

It follows that A is the mid-point of TB.

In the second part of the question, let the distance between the posts A and B be $2a$ (Fig. 56), and let O, the lowest point in the wire, be the origin, and the tangent at O to the curve the x-axis. Then, if d = sag in centre, B is the point (a, d) on the curve. The equation of the curve will be $y = kx^2$ where k = constant. Since the point (a, d) lies on the curve, $d = ka^2$ or $k = \frac{d}{a^2}$. Also, $\frac{dy}{dx} = 2kx = \frac{2dx}{a^2}$, and at B, $\frac{dy}{dx} = \frac{2d}{a^2} \cdot a = \frac{2d}{a}$. If the gradient at $B = t$ (known), then $\frac{2d}{a} = t$ and therefore the sag $d = \frac{at}{2}$.

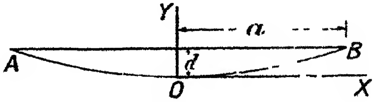


FIG. 56

EXAMPLE 2

At a point R of a straight rod PQ another rod RS is rigidly attached, the two rods being at right angles. The frame so formed is rotated about S in the plane of the rods. Prove that at any instant the direction of motion of any point in the rod PQ is tangential to the parabola having S as focus and R as vertex.

A parabola having S as focus and R as vertex will have PQ as the tangent at the vertex. Let any tangent TH to the parabola meet PQ at H (Fig. 57). Then by

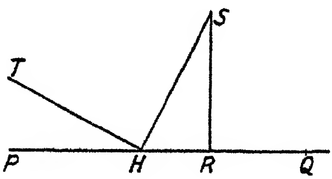


FIG. 57

(VII.7), SH is perpendicular to this tangent. Hence, if we regard H as any point in PQ, its direction of motion at any instant is perpendicular to SH, since S is the centre of rotation. The direction of motion of H is therefore tangential to the parabola.

EXAMPLE 3

$ABCD$ is a rectangle in which $AB = l$, $BC = b$. The side AB is divided at $Q_1, Q_2, Q_3, \ldots$, into n equal parts, and the side BC is divided at $R_1, R_2, R_3, \ldots$, into n equal parts. Lines are drawn through $Q_1, Q_2, Q_3, \ldots$, parallel to BC, and these lines intersect the joins AR_1, AR_2, $AR_3, \ldots$, in the points $P_1, P_2, P_3, \ldots$ Show that the points $P_1, P_2, P_3, \ldots$, lie on a parabola of which A is the vertex, and find the length of the latus-rectum of the parabola.

The diagram is shown in Fig. 58. Consider the point P_1. From the similar triangles AQ_1P_4, ABR_4, we have

$$\frac{Q_1P_1}{AQ_4} = \frac{BR_1}{AB}; \text{ and generally for the point } P_r,$$

$$\frac{Q_rP_r}{AQ_r} = \frac{BR_r}{AB} = \frac{\frac{rb}{n}}{l} \quad . \; . \; . \; (1)$$

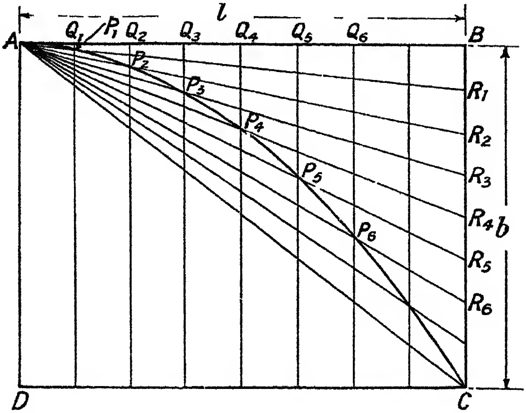


FIG. 58

Let P_r be the point (x, y) referred to AD, AB as axes of x and y; then $x = Q_rP_r$ and $y = AQ_r = \frac{rl}{n}$, so that $r = \frac{ny}{l}$. Substituting in (1), we have

$$\frac{x}{y} = \frac{\frac{ny}{l} \cdot \frac{b}{n}}{l} = \frac{by}{l^2} \text{ or } y^2 = \frac{l^2x}{b}$$

which is the equation of a parabola having A as vertex and latus-rectum equal to $\frac{l^2}{b}$.

EXAMPLE 4

A particle of mass m is projected with velocity V at an angle α below the horizontal. Assuming that gravity is the only force acting on the particle, show that it describes a parabola, and that the kinetic energy which it possesses at any point of its path is equal to the work that would be done on the particle if it fell to that point from rest at the directrix.

The horizontal and downward vertical components of the initial velocity V are $V \cos \alpha$ and $V \sin \alpha$ respectively. Since the horizontal motion is unaffected by gravity, the horizontal distance x described in time t is given by

$$x = V \cos \alpha \, . \, t \quad . \; . \; . \; (1)$$

The vertical distance y fallen in time t is given by

$$y = V \sin \alpha \,.\, t + \tfrac{1}{2} g t^2 \quad . \; . \; . \; (2)$$

where g = acceleration due to gravity.

Eliminating t between (1) and (2), we obtain

$$y = V \sin \alpha \left(\frac{x}{V \cos \alpha}\right) + \tfrac{1}{2} g \left(\frac{x}{V \cos \alpha}\right)^2$$

i.e.
$$y = xp + \frac{g}{2V^2}(1 + p^2)x^2, \text{ where } p = \tan \alpha$$

This equation may be written as

$$\frac{2V^2}{g(1+p^2)}\, y + \left[\frac{pV^2}{g(1+p^2)}\right]^2 = x^2 + x\,\frac{2pV^2}{g(1+p^2)} + \left[\frac{pV^2}{g(1+p^2)}\right]^2$$

or
$$\frac{2V^2}{g(1+p^2)}\left[y + \frac{p^2V^2}{2g(1+p^2)}\right] = \left[x + \frac{pV^2}{g(1+p^2)}\right]^2$$

Transferring to the point $\left(-\dfrac{pV^2}{g(1+p^2)}, -\dfrac{p^2V^2}{2g(1+p^2)}\right)$ as origin, we obtain the equation $x^2 = \dfrac{2V^2}{g(1+p^2)}\, y$, which represents a parabola with vertex at the origin and latus-rectum equal to $\dfrac{2V^2}{g(1+p^2)}$ or $\dfrac{2V^2 \cos^2 \alpha}{g}$

The distance of the directrix from the vertex $= \dfrac{V^2 \cos^2 \alpha}{2g}$; the distance below the directrix of the position of the particle at time t after projection

$$= (V \sin \alpha \,.\, t + \tfrac{1}{2} g t^2) + \frac{p^2 V^2}{2g(1+p^2)} + \frac{V^2 \cos^2 \alpha}{2g}$$

$$= V \sin \alpha \,.\, t + \tfrac{1}{2} g t^2 + \frac{V^2}{2g} \left(\text{since } \frac{p^2}{1+p^2} = \frac{\tan^2 \alpha}{\sec^2 \alpha} = \sin^2 \alpha\right)$$

Therefore the work that would be done on the particle in a fall from the directrix $= mg\left(V \sin \alpha \,.\, t + \tfrac{1}{2} g t^2 + \dfrac{V^2}{2g}\right) = \dfrac{m}{2}(2g\,V \sin \alpha \cdot t + g^2 t^2 + V^2)$

Again, the horizontal and vertical components of the velocity of the particle at time t are $V \cos \alpha$ and $V \sin \alpha + gt$ respectively. The actual velocity V of the particle then is given by

$$V_t^2 = (V \cos \alpha)^2 + (V \sin \alpha + gt)^2 = V^2 + 2gV \sin \alpha \,.\, t + g^2 t^2$$

and its kinetic energy $= \tfrac{1}{2} m\,[V^2 + 2gV \sin \alpha \,.\, t + g^2 t^2]$, which is the same as the expression found above for the work done.

89. **The Ellipse.** In the ellipse $e < 1$. Let S be the focus and ZZ' the directrix as in Fig. 53, and let the line SK (Fig. 59) be divided internally and externally at A and A' respectively, such that $SA/AK = e$ and $SA'/A'K = e$. A and A' are then points on the curve, and if points H and K' be taken on AA' as shown in the figure

such that $A'H = SA$ and $K'A' = AK$, it is evident from considerations of symmetry that we could trace out the ellipse with H as focus and a line through K' parallel to ZZ' as directrix. The ellipse has then two foci and two directrices. Let $AA' = 2a$ and C be the mid-point of AA'. AA' is called the *major-axis* of the ellipse.

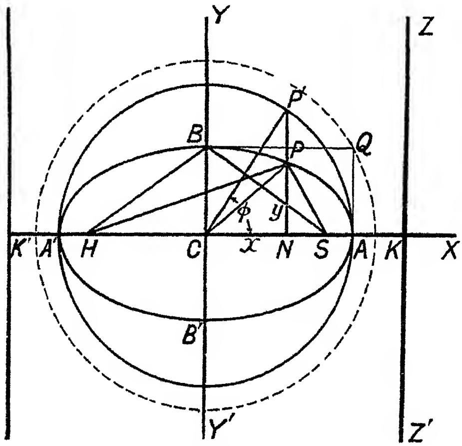


FIG. 59

Now
$$SA = e \,.\, AK. \qquad \text{(VII.12)}$$
$$SA' = e \,.\, A'K \qquad \text{(VII.13)}$$
$$HA = e \,.\, AK' \qquad \text{(VII.14)}$$

Subtracting (VII.12) from (VII.13), we have

$$SA' - SA = e(A'K - AK)$$

i.e.
$$HS = e \,.\, AA' = 2ae$$

$\therefore$
$$CS = ae \qquad \text{(VII.15)}$$

Adding (VII.12) and (VII.14), we have

$$SA + HA = e(AK + AK')$$

i.e.
$$2a = e \,.\, KK'$$

$\therefore$
$$CK = \frac{a}{e} \,. \qquad \text{(VII.16)}$$

Draw CY perpendicular to CK and let P be any point (x, y) on the curve referred to CK, CY as axes of x and y respectively; let N be the foot of the ordinate of P.

Then $\quad SP = e \,.\, NK = e(CK - CN) = e\left(\dfrac{a}{e} - x\right) = a - ex$ (VII.17)

and $\quad HP = e \,.\, NK' = e(CK' + CN) = e\left(\dfrac{a}{e} + x\right) = a + ex$ (VII.18)

$\therefore \; SP + HP = 2a.$

Hence, *the sum of the focal radii of any point on an ellipse is constant and equal to the major-axis.* (VII.19)

Again, $\quad SP^2 = (a - ex)^2$

i.e. $\quad y^2 + NS^2 = (a - ex)^2$

or $\quad y^2 = (a - ex)^2 - (ae - x)^2 = a^2(1 - e^2) - x^2(1 - e^2);$

whence $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1 - e^2)} = 1$

Put $a^2(1 - e^2) = b^2$, so that $e^2 = 1 - \dfrac{b^2}{a^2}$; then the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad \text{(VII.20)}$$

This is the standard form of the equation to an ellipse. Since, when $x = 0$, $y = \pm b$, the lengths CB, CB' in the figure are each equal to b. Now, if y is greater than b, $1 - \dfrac{y^2}{b^2}$ is negative and x is imaginary; similarly if x is greater than a, $1 - \dfrac{x^2}{a^2}$ is negative and y is imaginary. The ellipse is, therefore, a closed curve, its greatest diameter being AA' (the major-axis) and its least diameter being BB' (the minor-axis).

Since B is on the ellipse,

$$SB + HB = 2a, \;\therefore\; SB = HB = a \quad . \quad . \quad . \quad \text{(VII.21)}$$

Also $\quad AS \,.\, A'S = (a - ae)(a + ae) = a^2(1 - e^2) = b^2$ (VII.22)

If a circle be drawn having AA' as diameter and the ordinate NP be produced to meet the circle in P', then as in Art. 74 (*i*), $\dfrac{PN}{P'N} = \dfrac{b}{a}$. Join CP' and call angle NCP', ϕ; then

$$x = CN = CP' \cos \phi = a \cos \phi \qquad . \qquad . \quad \text{(VII.23)}$$

$$y = PN = \frac{b}{a} P'N = \frac{b}{a} a \sin \phi = b \sin \phi \qquad . \quad \text{(VII.24)}$$

Hence, the co-ordinate of any point on an ellipse can be expressed in terms of a single variable ϕ as *a cos* ϕ, *b sin* ϕ respectively. ϕ is

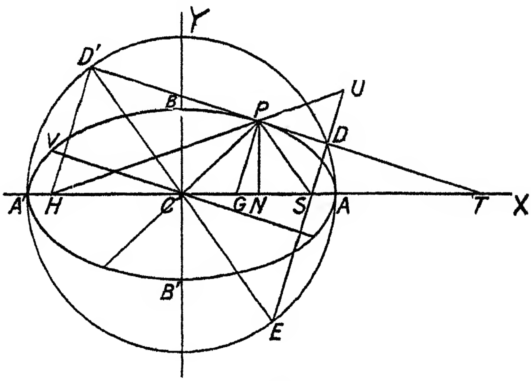


FIG. 60

called the *eccentric angle* of the point P and, as stated in Art. 74 (*i*), the circle on AA' as diameter is called the *auxiliary circle* of the ellipse.

We have proved (Art. 75, Example 1 (ii)) that the equation of the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $P\,(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad . \qquad . \qquad . \quad \text{(VII.25)}$$

and of the normal,

$$(y - y_1) \Big/ \frac{y_1}{b^2} = (x - x_1) \Big/ \frac{x_1}{a^2} \qquad . \qquad . \quad \text{(VII.26)}$$

Let the tangent and normal at P (Fig. 60) meet the x-axis in T and G respectively. Putting $y = 0$ in (VII.25) and (VII.26), we have

$$x = CT = \frac{a^2}{x_1} \text{ and } x = CG = x_1 \left(1 - \frac{b^2}{a^2}\right) = x_1 e^2$$

Hence, $CN \, . \, CT = x_1 \cdot \dfrac{a^2}{x_1} = a^2$. (VII.27)

Also $\qquad HG = HC + CG = ae + x_1e^2$

$\qquad\qquad = e(a + ex_1) = e \, . \, HP.$ . . . (VII.28)

and $\qquad GS = CS - CG = ae - x_1e^2$

$\qquad\qquad = e(a - ex_1) = e \, . \, SP$ . . (VII.29)

$$\therefore \qquad \frac{HG}{GS} = \frac{HP}{SP}$$

so that PG bisects the angle HPS of the triangle HPS.

Hence, *the normal and tangent at any point on an ellipse bisect the angles between the focal radii of the point.* (VII.30)

In Fig. 60 let SD, HD' be drawn perpendicular to the tangent at P. Produce SD to meet HP produced at U. By (VII.30) PD bisects $\widehat{SPU}$, and since $PD \perp SU$, the triangles SPD, UPD are congruent and $SP = PU$; $\therefore HU = HP + PS = 2a$.

Now, since C and D are the mid-points of SH, SU respectively, then $CD = \frac{1}{2}HU = a$; $\therefore$ D lies on the auxiliary circle and we can prove similarly that D' also lies on this circle.

Hence, *the feet of the perpendiculars drawn from the foci to any tangent to an ellipse lie on the auxiliary circle.* (VII.31)

If $D'C$ and DS meet in E, then E is a point on the auxiliary circle. Also the triangles $CD'H$, CES are congruent and $HD' = SE$. Now $SD \, . \, SE = AS \, . \, A'S = b^2$[by (VII.22)]

$$\therefore \qquad SD \, . \, HD' = b^2 \qquad . \quad . \quad . \text{ (VII.32)}$$

Let the eccentric angles of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be ϕ_1 and ϕ_2.

Then $\qquad x_1 = a \cos \phi_1 \qquad x_2 = a \cos \phi_2$

$\qquad\qquad y_1 = b \sin \phi_1 \qquad y_2 = b \sin \phi_2$

The equation of the line joining P and Q is

$$\frac{y - b \sin \phi_1}{b(\sin \phi_2 - \sin \phi_1)} = \frac{x - a \cos \phi_1}{a(\cos \phi_2 - \cos \phi_1)}$$

which reduces to

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$$

The gradient of this line is $-\frac{b}{a}\cot\frac{\phi_1+\phi_2}{2}$, and if the line moves parallel to itself the gradient will remain unaltered and accordingly $\phi_1+\phi_2$ will be constant.

Hence, *the sum of the eccentric angles of the ends of any chord of a system of parallel chords of an ellipse is constant.* (VII.33)

Let $R\ (\xi, \eta)$ be in the mid-point of PQ; then

$$\xi=\frac{a(\cos\phi_1+\cos\phi_2)}{2},\quad \eta=\frac{b(\sin\phi_1+\sin\phi_2)}{2}$$

so that
$$\frac{\eta}{\xi}=\frac{b}{a}\cdot\frac{2\sin\frac{\phi_1+\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}}{2\cos\frac{\phi_1+\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}}=\frac{b}{a}\tan\frac{\phi_1+\phi_2}{2}$$

Now, if m be the constant gradient of the system of chords parallel to PQ, $m=-\frac{b}{a}\cot\frac{\phi_1+\phi_2}{2}$, i.e. $\tan\frac{\phi_1+\phi_2}{2}=-\frac{b}{am}$; hence, $\frac{\eta}{\xi}=-\frac{b^2}{a^2m}$. The locus of R is, therefore, the straight line $y=-\frac{b^2}{a^2m}x=m'x$, where $m'=-\frac{b^2}{a^2m}$.

Hence, *the locus of the mid-points of a system of parallel chords of an ellipse is a diameter of the ellipse.* (VII.34)

Since $mm'=-\frac{b^2}{a^2}$, it follows from symmetry that the diameter $y=mx$ will bisect all chords parallel to the diameter $y=m'x$. Two diameters of an ellipse which are such that each bisects all chords parallel to the other, are said to be *conjugate*, and their gradients m and m' are connected by the relation

$$mm'=-\frac{b^2}{a^2}\quad . \quad . \quad . \quad \text{(VII.35)}$$

Let CP, CV be two conjugate diameters of an ellipse, and let the eccentric angles of P and V be ϕ_1 and ϕ_2 respectively. Then

the equation of CP is $\frac{y}{x} = \frac{b \sin \phi_1}{a \cos \phi_1} = \frac{b}{a} \tan \phi_1$, and that of CV is $\frac{y}{x} = \frac{b}{a} \tan \phi_2$.

By (VII.35) $$\frac{b}{a} \tan \phi_1 \cdot \frac{b}{a} \tan \phi_2 = -\frac{b^2}{a^2}$$

i.e. $$\frac{b^2}{a^2} (\tan \phi_1 \tan \phi_2 + 1) = 0,$$

whence $\tan \phi_1 = -\cot \phi_2$, and this implies that $\phi_1 = \frac{\pi}{2} + \phi_2$, or $\phi_1 = \phi_2 - \frac{\pi}{2}$

Hence, *the difference of the eccentric angles of two points at the ends of two conjugate diameters of an ellipse is a right angle.* (VII.36)

Again, $$CP^2 = a^2 \cos^2 \phi_1 + b^2 \sin^2 \phi_1$$
$$= a^2 \sin^2 \phi_2 + b^2 \cos^2 \phi_2 \left(\text{since } \phi_1 \sim \phi_2 = \frac{\pi}{2}\right)$$

and $$CV^2 = a^2 \cos^2 \phi_2 + b^2 \sin^2 \phi_2$$

$$\therefore CP^2 + CV^2 = a^2(\sin^2 \phi_2 + \cos^2 \phi_2) + b^2(\cos^2 \phi_2 + \sin^2 \phi_2)$$
$$= a^2 + b^2$$

Hence, *the sum of the squares of two conjugate semi-diameters of an ellipse is constant and equal to $a^2 + b^2$.* (VII.37)

Since CP bisects chords parallel to CV and a tangent is the limiting position of a chord, the tangent at P will be parallel to CV, and similarly the tangent at V will be parallel to CP.

By (VII.25) the equation of the tangent at V is

$$\frac{x(a \cos \phi_2)}{a^2} + \frac{y(b \sin \phi_2)}{b^2} = 1$$

i.e. $$\frac{x}{a} \cos \phi_2 + \frac{y}{b} \sin \phi_2 = 1$$

The perpendicular distance of C from this tangent is

$$\frac{1}{\sqrt{\frac{\cos^2 \phi_2}{a^2} + \frac{\sin^2 \phi_2}{b^2}}};$$

and $CP = \sqrt{a^2 \sin^2 \phi_2 + b^2 \cos^2 \phi_2}$, (from above)

Thus the area of the parallelogram formed by CP, CV, and the tangents at P and V is

$$\frac{ab}{\sqrt{b^2\cos^2\phi_2 + a^2\sin^2\phi_2}} \times \sqrt{a^2\sin^2\phi_2 + b^2\cos^2\phi_2} = ab$$

But the area of this parallelogram $= CP \,.\, CV \sin PCV$

$$\therefore \qquad CP \,.\, CV = ab \cdot \operatorname{cosec} P\hat{C}V \qquad . \qquad . \quad \text{(VII.38)}$$

Again, by (VII.17) and (VII.18),

$$SP \,.\, HP = (a - ex_1)(a + ex_1),$$

where P is the point (x_1, y_1)

$$= a^2 - e^2x_1^2$$

$$\begin{aligned}
\text{Now } CV^2 \qquad &= a^2\cos^2\phi_2 + b^2\sin^2\phi_2 \\
&= a^2 - (a^2 - b^2)\sin^2\phi_2 \\
&= a^2\left[1 - \left(1 - \frac{b^2}{a^2}\right)\cos^2\phi_1\right] \\
&= a^2[1 - e^2\cos^2\phi_1] \\
&= a^2 - e^2x_1^2
\end{aligned}$$

$$\therefore \qquad SP \,.\, HP = CV^2$$

Hence, *the product of the focal radii of any point on an ellipse is equal to the square on the semi-diameter which is parallel to the tangent at the point.* (VII.39)

90. **Director Circle.** The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is cut by the straight line $y = mx + c$, where $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$, i.e. where $(b^2 + a^2m^2)x^2 + 2a^2mc \,.\, x + a^2(c^2 - b^2) = 0$.

If the line is a tangent to the ellipse, the two roots of this equation will be equal; the condition required is

$$4a^4m^2c^2 = 4(b^2 + a^2m^2) \,.\, a^2(c^2 - b^2)$$

which reduces to $\qquad c^2 = a^2m^2 + b^2$

Hence, *the line $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all values of m.* (VII.40)

The condition that the tangent (VII.40) should pass through a given point (x', y') is $y' = mx' + \sqrt{a^2m^2 + b^2}$, or $m^2(x'^2 - a^2) -$

$2x'y'm + (y'^2 - b^2) = 0$; so that two tangents to the ellipse can be drawn from a given point (x', y').

Let the two values of m given by the above equation be m_1 and m_2; then the two tangents will be at right angles if $m_1m_2 = -1$, i.e. if $\dfrac{y'^2 - b^2}{x'^2 - a^2} = -1$, or $x'^2 + y'^2 = a^2 + b^2$.

Hence, *the locus of the point of intersection of two perpendicular tangents to an ellipse is the circle* $x^2 + y^2 = a^2 + b^2$. *This circle, shown dotted in Fig. 59, is called the "director-circle" of the ellipse.*

91. Polar Equation of Ellipse Referred to Centre as Pole. If in Fig. 60 P is the point (r, θ) referred to C as pole and CA as initial line, then $x = r\cos\theta$, $y = r\sin\theta$, and the polar equation of the ellipse is $\dfrac{r^2\cos^2\theta}{a^2} + \dfrac{r^2\sin^2\theta}{b^2} = 1$,

or $$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad . \quad . \quad . \quad \text{(VII.41)}$$

EXAMPLE 1

Prove that the normal at any point of an ellipse meets the axis at a distance from the focus which is in a constant ratio to the focal distance of the point.

If PG is the normal at P, prove that the circle on PG as diameter intercepts on the focal distances of P chords equal in length to the semi-latus rectum (U.L.)

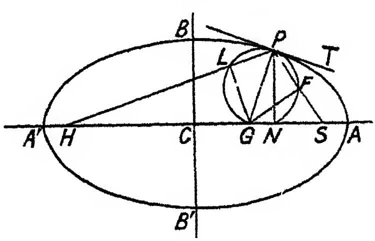


FIG. 61

Referring to Fig. 61, we have $GS = e\,.\,SP$ [by (VII.29)]; hence, $\dfrac{GS}{SP} = e$ (constant).

If the circle on PG as diameter cuts SP at F, then $PF = PG\cos G\hat{P}S$.

Now, since $\dfrac{GS}{SP} = e$, $\dfrac{\sin G\hat{P}S}{\sin P\hat{G}S} = e$; therefore $\sin G\hat{P}S = e\sin P\hat{G}S = e\,.\,\dfrac{PN}{PG}$,

and $\cos G\hat{P}S = \sqrt{1 - \sin^2 G\hat{P}S} = \sqrt{1 - e^2\dfrac{PN^2}{PG^2}} = \dfrac{\sqrt{PG^2 - e^2PN^2}}{PG}$

$\therefore\ PF = \sqrt{(y^2 + GN^2) - e^2y^2} = \sqrt{y^2(1 - e^2) + x^2(1 - e^2)^2}$
[since $GN = CN - CG = x(1 - e^2)$]

$$= \frac{b}{a}\sqrt{y^2 + x^2 \cdot \frac{b^2}{a^2}} \left(\text{since } 1 - e^2 = \frac{b^2}{a^2}\right)$$

$$= \frac{b}{a} \cdot \sqrt{b^2} = \frac{b^2}{a}$$

The length of the semi-latus rectum = value of y when $x = CS = ae$

$$= b\sqrt{1 - \frac{a^2e^2}{a^2}} = b\sqrt{1 - e^2} = \frac{b^2}{a}$$

Thus, PF = length of semi-latus rectum; and since PG bisects $H\hat{P}S$, the circle on PG as diameter will intercept on HP a chord PL equal to PF.

EXAMPLE 2

A particle moves in one plane in such a way that its co-ordinates at time t referred to rectangular axes through a fixed point O in the plane of motion are given by $x = a \cos kt$, $y = b \sin kt$.

Show that the particle describes an ellipse with O as centre and that its acceleration is directed towards O and proportional to its distance from O.

We have $\frac{x}{a} = \cos kt$ and $\frac{y}{b} = \sin kt$; hence,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 kt + \sin^2 kt = 1$$

Also, when $t = 0$, $x = a$ and $y = 0$. The particle therefore describes an ellipse whose major and minor axes are along the axes of x and y.

We find $\frac{d^2x}{dt^2} = -k^2a \cos kt = -k^2x$, and $\frac{d^2y}{dt^2} = -k^2b \sin kt = -k^2y$

If P is the position of the particle at time t and PM is drawn perpendicular to the x-axis, then, in vector notation,

Acceleration of particle = vector sum of accelerations $-k^2x$, $-k^2y$ parallel to Ox, Oy

$$= -k^2\overrightarrow{OM} - k^2\overrightarrow{MP}$$

$$= k^2\overrightarrow{MO} + k^2\overrightarrow{PM}$$

$$= k^2(\overrightarrow{MO} + \overrightarrow{PM}) = k^2\overrightarrow{PO}$$

i.e. acceleration of particle is directed towards O, and proportional to distance PO.

EXAMPLE 3

PHQ is a focal chord of an ellipse whose foci are S and H. Prove that the escribed circle of the triangle PSQ, which touches PQ externally, touches PQ in H and has its centre at the point where the tangents to the ellipse at P and Q meet. (U.L.)

Let the escribed circle, centre I (Fig. 62), touch SP and SQ produced at T and T' respectively and PQ at H'. Since tangents drawn from an external point to a circle are equal, $ST = ST'$, $PT = PH'$, $QT' = QH'$; also $ST + ST' = (SP + PH') + (SQ + QH') =$ perimeter of triangle SPQ.

But perimeter of triangle $SPQ = (SP + PH) + (SQ + QH) = 2a + 2a = 4a$ where $a =$ semi-major axis of the ellipse.

$\therefore ST = ST' = \frac{1}{2}(ST + ST') = 2a$; whence $SP + PH' = 2a$.

Therefore $PH' = PH$ and H and H' must be the same point.

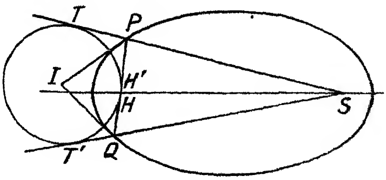


FIG. 62

Again, IP and IQ bisect the angles QPT, PQT' respectively; but by (VII.30) the tangents to the ellipse at P and Q bisect the exterior angles between the focal radii of the points. Hence, IP and IQ are tangents to the ellipse at P and Q, and I is the point where they meet.

EXAMPLE 4

A perpendicular CK is drawn from the centre C of an ellipse to the line joining the ends A and B of the semi-major and semi-minor axes. If $AK = x_1$ and BK

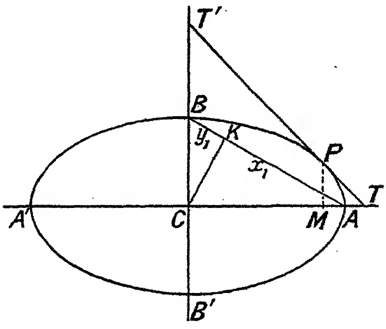


FIG. 63

$= y_1$, show that the point whose co-ordinates are (x_1, y_1) lies on the ellipse and that the tangent to the ellipse at this point is equally inclined to the axes.

From the similar triangles ACK, ABC (Fig. 63), we have

$$\frac{AK}{AC} = \frac{AC}{AB}$$

$$\therefore \qquad AK = \frac{AC^2}{AB} = \frac{a}{\sqrt{a^2 + b^2}} = x_1$$

Also $\quad BK \quad AB \quad AK \quad \sqrt{a^2+b^2} \quad \frac{a^2}{\sqrt{a^2+b^2}} \quad \frac{b^2}{\sqrt{a^2+b^2}} \quad y_1$

Now $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\frac{a^4}{a^2(a^2+b^2)}+\frac{b^4}{b^2(a^2+b^2)}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}$ $=1$; so that the point (x_1, y_1) lies on the ellipse.

The equation of the tangent at (x_1, y_2) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \text{ or } \frac{x}{a^2}\cdot\frac{a^2}{\sqrt{a^2+b^2}}+\frac{y}{b^2}\cdot\frac{b^2}{\sqrt{a^2+b^2}}=1$$

i.e.,
$$x+y=\sqrt{a^2+b^2}$$

The tangent is of gradient -1, and is therefore equally inclined to the axes. P is the point (x_1, y_1), and $T'T$ is the tangent.

92. **The Hyperbola.** In the hyperbola, $e > 1$. Let the line SK (Fig. 64) be divided internally and externally at A and A' respectively,

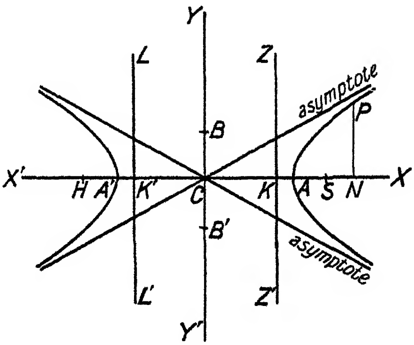


Fig 64

such that $SA/AK = e$ and $SA'/A'K = e$. The point A' will be on the left of the directrix ZZ'. If points H and K' be taken on AA' as shown such that $A'H = AS$ and $A'K' = KA$, it is evident from symmetry that we could trace out the hyperbola with H as focus and a line LL' through K' parallel to ZZ' as directrix. Let $AA' = 2a$ and let C be the mid-point of AA'. C is the *centre* and AA' the *transverse axis* of the hyperbola.

Now
$$SA = e\,.\,AK \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$SA' = e\,.\,A'K \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$HA = e\,.\,AK' \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Subtracting (1) from (2), we have

$$SA' - SA = e(A'K - AK)$$

$$AA' = e \, . \, KK'$$

$$CK = \frac{a}{e} \quad . \qquad \text{(VII.42)}$$

nd (3), we have

$$SA + HA = e(AK + AK')$$

$$SH = e \, . \, AA'$$

$$CS = ae \quad . \quad . \quad . \quad . \qquad \text{(VII.43)}$$

erpendicular to CK and let P be any point (x, y) on
red to CK, CY as axes of x and y respectively; let N
the ordinate of P. Then

$$= e(CN - CK) = e\left(x - \frac{a}{e}\right) = ex - a \qquad \text{(VII.44)}$$

$$= e \, . \, NK' = e(CN + CK') = e\left(x + \frac{a}{e}\right) = ex + a$$

$$\text{(VII.45)}$$

$$= 2a$$

ifference of the focal radii of any point on
constant and equal to the transverse axis. (VII.46)

$$SP^2 = (ex - a)^2$$

$$\cdot \, SN^2 = (ex - a)^2$$

$$y^2 = (ex - a)^2 - (x - ae)^2 = (e^2 - 1)x^2 - (e^2 - 1)a^2$$

$$\frac{y^2}{- 1)} = 1$$

) $= b^2$, so that $e^2 = 1 + \frac{b^2}{a^2}$; then the equation of
becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \qquad \text{(VII.47)}$$

n a numerically, $\frac{x^2}{a^2} - 1$ is negative and y is imaginary,
of the graph lies between the lines given by $x = \pm a$.
than a numerically, y is real and for any given value
two values of y differing only in sign. The transverse
refore, an axis of symmetry. As x increases numeric-
alue a, $|y|$ increases, so that the curve consists of two

open infinite branches. When $x = 0$, y is imaginary; but if we choose points B and B' on the y-axis such that $CB = b$, $CB' = -b$, BB' is called the *conjugate axis* of the hyperbola. BB' will be the transverse axis and AA' the conjugate axis of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, which is said to be *conjugate* to the original hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

By simply changing $+ b^2$ to $- b^2$ we can derive many properties of the hyperbola from the corresponding properties of the ellipse. We give the most important of these in the list below, and also certain geometrical properties which hold for both curves, the notation being that used in the case of the ellipse.

	Ellipse	Hyperbola
Equation of curve	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Tangent at point (x_1, y_1)	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
Normal at point (x_1, y_1)	$\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}$	$\frac{y - y_1}{-y_1/b^2} = \frac{x - x_1}{x_1/a^2}$
Tangent of gradient m	$y = mx \pm \sqrt{a^2m^2 + b^2}$	$y = mx \pm \sqrt{a^2m^2 - b^2}$
Director-circle .	$x^2 + y^2 = a^2 + b^2$	$x^2 + y^2 = a^2 - b^2$
Locus of mid-points of chords of gradient m .	$y = m'x$, where $m' = -\frac{b^2}{a^2m}$	$y = m'x$, where $m' = \frac{b^2}{a^2m}$
Sum of squares of two conjugate semi-diameters	$a^2 + b^2$	$a^2 - b^2$
Product $CN \, . \, CT$	a^2	a^2
Length CG . .	e^2x_1	e^2x_1
Angle between focal radii of any point on the curve .	Bisected internally by normal and externally by tangent.	Bisected internally by tangent and externally by normal.
Product $SD \, . \, HD$	b^2	b^2
Locus of D and D'	Circle $x^2 + y^2 = a^2$	Circle $x^2 + y^2 = a^2$
Polar equation of curve with pole at centre .	$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$	$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}$

In the hyperbola two conjugate diameters cannot both cut the curve in real points. For if $y = mx$ and $y = m'x$ are conjugate diameters and $m < m'$ (for m cannot equal m'), then since $mm' = \frac{b^2}{a^2}$,

m must be $< \frac{b}{a}$ and $m' > \frac{b}{a}$. Now the line $y = m'x$ cuts the curve where $\frac{x^2}{a^2} - \frac{m'^2x^2}{b^2} = 1$, or $x^2 = \frac{a^2b^2}{b^2 - a^2m'^2}$, which is negative since $m' > \frac{b}{a}$. Hence, x is imaginary. Also since $m < \frac{b}{a}$, the line $y = mx$ cuts the curve in real points.

93. Asymptotes. When the point of contact of a tangent to a curve moves off to an infinite distance from the origin, the limiting position to which the tangent tends is called an *asymptote* of the curve (Fig. 64).

The line $y = mx + c$ cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$, or

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0 \qquad \text{(VII.48)}$$

Put $x = \frac{1}{z}$, so that the equation becomes

$$a^2(b^2 + c^2)z^2 + 2a^2mcz - (b^2 - a^2m^2) = 0 \quad . \quad \text{(VII.49)}$$

Now, if $y = mx + c$ is an asymptote to the hyperbola, equation (VII.48) will have two infinite roots, and therefore equation (VII.49) will have two zero roots; the conditions required are $b^2 - a^2m^2 = 0$ and $a^2mc = 0$, i.e. $m = \pm \frac{b}{a}$ and $c = 0$ (since a and m are not zero).

Hence, the lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (see Fig. 64).

The combined equation of the two asymptotes of the hyperbola is then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$ (VII.50)

If $2\alpha =$ angle between the asymptotes, then

$$\tan \alpha = \frac{b}{a} \quad . \quad . \quad . \quad . \quad \text{(VII.51)}$$

and
$$\sec \alpha = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{e^2} = e \qquad \text{(VII.52)}$$

This method of finding the asymptotes is applicable to any algebraical curve. From the definition of an asymptote we note that only curves with infinite branches can have real asymptotes.

The lengths of the perpendiculars from the point (x', y') on the hyperbola to its asymptotes $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$ are $\left(\frac{x'}{a} + \frac{y'}{b}\right) \Big/ \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$ and $\left(\frac{x'}{a} - \frac{y'}{b}\right) \Big/ \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$. Their product equals $\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) \Big/ \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{a^2b^2}{a^2 + b^2}$

Hence, *the product of the perpendiculars from any point on a hyperbola to its asymptotes is constant and equal to* $\frac{a^2b^2}{a^2 + b^2}$ (VII.53)

94. **Rectangular Hyperbola.** When $b = a$ the hyperbola is said to be rectangular. Its equation is then

$$x^2 - y^2 = a^2 \quad . \quad . \quad . \quad . \quad \text{(VII.54)}$$

The equations of its asymptotes are $y = x$ and $y = -x$, so that the asymptotes of a rectangular hyperbola are at right angles to each other and bisect the angles between the transverse and conjugate axes. If P be any point (x, y) on a rectangular hyperbola referred to its asymptotes as axes of reference, then by (VII.53) the product of the perpendiculars from P to the asymptotes $= xy = \frac{a^4}{2a^2} = \frac{a^2}{2}$.

Thus, the equation of a rectangular hyperbola referred to its asymptotes as axes of reference is

$$xy = \text{constant} \quad . \quad . \quad . \quad \text{(VII.55)}$$

For example, the pressure-volume relation of a gas $pv = c$ gives a rectangular hyperbola.

EXAMPLE 1

Find the point of intersection in the first quadrant of the hyperbola $3x^2 - 4y^2 = 12$ and the parabola $y^2 = 4x$, and find also the angle at which the curves intersect.

The two curves cut where $3x^2 - 4(4x) = 12$, or $3x^2 - 16x - 12 = 0$; this gives $(3x + 2)(x - 6) = 0$; i.e. $x = -\frac{2}{3}$ or 6.

When $x = 6$, $y^2 = 4 \times 6$ and $y = \sqrt{24}$; the required point of intersection is

then $(6, 2\sqrt{6})$. The tangents at the point (x_1, y_1) on the curves are $3xx_1 - 4yy_1 = 12$ and $yy_1 = 2(x + x_1)$ respectively. At the point $(6, 2\sqrt{6})$ the tangents are

$$3x(6) - 4y(2\sqrt{6}) = 12, \quad \text{and} \quad y(2\sqrt{6}) = 2(x + 6);$$

i.e.
$$y = \frac{3\sqrt{6}}{8}x - \frac{\sqrt{6}}{4} \text{ and } y = \frac{\sqrt{6}}{6}x + \sqrt{6}$$

The gradients of these tangents are $m_1 = \frac{3\sqrt{6}}{8}$ and $m_2 = \frac{\sqrt{6}}{6}$; the angle θ between the tangents is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \text{ (Art. 74 (c))}$$

$$\therefore \tan\theta = \left(\frac{3\sqrt{6}}{8} - \frac{\sqrt{6}}{6}\right)\Big/\left[1 + \frac{3\sqrt{6}}{8}\cdot\frac{\sqrt{6}}{6}\right] = \frac{5\sqrt{6}}{24}\Big/\frac{11}{8} = \frac{5\sqrt{6}}{33} = 0.3711$$

$\therefore \quad \theta = 20° \, 22'$ nearly.

EXAMPLE 2

(a) OX, OY (Fig. 65) are two lines at right angles. A third line PQ cuts them at P and Q respectively, such that the triangle OPQ is of constant area. Prove that the locus of the mid-point R of PQ is a rectangular hyperbola.

(b) Prove that the eccentricity of the hyperbola conjugate to the hyperbola $x^2 - 3y^2 = 3$ is 2.

(a) Let the co-ordinates of R be (x, y) referred to OX, OY as axes of reference. Then $OP = 2x$ and $OQ = 2y$; so that area $OPQ = \frac{1}{2} \cdot 2x \cdot 2y = 2xy$ = constant (by hypothesis).

The locus of R is therefore the rectangular hyperbola $xy = c$ (where c is some constant).

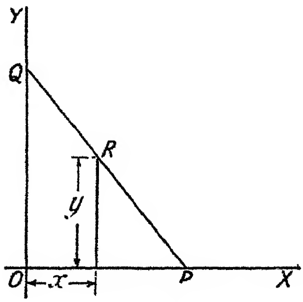


FIG. 65

(b) The equation $x^2 - 3y^2 = 3$ can be written as $\frac{x^2}{3} - \frac{y^2}{1} = 1$; the equation of the conjugate hyperbola is $\frac{y^2}{1} - \frac{x^2}{3} = 1$ (p. 214), and its eccentricity e' is given by $e'^2 = 1 + \frac{3}{1} = 4$; whence $e' = 2$.

EXAMPLE 3

Find the condition that the equation $y^2 = px^2 + qx + r$ should represent a hyperbola. This condition being satisfied, find the equations of the asymptotes of the hyperbola.

We can write the given equation in the form

$$\left[x^2 + \frac{q}{p}x + \frac{q^2}{4p^2}\right] = y^2 + \frac{q^2}{4p} - r$$

or
$$\frac{\left(x + \dfrac{q}{2p}\right)^2}{\dfrac{q^2 - 4pr}{4p^2}} - \frac{y^2}{\dfrac{q^2 - 4pr}{4p}} = 1$$

Put $\dfrac{q^2 - 4pr}{4p} = k$ and transfer the origin to the point $\left(-\dfrac{q}{2p}, 0\right)$; the equation then becomes $\dfrac{x^2}{k/p} - \dfrac{y^2}{k} = 1$. Now, whatever be the sign of k, this equation will represent a hyperbola if p is positive.

The combined equation of the asymptotes is

$$\frac{x^2}{k/p} - \frac{y^2}{k} = 0 \text{ [by (VII.50)]},$$

or, reverting to the original axes,

$$\frac{\left(x + \dfrac{q}{2p}\right)^2}{\dfrac{k}{p}} - \frac{y^2}{k} = 0, \text{ i.e. } p\left(x + \frac{q}{2p}\right)^2 = y^2$$

or
$$px^2 + qx + \frac{q^2}{4p} = y^2$$

95. **Catenaries.** Fig. 66 shows a chain or string ACB hanging freely under gravity, its ends A and B being fixed to two points not in the same vertical line. Let T_C be the tension at the lowest point C, and T_P the tension at any point P on the chain. Then, if the angle between the tangent to the curve at P and the horizontal be ψ and if the weight of the portion CP of the chain be W_P, we have by resolving vertically and horizontally for the equilibrium of the portion CP

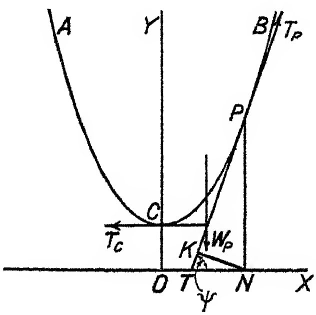


FIG. 66

$$T_P \sin \psi = W_P \quad . \quad \text{(VII.56)}$$

$$T_P \cos \psi = T_C \quad . \quad \text{(VII.57)}$$

Squaring and adding,

$$T_P^2 = T_C^2 + W_P^2 \quad . \quad . \quad . \quad \text{(VII.58)}$$

Dividing (VII.56) by (VII.57) and writing $\dfrac{dy}{dx}$ for $\tan \psi$, we have

$$\frac{dy}{dx} = \frac{W_P}{T_C} \quad . \quad . \quad . \quad . \quad \text{(VII.59)}$$

The relation (VII.57) shows that the horizontal component of the tension is constant.

UNIFORM CATENARY. If the chain is uniform, i.e. if its weight per unit length is constant, the curve in which it hangs is called the *uniform* or *common catenary*.

Let w = weight per unit length and let T_C be equal to the weight of a length c of the chain. The relations (VII.56) and (VII.57) now become

$$T_P \sin \psi = ws \text{ (where } s = \text{length of arc } CP) \qquad \text{(VII.60)}$$

and $$T_P \cos \psi = wc \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VII.61)}$$

Dividing (VII.60) by (VII.61), we have $\tan \psi = \frac{s}{c}$,

i.e. $$s = c \tan \psi, \text{ (the intrinsic equation of the curve)} \qquad \text{(VII.62)}$$

Let P be the point (x, y) on the curve referred to rectangular axes OX, OY, where OY is the vertical through C, and O is distant c below C. The quantity c is called the *parameter* of the catenary. If Δs be an element of arc at the point (x, y) on the curve, then $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$ (Art. 111), and in the limit

$$\frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

From (VII.62), since $\tan \psi = \frac{dy}{dx}$, we have $s = c\frac{dy}{dx}$, and hence,

$$\frac{dx}{dy} = \frac{c}{s}$$

$$\therefore \qquad \frac{ds}{dy} = \sqrt{\frac{c^2}{s^2} + 1}, \text{ or } \frac{s\,ds}{\sqrt{c^2 + s^2}} = dy$$

Integrating, $\sqrt{c^2 + s^2} = y +$ constant.

Now, when $y = c$, $s = 0$ by our choice of axes, so that the constant $= 0$. Hence,

$$y = \sqrt{c^2 + s^2}, \text{ or } y^2 = c^2 + s^2 \quad . \quad . \quad \text{(VII.63)}$$

Since $$s = c\frac{dy}{dx}, \text{ we have from (VII.63)}$$

$$y^2 = c^2\left[1 + \left(\frac{dy}{dx}\right)^2\right], \text{ and therefore } \frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\therefore \qquad \frac{c\,dy}{\sqrt{y^2 - c^2}} = dx$$

Integrating, $c \cosh^{-1}\frac{y}{c} = x +$ constant.

Now, when $x = 0$, $y = c$, and the constant $= c \cosh^{-1} 1 = 0$.

Hence, $\cosh^{-1}\frac{y}{c} = \frac{x}{c}$, or

$$y = c \cosh\frac{x}{c} \qquad . \quad . \quad . \quad . \quad \text{(VII.64)}$$

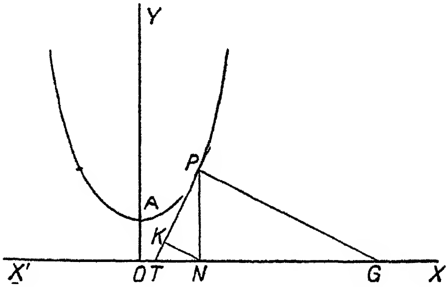


FIG. 67

This is the Cartesian equation of the curve.

Differentiating (VII.64), we have $\frac{dy}{dx} = \sinh\frac{x}{c}$; but

$$\frac{dy}{dx} = \frac{s}{c}, \text{ so that } s = c \sinh\frac{x}{c} \qquad . \quad . \quad \text{(VII.65)}$$

If A and B are at the same level and the span $AB = 2a$, then by (VII.65) the total length of chain $= 2c \sinh\frac{a}{c}$. From (VII.63) we have $y^2 = c^2 + s^2 = c^2 + c^2 \tan^2 \psi = c^2 \sec^2 \psi$, or

$$y = c \sec \psi \qquad . \quad . \quad . \quad \text{(VII.66)}$$

The relation (VII.61) gives $T_P = wc \sec \psi = wy$

$$\left.\begin{array}{l}\text{Hence, \textit{the tension at any point in a uniform catenary}}\\ \text{\textit{is equal to wy, where w = weight per unit length and y is}}\\ \text{\textit{the ordinate of the point.}}\end{array}\right\} \qquad \text{(VII.67)}$$

From N, the foot of the ordinate of P, draw NK perpendicular to the tangent at P. Then $PN = y$ and $P\hat{N}K = \psi$, so that $NK = \frac{y}{\sec \psi}$

$= c$, and $PK = NK \tan\psi = c \tan\psi = s$. The relations connecting y, s, c can all be derived from the right-angled triangle PNK (Fig. 67).

We have proved in Art. 80, Example 1, that the radius of curvature and the normal at any point of the uniform catenary are both equal

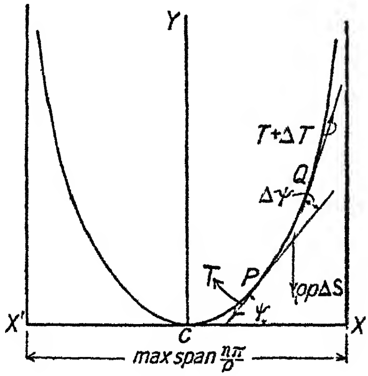


FIG 68

to $\frac{y^2}{c}$. Thus, if the normal PG be drawn (Fig. 67), $NPG = \psi$ and PG $= y \sec\psi = y\left(\frac{y}{c}\right) = \frac{y^2}{c}$.

CATENARY OF UNIFORM STRENGTH (FIG. 68). When the cross-section at any point of the chain is proportional to the tension at that point, the form which the curve assumes is called the *catenary of uniform strength.*

Let the axes of x and y be taken through C, the lowest point, as shown. Consider the equilibrium of a portion PQ of the chain of length Δs, P being any point (x, y) and Q the point $(x + \Delta x, y + \Delta y)$. The weight of $PQ = \rho p\Delta s$, where p = area of cross-section at P and ρ = weight of unit volume of the chain. If T be the tension at P and $T + \Delta T$ the tension at Q, we have, by resolving along the normal at P,

$$(T + \Delta T)\sin\Delta\psi = \rho p\,\Delta s\cos\psi = \rho p\,\Delta x$$

(since $\Delta x = \Delta s\cos\psi$).

Since $\Delta\psi$ is small, we can replace $\sin\Delta\psi$ by $\Delta\psi$, and then we have $T\Delta\psi = \rho p\,\Delta x - \Delta T\,\Delta\psi$.

In the limit $\frac{d\psi}{dx} = \frac{\rho p}{T} = \frac{\rho p}{np} = \frac{\rho}{n}$ (for by hypothesis $T = np$ where n is some constant).

Integrating, $\psi = \frac{\rho}{n}x$ (the constant of integration being zero, since $\psi = 0$ when $x = 0$)

Hence, $$\frac{dy}{dx} = \tan\psi = \tan\left(\frac{\rho}{n}x\right)$$

Integrating we obtain the Cartesian equation of the curve, namely,

$$y = \frac{n}{\rho}\log_e \sec\left(\frac{\rho}{n}x\right) \qquad \text{(VII 68)}$$

the constant of integration being zero

When $\psi = \pm\frac{\pi}{2}$, i e when $\frac{\rho}{n}x = \pm\frac{\pi}{2}$, is infinite The limits for x are then $\pm\frac{n\pi}{2\rho}$, and the maximum span is therefore $\frac{n\pi}{\rho}$

The case of a parabolic catenary is treated in Art 127, Example 4. We shall, however, show here that in the vicinity of the lowest part C (Fig 66) the shape of the common catenary is approximately that of a parabola Since, when x is small compared with c

$$e^{\frac{x}{c}} = 1 + \frac{x}{c} + \frac{x^2}{2c^2}$$

and $$e^{-\frac{x}{c}} = 1 - \frac{x}{c} + \frac{x^2}{2c^2}$$

then $$\cosh\frac{x}{c} = 1 + \frac{x^2}{2c^2} \text{ approximately}$$

Hence (VII 64) becomes

$$y = c + \frac{x^2}{2c} \qquad \text{(VII 69)}$$

which is the equation to a parabola with axis vertical and the lowest part of the catenary as vertex In dealing with any catenary in which the sag is small compared with the span, we may, therefore, assume that the chain hangs in the form of a parabola

EXAMPLE 1

In the catenary, prove that the tension T at any point P, the tension T_0 at the lowest point, and the weight W of the chain from the lowest point up to P, are connected by $T^2 = T_0^2 + W^2$

If the total length of the chain be 100 ft, the total weight 40 lb, and the sag 10 ft show that the greatest tension is 52 lb, and that the distance apart of the supports is X where $\cosh\frac{X}{240} = \frac{13}{12}$ (U L)

The catenary here means the uniform catenary. The relation T ... w which is true for any catenary, is proved in Art. 95 [see (VII 55)].

Let the co-ordinates of a point of support be (x, y); then y ... 10 ... c, and the relation y^2 ... c ... s ... (VII 63) gives $(10$ ... $c)$ ... c ... 50 ... i.e. 100 ... 20 ... 2 500, whence c ... 120 ft ... T ... w ... $\frac{40}{100}$ ... 120 ... 48 lb. The relation $T^2 - T_0^2$ ... W^2 now gives

$$T^2 \quad 48 \quad 20 \quad 2\,304 \quad 100 \quad 2\,704$$

$$T \quad 52 \text{ lb}$$

Otherwise thus $\quad T \quad wy \quad \frac{40}{100} \quad (120 \quad 10) \quad 52 \text{ lb}$

The equation of the catenary is y ... $c \cosh \frac{x}{c}$, so that at a point of support

$$y \quad c \cosh \frac{x}{c}$$

i.e. $$130 \quad 120 \cosh \frac{x}{120}$$

Now X ... distance apart of the supports ... $2x$

Hence, $$\frac{13}{12} \quad \cosh \frac{X}{240}$$

EXAMPLE 2

A uniform chain is suspended from two points A and B, A being h ft higher than B. The inclinations of the chain to the horizontal at A and B are ψ_1 and ψ_2. Show that the total length of the chain is $h \cos \frac{\psi_1 + \psi_2}{2} \operatorname{cosec} \frac{\psi_1 \quad \psi_2}{2}$ ft

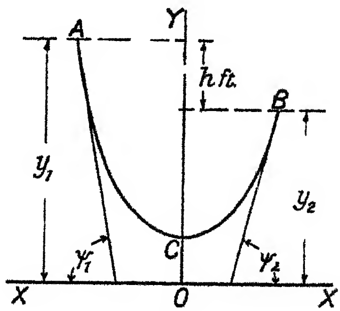


FIG 69

Referring to Fig 69, let s_1 ... arc CA and s_2 ... arc CB; also let the ordinates of A and B be y_1 and y_2.

Now by (VII 62) $\quad s_1 - c \tan \psi_1$ and s ... $c \tan \psi$

and by (VII 66) $\quad y_1 \quad c \sec \psi_1$ and y ... $c \sec \psi_2$

Hence, $\quad h \quad y_1 - y_2 \quad c(\sec \psi_1 - \sec \psi_2) \quad c \frac{\cos \psi_2 \quad \cos \psi_1}{\cos \psi_2 \quad \cos \psi_1} \qquad (1)$

If $\quad l =$ total length of chain, then

$$l = s_1 + s_2 = c(\tan\psi_1 + \tan\psi_2) = c\,\frac{\sin(\psi_1 + \psi_2)}{\cos\psi_1 \,.\, \cos\psi_2} \quad . \quad . \quad . \quad (2)$$

Dividing (2) by (1), we have

$$\frac{l}{h} = \frac{\sin(\psi_1 + \psi_2)}{\cos\psi_2 - \cos\psi_1} = \frac{2\sin\frac{\psi_1 + \psi_2}{2} \,.\, \cos\frac{\psi_1 + \psi_2}{2}}{2\sin\frac{\psi_1 + \psi_2}{2} \,.\, \sin\frac{\psi_1 - \psi_2}{2}} = \frac{\cos\frac{\psi_1 + \psi_2}{2}}{\sin\frac{\psi_1 - \psi_2}{2}}$$

Hence, $$l = h\cos\frac{\psi_1 + \psi_2}{2}\operatorname{cosec}\frac{\psi_1 - \psi_2}{2} \text{ ft.}$$

EXAMPLE 3

A uniform chain of length l is stretched nearly horizontal between two points in the same horizontal plane. Prove that the excess of the length over the span is approximately $\dfrac{w^2l^3}{24T^2}$, where w = weight per unit length of chain and T = tension.

A telegraph wire weighing 0·12 lb per foot length is stretched tightly between two points 200 ft apart. Find the sag in the middle in order that the maximum tension may be 180 lb.

Let the equation of the curve in which the chain hangs be $y = c\cosh\dfrac{x}{c}$ and let the span be d. If ψ be the slope at a point of support, then $\dfrac{l}{2} = c\tan\psi$; but ψ, and therefore $\tan\psi$, are small, so that c must be large.

We have $\quad \dfrac{l}{2} = c\sinh\dfrac{d}{2c}$ from (VII.65);

or
$$l = 2c\left(\frac{e^{\frac{d}{2c}} - e^{-\frac{d}{2c}}}{2}\right)$$
$$= c\left[\left\{1 + \frac{d}{2c} + \frac{1}{\lfloor 2}\left(\frac{d}{2c}\right)^2 + \frac{1}{\lfloor 3}\left(\frac{d}{2c}\right)^3 + \ldots\right\}\right.$$
$$\left. - \left\{1 - \frac{d}{2c} + \frac{1}{\lfloor 2}\left(\frac{d}{2c}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{d}{2c}\right)^3 + \ldots\right\}\right]$$

whence $$l = c\left[\frac{d}{c} + \frac{1}{3}\left(\frac{d}{2c}\right)^3\right] \text{ approx.}$$

$$\therefore \quad l - d = \frac{d^3}{24c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the chain is nearly horizontal, the tension is very nearly the same throughout. We can take then $T = wc$, and since l and d are nearly equal, we shall introduce only a small error in a small term by writing l^3 for d^3 on the right-hand side of (1).

$$\therefore \quad \text{Excess of length over span} = l - d = \frac{l^3}{24\left(\frac{T}{w}\right)^2} = \frac{w^2l^3}{24T^2} \text{ approx.}$$

e the sag in the middle of the telegraph wire (Fig. 56). Assuming the onstant throughout and taking moments about B for the equilibrium of on OB, we have

$$Td = \frac{wl}{2} \times 50 \text{ very nearly}$$

$$d = \frac{25wl}{T} = \frac{25 \times 0.12 \times 200}{180}$$

have taken l equal to the span, 200 ft, the error introduced in d being ll.

$$d = \frac{10}{3} = 3\tfrac{1}{3} \text{ ft}$$

3 4

hat in the catenary of uniform strength (1) the vertical projection of the curvature at any point is constant, (2) $s = \frac{n}{\rho}\log\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$, (3) the t any point is proportional to the radius of curvature at the point.

Art. 95, we proved that $\psi = \frac{\rho}{n}x$. If R is the radius of curvature, then

$$R = \frac{ds}{d\psi} = 1\Big/\frac{d\psi}{ds} = 1\Big/\left(\frac{\rho}{n}\cdot\frac{dx}{ds}\right) = \frac{n}{\rho\cos\psi}$$

$$R\cos\psi = \frac{n}{\rho} = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

e radius of curvature is inclined to the vertical at an angle ψ, so that = vertical projection of radius of curvature, which is therefore constant

ce $\frac{ds}{d\psi} = \frac{n}{\rho\cos\psi} = \frac{n}{\rho}\sec\psi$, then, integrating between the limits 0 to ψ) we have

$$s = \frac{n}{\rho}\int_0^{\psi}\sec\psi\,d\psi = \frac{n}{\rho}\log_e\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right) \text{ (see Art. 48)}$$

horizontal component of the tension is constant,

$$T\cos\psi = K, \text{ where } K \text{ is constant}$$

rom above, $\cos\psi = \frac{n}{\rho R}$

$$\frac{Tn}{\rho R} = K \text{ or } T = \frac{\rho K}{n}\cdot R$$

e tension varies as the radius of curvature.

3 5

rm chain 100 ft long, weighing 150 lb, is suspended between two points the vertical lines through A and B being 75 ft apart. A is 25 ft higher Find c if the equation to the catenary in which the chain hangs is

$$y = c\cosh\frac{x}{c}$$

Let the co-ordinates of A referred to the principal axes OX and OY (as in Fig. 69) be $x = -d$, $y = y_1$.

Then those of B are $x = 75 - d$, $y = y_1 - 25$.

By (VII.64) $$y_1 = c \cosh \frac{-d}{c}$$

i.e. $$y_1 = c \cosh \frac{d}{c} \qquad . \qquad . \qquad . \qquad (1)$$

and $$y_1 - 25 = c \cosh \frac{75 - d}{c} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (2)$$

Subtracting (2) from (1)

$$25 = c\left(\cosh \frac{d}{c} - \cosh \frac{75 - d}{c}\right)$$

$$\therefore \qquad \frac{25}{2c} = \sinh \frac{75}{2c} \sinh \frac{2d - 75}{2c} \qquad . \qquad . \qquad . \qquad (3)$$

$$\left[\text{since } \cosh \theta - \cosh \phi = 2 \sinh \frac{\theta + \phi}{2} \sinh \frac{\theta - \phi}{2}\right]$$

Again, the length of CB is by (VII.65)

$$S_1 = c \sinh \frac{75 - d}{c}$$

and that of CA is

$$S_2 = c \sinh \left(-\frac{d}{c}\right) = - c \sinh \frac{d}{c}$$

This is given as a negative quantity by the formula, and so the length of ACB is $S_1 - S_2$. Hence,

$$100 = c\left(\sinh \frac{75 - d}{c} + \sinh \frac{d}{c}\right)$$

$$\therefore \qquad \frac{50}{c} = \sinh \frac{75}{2c} \cosh \frac{2d - 75}{2c} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (4)$$

$$\left[\text{since } \sinh \theta + \sinh \phi = 2 \sinh \frac{\theta + \phi}{2} \cosh \frac{\theta - \phi}{2}\right]$$

Squaring (3) and (4) and subtracting the first result from the second

$$\left(\frac{50}{c}\right)^2 - \left(\frac{25}{2c}\right)^2 = \sinh^2 \frac{75}{2c}$$

Or, $$\sinh \frac{75}{2c} = \frac{25}{2c}\sqrt{15} = \frac{96{\cdot}83}{2c} = \frac{48{\cdot}42}{c}$$

i.e. $$\sinh u = u \times \frac{2 \times 48{\cdot}42}{75}, \text{ where } u = \frac{75}{2c}$$

Or, $$\sinh u = 1{\cdot}291u$$

Solving by the method of Art. 61, or by Newton's method, we find $u = 1{\cdot}27$ from which $c = 29{\cdot}5$.

EXAMPLES VII

(1) If $u = 0$ and $v = 0$ are the equations of two straight lines, what locus is represented by the equation $u - \lambda v = 0$? The equations of the sides of a triangle are $11x + 3y + 23 = 0$, $7x - 9y - 64 = 0$, $3x + 5y - 23 = 0$. Find the equations of the lines through the vertices perpendicular to the opposite sides, and show that the lines meet in the point $(3, -2)$. (U.L.)

(2) Find the co-ordinates of the point which divides the line joining the points (x_1, y_1), (x_2, y_2), internally in the ratio $l : m$. Find the ratio of the segments into which the lines joining $(1, 3)$ to $(5, -3)$ and $(4, 5)$ to $(-1, -4)$ are divided by their point of intersection. (U.L.)

(3) For what value of k does the equation $6x^2 - 42xy + 60y^2 - 11x + 10y + k = 0$ represent two straight lines? Prove that the lines are inclined to each other at an angle $\tan^{-1} (\frac{3}{11})$.

(4) Find the equation of the circle which passes through the vertices of the triangle formed by the lines $x + y - 5 = 0$, $x - y + 1 = 0$, $y + 1 = 0$. Find also the co-ordinates of the centre of the circle.

(5) The circle $x^2 + y^2 - 6x - 4y - 12 = 0$ is cut by the line $3y + 4x - 18 = 0$. Find the co-ordinates of the points of intersection and the equations of the tangents to the circle at these points.

(6) Show that the polar equation of a circle of radius a referred to a diameter as initial line and an end of that diameter as pole is $r = 2a \cos \theta$.

For what value of k is the straight line $r = k \sec (\theta - \alpha)$ a tangent to the circle $r = 2a \cos \theta$?

(7) Prove that the foot of the perpendicular from the focus of a parabola on any tangent to the curve lies on the tangent at the vertex.

Given the focus and directrix of a parabola, construct geometrically the two tangents to the curve from any point, and also their points of contact. (U.L.)

(8) Prove that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix. Prove that the length of the chord which is the normal at an extremity of the latus-rectum is $2\sqrt{2}l$ where l is the length of the latus-rectum. (U.L.)

(9) A point moves in a plane in such a way that its co-ordinates at time t are $x = a \sin kt$, $y = a \cos 2kt$. Show that it describes a parabola.

(10) Write down the equation of the tangent at any point (x', y') on the parabola $y^2 = 4ax$.

The tangents at the points $P(x_1, y_2)$ and $Q(x_2, y_2)$ on the parabola intersect in the point T. The vertex A is joined to the points P, Q, T, and the joins meet the directrix in P', Q', T', respectively. Prove that $P'T' = T'Q'$.

(11) The points (x_1, y_1), (x_2, y_2), (x_3, y_3) are on the parabola $y^2 = 4ax$. If $x_1x_3 = x_2^2$, show that the point of intersection of the tangents at (x_1, y_1) and (x_3, y_3) lies on the line through (x_2, y_2) perpendicular to the axis of the parabola.

(12) A particle is projected with velocity V at an angle α to the horizontal. Find (1) the greatest height attained, (2) the range on the horizontal plane through the point of projection, (3) the time of flight.

(13) Two particles are projected simultaneously from the same point, and in the same vertical plane, with equal velocities V, the angles of elevation being θ_1 and θ_2 respectively. Taking the origin at the point of projection and the axes of x and y horizontal and vertical respectively, find the co-ordinates of each particle at the end of time t, and show that the line joining the particles at any instant is inclined at a constant angle to the horizontal.

(14) Prove that the focal distances of the point (x', y') on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are $a \pm ex'$.

Prove that the product of the focal distances is equal to the square on the semi-diameter of the ellipse which is parallel to the tangent at (x', y'). (U L.)

(15) Prove that an ellipse can be projected orthogonally into a circle. Hence, or otherwise, prove that if CP, CD are conjugate diameters, the sum of the squares of the abscissae of P and D is equal to the square on the semi-axis major. (U.L.)

(16) Prove that the locus of a point which is equidistant from two given circles, one of which lies entirely within the other, is an ellipse. Find the eccentricity of the ellipse in terms of the radii of the circles and the distance between their centres. (U.L.)

(17) Find the equation to the ellipse which has the point (1, 2) as focus and the line $2x - 3y + 6 = 0$ as the corresponding directrix, and which is of eccentricity $\frac{2}{3}$.

Determine the value or values of m, so that $y = mx + 2$ may be a tangent, and find the point or points of contact. (U.L.)

(18) Prove that the tangent at any point on an ellipse cuts the major axis at the same point as the tangent to the auxiliary circle at the corresponding point. Show also that the locus of the foot of the perpendicular from a focus to a tangent is the auxiliary circle. (U.L.)

(19) The focal distances SP, HP of a point P on an ellipse make angles α and β respectively with the major axis. Prove that $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{1 - e}{1 + e}$, where e is the eccentricity of the ellipse.

(20) P and Q are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, their eccentric angles being ϕ_1 and ϕ_2 respectively. If the join PQ cuts the major axis at the point $(x', 0)$, show that x' is given by the relation

$$(x' + a) \tan \frac{\phi_1}{2} = (x' - a) \cot \frac{\phi_2}{2}$$

(21) OA and OB are two lines at right angles; the circles with O as centre and these lines as radii are drawn, and any line through O meets the circles in P and Q; lines are drawn through P and Q parallel to OA and OB, meeting in R and S. Prove that R and S move on two ellipses. (U.L.)

(22) Find the condition that the straight line $px + qy = 1$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(23) Find the eccentricity of a hyperbola the angle between whose asymptotes is 30°.

(24) If P is any point on a hyperbola and N is the foot of the ordinate of P, prove that $\frac{PN^2}{AN \cdot A'N} = \frac{b^2}{a^2}$

(25) P is any point on a hyperbola; the tangent at P cuts the asymptotes at Q and R respectively. Prove that $QP = PR$ and that the area of the triangle CQR is constant, C being the centre of the hyperbola.

(26) Show that the co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be expressed in terms of a single parameter θ as $x = a \sec \theta$, $y = b \tan \theta$,

Find the equations of the tangents at the points θ_1 and θ_2 on the hyperbola, and prove that the locus of their point of intersection is the curve

$$a^2y^2 = b^2\left[x^2 \sec^2 \frac{\theta_1 - \theta_2}{2} - a^2\right]$$

(27) The point whose co-ordinates are $x = 3t$, $y = \frac{4}{t}$, lies on the rectangular hyperbola $xy = 12$. Find the equations of the tangent and normal to the curve at a point whose parameter is t_1. Show that the normal cuts the curve again in the point whose parameter t_2 is given by $t_2 = -\frac{16}{9t_1^3}$.

(28) S is the centre of a circle of radius a and PQ is a given straight line outside the circle. From M, the foot of the perpendicular from S to PQ, two tangents are drawn to the circle, their points of contact being A and A'. Show that PQ is a tangent to the parabola having S as focus, A (or A') as vertex, and latus-rectum equal to $4a$.

(29) What does the relation (VII.1) become in the case of a parabola? Use this relation to prove that, if d_1 and d_2 are the lengths of two perpendicular focal chords of a parabola, then $\frac{1}{d_1} + \frac{1}{d_2}$ is constant.

(30) A straight rod PQ, 7 in. long, moves with its ends P and Q on two straight perpendicular grooves OX and OY. R is a point on the rod distant 3 in. from P. Find the eccentricity of the ellipse which R traces out. Find also the position of a point in PQ which traces out an ellipse of eccentricity [illegible].

(31) AB is a diameter of a circle of radius 5 in. All ordinates of the circle perpendicular to AB are shortened in the ratio 7 : 10. Find (i) the eccentricity, (ii) the length of the latus-rectum, of the ellipse thus obtained.

(32) A straight line cuts the rectangular axes OX, OY at P and Q respectively and always passes through a fixed point R (a, b). Prove that the locus of the mid-point of PQ is a rectangular hyperbola whose asymptotes are parallel to OX, OY.

(33) S is a fixed point distant 1 in. from a fixed straight line AB, and P is a point $\frac{1}{2}$ in. from AB and $3\frac{1}{2}$ in. from S. The perpendicular SK to AB is produced its own length to Q. With centres Q and P and radii 5 in. and $1\frac{1}{2}$ in. respectively circular arcs are drawn cutting at H. C is the mid-point of the join SH. Show that C is the centre of an ellipse of major axis 5 in. which has S and H as foci, passes through P, and touches the line AB.

(34) Show that the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ have the same foci. Find the co-ordinates of the points of intersection of the two curves. Write down the equations of the asymptotes of the hyperbola. Prove that if the ordinate of one of the above points of intersection is produced to cut an asymptote at P, then P lies on the auxiliary circle of the ellipse.

(35) Prove that $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C$

Hence, or otherwise, obtain the equation of the curve assumed by a heavy uniform chain hanging in equilibrium with its ends attached to two fixed points, in the form $y = c \cosh \frac{x}{c}$

A telephone wire weighing 100 lb to the mile length has a span of 80 yd and a sag of 12 in. Find approximately the tension at the supports. (U.L.)

(36) Obtain the equations of equilibrium of a perfectly uniform flexible chain hanging freely under gravity in the form $T\cos\phi = wc$, $T\sin\phi = ws$, where T denotes the tension, ϕ the inclination of the tangent to the horizontal, w the weight per unit length of the chain, c a constant, and s the length of the chain measured from the lowest point. Deduce the formulae

$$y = c\cosh\frac{x}{c},\ s = c\sinh\frac{x}{c} \qquad \text{(U.L.)}$$

(37) Write down the equations of equilibrium of an element of a flexible cord hanging between two points under no force but its own weight, and deduce that (1) $\dfrac{dT}{dy} = w$, (2) $T = w\dfrac{dx}{d\phi}$, where T denotes the tension, w the weight per unit length, and ϕ the angle of slope at the point x, y.

Find the relation between y and x for the catenary of uniform strength, in which $T = cw$, where c is a constant. (U.L.)

(38) A uniform chain of length l is suspended from two points at the same level. If the greatest tension which the chain can bear is wk, where w is the weight per unit length of chain, prove that the span cannot exceed

$$\frac{\sqrt{4k^2 - l^2}}{2}\log_e\frac{2k + l}{2k - l}$$

(39) A telegraph wire has a span of 88 ft. Find the excess of the length of the wire over the span if the sag in the middle is 12 in.

(40) If two uniform chains of lengths l_1 and l_2 and weights W_1 and W_2 respectively are joined together so as to form a single chain, and the whole is suspended freely from two fixed points, show that the radii of curvature ρ_1 and ρ_2 of the two portions at their junction are such that $\dfrac{\rho_1}{\rho_2} = \dfrac{W_1 l_2}{W_2 l_1}$

(41) A chain hangs in the form of a parabola between two points on the same level l ft apart. The sag in the middle is s ft. If s is small compared with l, find an approximate expression for the total length of the chain. If the coefficient of linear expansion for the chain is α and the temperature increases by ΔT, find the increase in the sag due to this increase, and, if the tension at the middle is H lb, find the change in the tension.

(42) A uniform chain, 60 ft long, weighing 100 lb, is suspended between two points at the same level 30 ft apart. Find the greatest and least tensions in the chain.

(43) If one support in the previous example is 6 ft higher than the other, but they are still 30 ft apart (horizontally), find the greatest and least tensions.

(44) Mohr's rule for finding the central deflection of a freely supported beam states that the deflection is equal to the sag at the mid-point of a chain of the same span, carrying a load represented by the bending moment diagram on the beam, and having a horizontal tension EI (E and I are defined in Ex. 3, Art. 80). Prove this rule, and apply it to find the deflection of a freely supported beam of span l ft carrying W lb at its centre.

(45) A uniform chain is hung from two points A and B on the same horizontal line and at a distance $2a$ apart. The inclination of the chain to the horizontal at

A or B is α. If $2s$ is the total length of the chain, w its weight per unit length, T_0 the tension at the lowest point, and d the sag, show that

$$\frac{\tan\alpha}{s} = \frac{w}{T_0} = \frac{\sec\alpha - 1}{d} = \frac{\log_e(\tan\alpha + \sec\alpha)}{a}$$

(U.L.)

(46) Show that in the common catenary, $y^2 = s^2 + c^2$, and $T = wy$, where y is the height of any point above the directrix, c the parameter, s the arc, T the tension, and w the weight per unit length.

A kite is flown at the end of 100 ft of string. The tension at the hand is equal to the weight of 40 ft of string, and is inclined at 30° to the horizontal. Show that the kite is about 85 ft above the hand. (U.L.)

(47) A uniform chain of length $2l$ runs over a small light pulley with a smooth horizontal axis. Let $2x$ denote the difference in length of the two hanging portions at any instant, and suppose that $x = a$ and $\dot{x} = u$ when $t = 0$. Find x at time t.

Show briefly that the inertia of the pulley could be taken into account merely by substituting ng for g in your results, n being a proper fraction. (U.L.)

CHAPTER VIII

GRAPHS—LAWS OF GRAPHS

96. **Graphs.** In Chapter IV we used graphs in order to obtain solutions of equations, and we showed how to tabulate values of functions in such a way as to facilitate the numerical work involved. We are here concerned with the use of graphs as an aid to the study of functions. A statement such as $y = f(x)$, where $f(x)$ represents any particular function of x is just a symbolic relation. By substituting various values of x we obtain corresponding values of y and, of course, this gives us an idea of how changes in the value of y are produced by those in the value of x. By finding values of $\frac{dy}{dx}$ and of $\frac{d^2y}{dx^2}$ we can still further increase our knowledge of the way in which y varies, but it is difficult in most cases to form by these methods a complete idea of the relationship between the two variables. A graph, however, gives us at once a picture of the relationship and enables us to think clearly and comprehensively of the values of the function for all values of x, as well as of the ways in which the value of the function varies as x changes. It is essential that students of applied science be familiar with the graphs of important functions of common occurrence, such as x^n, e^{kx}, e^{-kx}, $\log x$, $\sin(nx + b)$, etc. This is the more essential as in some cases the graph is actually a scale drawing of the centre line of some object whose shape is required to be known. Thus, a long thin column with free ends bends under a critical load so that its centre line forms part of the graph of $y = A \sin mx$, whilst a beam under certain conditions deflects so that its centre line forms part of a curve whose equation is $y = ax^3 + bx^2 + cx + d$. We often need to draw a graph before we can understand the fullest implications of an analytical result.

97. **Graph of $y = x^n$.** Fig. 70 shows the graphs of $y = x^n$ for the values $n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}, -1, -2, -\frac{1}{2}$. We have shown only those parts of the graphs for which x is positive. The graphs may be completed by making use of considerations of symmetry.

The gradient of the graph of $y = x^n$ is given by $\frac{dy}{dx} = nx^{n-1}$. Since $y = 1$ when $x = 1$, for all values of n, all the graphs pass

through the point P whose co-ordinates are $x = 1$, $y = 1$. We will first discuss the shape of the graph when n is positive.

When $x = 0$, $y = 0$ and the graphs pass through O.

If $n > 1$, $n - 1$ is positive, and $\frac{dy}{dx} = 0$ when $x = 0$. Also $\frac{dy}{dx}$ increases as x increases and $\underset{x \to \infty}{Lt.} \frac{dy}{dx} = \infty$. Thus the graph touches OX at the origin, and the gradient increases as x increases, being large for large values of x. The graph is concave upwards for all

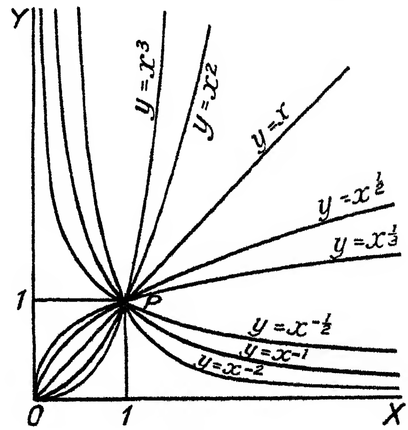


FIG. 70

positive values of x. These are the characteristics of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, etc. (Fig. 70).

If $1 > n > 0$, $n - 1$ is negative and the gradient is $\frac{n}{x^{1-n}}$, which approaches the limit ∞ as x decreases towards the value $x = 0$. The value of $\frac{dy}{dx}$ decreases continually as x increases, and $\underset{x \to \infty}{Lt.} \frac{dy}{dx} = \underset{x \to \infty}{Lt.} \frac{n}{x^{1-n}} = 0$. The graph of $y = x^n$ $(1 > n > 0)$ therefore touches the axis of y at the origin, and its gradient decreases continually as x increases. For large values of x the gradient is small. These are seen from Fig. 70 to be characteristics of the graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$, $y = x^{\frac{1}{4}}$, etc.

If $n < 0$, the gradient $\frac{dy}{dx} = \frac{n}{x^{-n+1}}$ is negative for all positive values of x and decreases numerically as x increases. If $x = 0$, $\frac{dy}{dx} = \infty$,

and $\underset{x\to\infty}{Lt.}\,\frac{dy}{dx} = \underset{x\to\infty}{Lt.}\,\frac{n}{x^{1-n}} = 0$. Also $\underset{x\to 0}{Lt.}\,y = \underset{x\to 0}{Lt.}\,\frac{1}{x^{-n}} = \infty$. Hence, the graphs have the axes as asymptotes. Since $\frac{dy}{dx}$ is negative and decreases numerically as x increases, the graphs are all concave upwards for positive values of x.

For values of x between $x = 0$ and $x = 1$, the value of x^{n+1} is less than that of x^n. Hence, if a point be supposed to move vertically upwards from a point on the x-axis between $x = 0$ and $x = 1$, it

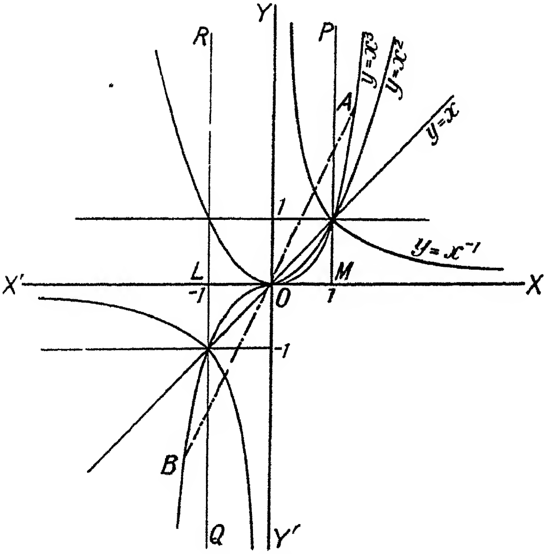


FIG. 71

will meet the members of the family of curves representing $y = x^n$ (for all values of n) in the order of n decreasing. If the moving point starts from a point on OX to the right of P it will meet the graphs in the reverse order.

EXTENSION OF THE GRAPHS. If n is even $(-x)^n = (+x)^n$ and the graph of $y = x^n$ is symmetrical about OY and lies entirely above OX. If n is odd $(-x)^n = -(+x)^n$ and the graph of $y = x^n$ is symmetrical about the origin O; this means that a line drawn through O to cut the graph will cut it in a pair of points A and B on opposite sides of the origin O, and such that $BO = OA$ (Fig. 71). The extended graphs of $y = x$, $y = x^2$, $y = x^3$, and $y = x^{-1}$ are shown in Fig. 71.

PMLR and *PMLQ* are the limiting positions of the graphs of $y = x^{2n}$ and $y = x^{2n+1}$ respectively as n approaches infinity, n being a positive integer.

The graph of $y = x^p$, where p is not an integer, lies between the graphs of $y = x^n$ and $y = x^{n+1}$, where n and $n + 1$ are the two consecutive integers between which p lies.

The graphs in Figs. 70 and 71 can be converted into the graphs of $y = ax^n$, where a is positive, by changing the scale on which y is plotted, so that a length which originally measured unity is made to measure a units.

98. **The Graphs of e^{kx} and e^{-kx}. Growth and Decay Functions.** The graphs of e^x and e^{-x} are drawn in Fig. 72.

GRAPH OF e^x. We have seen that if $y = e^x$, $\dfrac{dy}{dx} = e^x$ also. For positive values of x, e^x is obviously positive. For negative values of x we have $e^x = \dfrac{1}{e^{-x}} = \dfrac{1}{e^n}$, where n is positive, so that e^n, and therefore e^x, is positive. Thus, e^x is positive for all values of x.

The gradient, being e^x, is also positive for all values of x, and therefore both the ordinate and the gradient are positive everywhere, and increase as x increases.

$\underset{x \to -\infty}{Lt.}\ e^x = 0$, so that both the ordinate and the gradient are very small for large negative values of x.

$e^0 = 1$, hence both the ordinate and the gradient are equal to 1 when $x = 0$.

Thus, starting with large negative values of x the graph of $y = e^x$ in Fig. 72 first runs along just above the axis of x. The height and the gradient gradually increase with x, both passing through the value 1 when $x = 0$. After the curve has crossed the y-axis the height and gradient continue to increase with x until they are both very large for large positive values of x.

If we give to x the series of values a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, etc., where a is any number and d is a positive number, the ordinates are respectively e^a, $e^{a+d} = e^d e^a$, $e^{a+2d} = (e^d)^2 e^a$, $e^{a+3d} = (e^d)^3 e^a$, etc. These numbers form a geometrical progression of which the first term is e^a and the common ratio is e^d, $e^d > 1$. Thus, the graph is such that starting from any point on it a series of equidistant ordinates extending to the right have lengths which increase in the same way as the terms of an increasing geometrical progression.

GRAPH OF e^{-x}. This graph is symmetrical about OY with the graph of e^x for $e^{-(a)} = e^{+(-a)}$. The graph is shown in Fig. 72 and needs no further description.

The graphs of $y = e^{kx}$ and $y = e^{-kx}$ can be obtained from those of $y = e^x$ and $y = e^{-x}$ respectively by changing the scale for values of x so that what was previously unit length, now measures $\frac{1}{k}$ new units of length. Thus the graph of e^{2x} would be obtained from that of e^x by changing all the numbers representing values of x to one-half

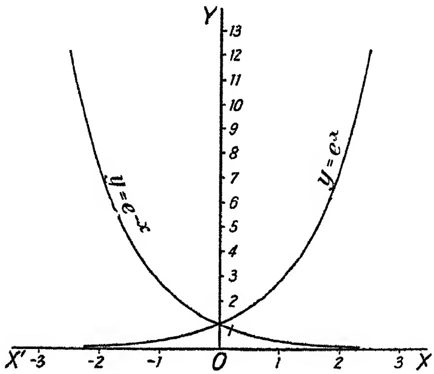


FIG. 72

of their original values. By writing $k = \log_e a$ we convert the graph of $y = e^{kx}$ into that of $y = e^{x \log_e a} = a^x$.

The function $y = Ae^{kx}$, where k is a positive number, is known as the *growth* function. Since $\frac{dy}{dx} = kAe^{kx} = ky$, the rate of growth of the function is always k times the value of the function. A sum of money invested at compound interest increases in a somewhat similar manner to this. If r is the rate per cent, the sum S pounds increases by $\frac{r}{100} \times S$ pounds every year if the interest is paid yearly, or by $\frac{r}{400} \times S$ every quarter year if the interest is paid quarterly; in each case the rate of growth is proportional to S. To pursue this matter further we will assume that interest is paid n times per year. Let A be the amount originally invested in pounds.

After the first payment of interest $S = A\left(1 + \frac{0{\cdot}01r}{n}\right)$

,, second ,, ,, ,, $S = A\left(1 + \frac{0{\cdot}01r}{n}\right)\left(1 + \frac{0{\cdot}01r}{n}\right)$

$= A\left(1 + \frac{0{\cdot}01r}{n}\right)^2$

,, third ,, ,, ,, $S = A\left(1 + \frac{0{\cdot}01r}{n}\right)^3$

,, fourth ,, ,, ,, $S = A\left(1 + \frac{0{\cdot}01r}{n}\right)^4$

,, nth ,, ,, ,, $S = A\left(1 + \frac{0{\cdot}01r}{n}\right)^n$ (VIII.1)

If the money remains invested for N years there will be Nn payments of interest, and the amount will then be

$$S = A\left(1 + \frac{0{\cdot}01r}{n}\right)^{nN} \quad . \quad . \quad . \quad \text{(VIII.2)}$$

If, now, n is made very large S will be the total amount on the assumption that the interest is added at the end of each of a large number of very short intervals of time covering the period of N years. If we find $\underset{n\to\infty}{Lt.}\left(1 + \frac{0{\cdot}01r}{n}\right)^{nN}$, this quantity becomes one which is continually increasing, its rate of growth at any instant of time being proportional to its value at that instant. Proceeding to the limit, S becomes

$$\begin{aligned} S &= A \underset{n\to\infty}{Lt.}\left(1 + \frac{0{\cdot}01r}{n}\right)^{nN} \\ &= A \underset{n\to\infty}{Lt.}\left(1 + \frac{1}{n/0{\cdot}01r}\right)^{\frac{n}{0{\cdot}01r} \times 0{\cdot}01rN} \\ &= A \underset{m\to\infty}{Lt.}\left(1 + \frac{1}{m}\right)^{m \times 0{\cdot}01rN}, \text{ where } m = \frac{n}{0{\cdot}01r} \end{aligned}$$

$$\therefore \qquad S = Ae^{0 \cdot 01rN} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VIII.3})$$

which agrees with $y = Ae^{kx}$ if we write y for S, x for N, and k for $0{\cdot}01r$. This is why we call $y = Ae^{kx}$ "the true compound interest law," or the *growth* function.

The function $y = Ae^{-kx}$, where k is a positive number, has a negative rate of increase for $\frac{dy}{dx} = -kAe^{-kx} = -ky$. Hence, the function decreases continually at a rate proportional to itself. For this reason $y = Ae^{-kx}$ is known as the *decay* function. The growth and decay functions enter into a large number of investigations in applied science. If a belt is just slipping over the surface of a pulley and T_1 and T_2 are the tensions in the parts of the belt on opposite sides of the pulley, and θ the angle subtended at the axis of the pulley by the arc of contact between belt and pulley, $T_1 = T_2e^{\mu\theta}$. When a body whose temperature is θ degrees above its surroundings is left to cool, its temperature at any time t is given by $\theta = \theta_0e^{-kt}$ where θ_0 and k are constants.

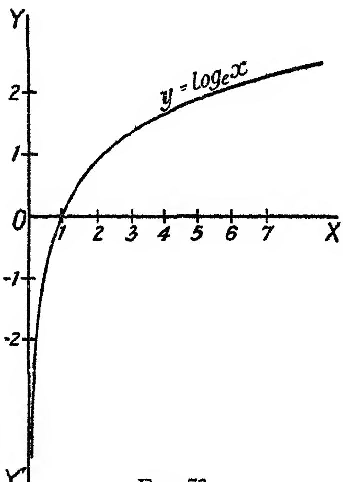


FIG. 73

The reader will probably be familiar with many problems in which the decay function occurs. It occurs in connection with (1) damped oscillations, both mechanical and electrical, (2) the convective equilibrium of the atmosphere, (3) the area of a healthy wound as a function of the time, (4) the resisted motion of a particle, and (5) the flow of heat, etc.

99. **Graph of $y = A \log_a x$.** If $y = e^x$, then $x = \log_e y$. Interchanging x and y we have $y = \log_e x$. Thus, if we interchange the axes of x and y in Fig. 72, the graph of $y = e^x$ will be converted into that of $y = \log_e x$. We have drawn the figure with its axes interchanged, thus obtaining the graph of $y = \log_e x$ (Fig. 73). We discussed the shape of the graph in the last section; it is only necessary here to state that as the logarithm of a negative number is imaginary there is no part of the graph to the left of the y-axis.

To obtain the graph of $\log_a x$ we notice that $\log_a x = k \log_e x$ where $k = \log_a e = \frac{1}{\log_e a}$. The graph in Fig. 73 will, therefore, become the graph of $y = \log_a x$ if we change the scale for values of y so that each unit of length along OY represents the number k. If each unit of length along OY is taken to represent the number kA, the graph becomes that of $y = A \log_a x$.

100. The Graphs of $y = \sin (px + q)$ and $y = ae^{-kx} \sin (px + q)$. We assume the reader to be familiar with the graphs of the simple

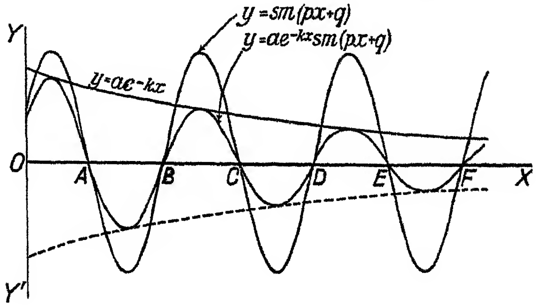


FIG. 74

trigonometrical functions $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, etc. The graph of $y = \sin (px + q)$ crosses the axes of x for every value of x given by $\sin (px + q) = 0$. The values of x which satisfy this are given by $x = \frac{n\pi - q}{p}$, where n is zero or any positive or negative integer.

When $x = 0$, $y = \sin q$. Maximum and minimum values of y equal respectively to 1 and -1, occur alternately for values of x half-way between pairs of consecutive values of x for which $y = 0$. We show the graph in Fig. 74. The abscissae of the points A, B, C, D, E, etc., are respectively $\frac{\pi - q}{p}$, $\frac{2\pi - q}{p}$, $\frac{3\pi - q}{p}$, etc. Any complete undulation of the curve such as that between B and D may be made to coincide with any other complete undulation such as that between D and F, by sliding it along parallel to the axes of x through a distance $\frac{2\pi n}{p}$, where n is an integer. Since $\frac{d^2y}{dx^2} = -p^2 \sin (px + q) = -p^2y$, the points where $y = 0$, i.e. A, B, C, D, E, etc., are points of inflexion. The value of $\cos (px + q)$ at these points is $+1$ or -1

alternately, hence $\frac{dy}{dx} = p \cos (px + q)$ has the values p and $-p$ alternately. With the values of p and q given it is an easy matter to sketch in the curve. A more accurate graph can be obtained by careful tabulation and plotting, but this method is rather tedious. We shall show below a graphical method of constructing the graph.

To draw the graph of $y = ae^{-kx} \sin (px + q)$ we first draw the graphs of $y = ae^{-kx}$ and $y = \sin (px + q)$, and then by multiplying ordinates we obtain the required graph. This method is often exceedingly tedious when an accurate graph is to be drawn, and a geometrical method is used in the example below. If, however, we only require a sketch of the graph the above method is the best, as we only need to take values of x for which $\sin (px + q)$ is -1, 0, or $+1$. Fig. 74 shows a sketch of the graph of $y = ae^{-kx} \sin (px + q)$. (We have sketched the graph of $y = -ae^{-kx}$ in order to locate the points on the graph below OX.) The graph is a wavy curve which touches the graphs of $y = ae^{-kx}$ and $y = -ae^{-kx}$ as shown.

EXAMPLE

Draw the graphs of (1) $y = 2\cdot 2 \sin (0\cdot 3t + 0\cdot 7)$ and (2) $y = 2\cdot 2\, e^{-0\cdot 1t} \sin (0\cdot 3t + 0\cdot 7)$ showing two complete undulations in each case.

The graph (1) is shown in Fig. 75. The circle is drawn with radius CP of length

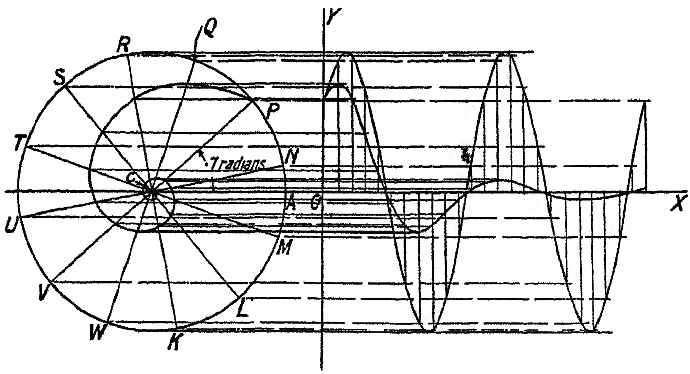


FIG. 75

2·2 units. As the graph is to include two complete undulations, the angle $0\cdot 3t + 0\cdot 7$ must increase by 4π radians, and so t will increase by $\frac{4\pi}{0\cdot 3} = 41\cdot 89$ units. The twelve radii CP, CQ, CR, etc., are equally spaced. CA is a horizontal line through C and $\widehat{ACP} = 0\cdot 7$ radians. The axis of x is taken along CA, the origin being at O. The vertical projections of the lines CP, CQ, CR, etc., are respectively

$2{\cdot}2 \sin 0{\cdot}7,\ 2{\cdot}2 \sin\left(\frac{2\pi}{12} + 0{\cdot}7\right),\ 2{\cdot}2 \sin\left(2 \times \frac{2\pi}{12} + 0{\cdot}7\right),\ 2{\cdot}2 \sin\left(3 \times \frac{2\pi}{12} + 0{\cdot}7\right)$, etc. These are the values of $y = 2{\cdot}2 \sin (0{\cdot}3t + 0{\cdot}7)$ for the values of t, $t = 0$, $t = \frac{2\pi}{12 \times 0{\cdot}3}$, $t = 2 \times \frac{2\pi}{12 \times 0{\cdot}3}$, $t = 3 \times \frac{2\pi}{12 \times 0{\cdot}3}$, etc., respectively.

In order to construct the graph, therefore, we erect ordinates at each of the points on the x axis represented by the above values of t, and then draw horizontal lines through the points P, Q, R, etc.; the intersection of each horizontal line with the corresponding ordinate gives a point on the graph. There will be two points of intersection on each horizontal line, the points being separated by the distance which represents $t = \frac{2\pi}{0{\cdot}3}$. By drawing a curve through the points taken in order, we obtain the required graph.

The graph (2) is also shown in Fig. 75. From the equation $y = 2{\cdot}2e^{-0{\cdot}1t} \sin (0{\cdot}3t + 0{\cdot}7)$ we see that y may be looked upon as the vertical projection of a line $2{\cdot}2e^{-0{\cdot}1t}$ units in length, which makes an angle of $0{\cdot}3t + 0{\cdot}7$ radians with the horizontal. We can construct the graph by the method similar to that used with graph (1), the only modification needed being that the lengths of CP, CQ, CR, etc., must be $2{\cdot}2$, $2{\cdot}2e^{-0{\cdot}1 \times \frac{2\pi}{12 \times 0{\cdot}3}}$, $2{\cdot}2e^{-0{\cdot}1 \times \frac{4\pi}{12 \times 0{\cdot}3}}$, etc., respectively.

The ratio of the second to the first of any two successive values of $e^{-0{\cdot}1t}$ is $e^{-0{\cdot}1 \times \frac{2\pi}{3{\cdot}6}} = e^{-0{\cdot}1745}$. Let α represent this ratio. Then $\log \alpha = -0{\cdot}1745 \times 0{\cdot}4343 = -0{\cdot}07579$.

If y_1 and y_2 are successive values of $2{\cdot}2e^{-0{\cdot}1t}$, then since $\frac{y_2}{y_1} = \frac{y_3}{y_2}$ = etc., $= \alpha$,

$$\log y_2 = \log y_1 + \log \alpha$$

$$\therefore \quad \log y_2 = \log y_1 - 0{\cdot}07579 \text{ and } \log y_n = \log y_{n-1} - 0{\cdot}07579.$$

By means of this relation we obtain the values tabulated below.

Radius	l = Length along radius	Log l	l = Length along radius	Log l
CP	2·2	0·3424	0·271	$\bar{1}{\cdot}4328$
CQ	1·848	0·2666	0·228	$\bar{1}{\cdot}3570$
CR	1·552	0·1908	0·191	$\bar{1}{\cdot}2812$
CS	1·303	0·1150	0·160	$\bar{1}{\cdot}2054$
CT	1·094	0·0392	0·135	$\bar{1}{\cdot}1296$
CU	0·919	$\bar{1}{\cdot}9634$	0·113	$\bar{1}{\cdot}0538$
CV	0·772	$\bar{1}{\cdot}8876$	0·095	$\bar{2}{\cdot}9780$
CW	0·648	$\bar{1}{\cdot}8118$	0·080	$\bar{2}{\cdot}9022$
CK	0·545	$\bar{1}{\cdot}7360$	0·067	$\bar{2}{\cdot}8264$
CL	0·457	$\bar{1}{\cdot}6602$	0·056	$\bar{2}{\cdot}7506$
CM	0·384	$\bar{1}{\cdot}5844$	0·047	$\bar{2}{\cdot}6748$
CN	0·323	$\bar{1}{\cdot}5086$	0·040	$\bar{2}{\cdot}5990$
			0·033	$\bar{2}{\cdot}5232$

The numbers in the column headed "log l" are each obtained by subtracting 0·0758 from the preceding number. The numbers in the column headed "l" are the antilogarithms of the numbers in the "log l" column.

The values of l are marked off from the centre along the corresponding radii in the figure and the vertical projections of these are the ordinates to the graph of $2{\cdot}2e^{-0{\cdot}1t}\sin(0{\cdot}3t + 0{\cdot}7)$. The spiral curve inside the circle has the equation $r = 2{\cdot}2e^{-\frac{1}{3}(\theta - 0{\cdot}7)}$, the centre being the pole and CA the initial line.

101. **Laws of Graphs.** When we have obtained experimentally a number of pairs of corresponding values of two variables, it is often necessary to find a mathematical relation between the variables. This relation is known as the *law* of the graph. If, for example, we have found experimentally, using a certain load raising machine, a number of pairs of values of the force P lb applied to the machine and the load W lb lifted by it, we shall find on plotting P (vertically) and W (horizontally) that the plotted points lie approximately on a straight line. In this case the *law of the graph*, sometimes called *the law of the machine*, is $P = aW + b$; a and b are found from the graph, a is the gradient and b is the value of P at the point in which the graph cuts the axis of P. The reader should know how to find the values of a and b, and we assume that he has worked many examples on finding the laws of straight line graphs. The reader must remember that the laws found from graphs are only approximately true, and that outside of the range of values of the independent variable covered by the experiment the law may deviate very largely from the truth.

When the graph plotted between the experimental values of the variables is not a straight line, it is necessary to change the relation into one between two other variables, so that the graph plotted between these latter is a straight line. Thus, the graph between y and x when $y = a + bx^2$ is a parabola (Art 88). If, however, we put $X = x^2$ the relation becomes $y = a + bX$, and so the graph between X and y is a straight line.

EXAMPLE

The following values of x and y can be represented approximately by the law $y = a + bx^2$. Test this statement, and if it is true, find the approximate values of a and b.

x	0	2	4	6	8	10
y	7·76	11·8	24·4	43·6	71·2	107·0

Here we introduce a new variable $X = x^2$, which turns the equation $y = a + bx^2$ into the equation $y = a + bX$.

$X = x^2$	0	4	16	36	64	100

We have drawn the graph between y and $X = x^2$ in Fig. 76. From this graph we have

$$b = \text{gradient of graph}$$

$$= \frac{67 \cdot 8}{70} = 0.969$$

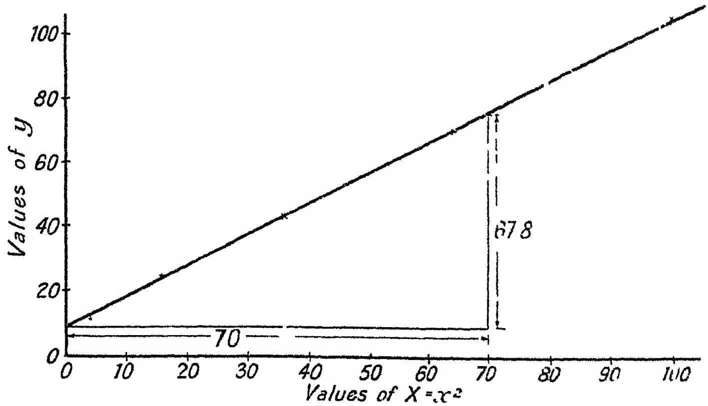


FIG. 76

and $\quad a$ = intercept on y axis, or ordinate of point in which graph cuts y axis
$= 9{\cdot}20$

Hence the law is

$$y = a + bX$$

or $\quad y = 0{\cdot}969x^2 + 9{\cdot}20$

102. **Method of Procedure in Finding the Law Connecting y and x from Tabulated Values of the Variables.** Plot the values of y and x, taking y vertical and x horizontal. If a straight line can be drawn which passes through or near the points which have been plotted it should be drawn, and its equation $y = mx + c$ is the law of the graph. The reader knows that we draw a smooth graph through points plotted from experimental measurements, so as to cancel out in part any inaccuracies which may be, and are usually, present. The straight line should be drawn so that the points are distributed evenly about it, some on one side, some on the other.

No.	Original Relation	Altered Form if necessary	Substitutions	Modified Relation	Gradient	Intercept
1	$y = a + bx^2$	—	$Y = y,\ X = x^2$	$Y = bX + a$	b	a
2	$xy = a + bx$	$y = a\dfrac{1}{x} + b$	$Y = y,\ X = \dfrac{1}{x}$	$Y = aX + b$	a	b
3	$xy = ax + by$	$y = b\dfrac{y}{x} + a$	$Y = y,\ X = \dfrac{y}{x}$	$Y = bX + a$	b	a
4	$y = ax + bx^2$	$\dfrac{y}{x} = bx + a$	$Y = \dfrac{y}{x},\ X = x$	$Y = bX + a$	b	a
5	$y = ax^n$	$\log_{10}y = n\log_{10}x + \log_{10}a$	$Y = \log_{10}y,\ X = \log_{10}x$	$Y = nX + \log_{10}a$	n	$\log_{10}a$
6	$y = ae^{kx}$	$\log_{10}y = 0{\cdot}4343kx + \log_{10}a$	$Y = \log_{10}y,\ X = x$	$Y = 0{\cdot}4343kX + \log_{10}a$	$0{\cdot}4343k$	$\log_{10}a$
7	$y + a\sqrt{x^2 + y^2} = b$	$y = -a\sqrt{x^2 + y^2} + b$	$Y = y,\ X = \sqrt{x^2 + y^2}$	$Y = -aX + b$	$-a$	b

If the points do not lie evenly about a straight line a smooth curve should be drawn through them. The shape of this graph may be like some graph with which we are familiar and whose equation or law we know. If so, we proceed to test if that law will fit the results. If we do not recognize the shape of the graph, theory may give us some clue as to the nature of the law. We then take this law and alter its form so that, by taking new variables, we may put the law in the form $y = mx + c$.

On plotting values of the new variables and drawing the straight line which passes evenly through the points, we are enabled to determine m, which is the gradient of the line, and c, which is the value of y for the point of intersection of the line and the line $x = 0$.

We give above some examples of relations in common use showing the changes necessary in each case in order to produce a straight-line graph; x and y are the old variables and X and Y the new ones. The modified relation is compared with $y = mx + c$, and the quantities corresponding to m and c are respectively the gradient and intercept on OY.

n, k, a, and b are constants whose values are to be found from the values of the gradient and intercept. We had an example on No. 1 in the last section. We shall work examples on Nos. 5, 6, and 7.

EXAMPLE 1

The table below gives the pressure P lb per ft^2 and the volume V ft^3 of 1 lb of steam at maximum density. Assuming that $PV^n = C$, find the values of the constants n and C. Writing y for P and x for V, the law becomes $y = Cx^{-n}$, which is of the type 5 above. Taking logs of both sides of $PV^n = C$, we have on transposing

$$\log_{10} P = -n \log_{10} V + \log_{10} C \quad . \quad . \quad . \quad \text{(VIII.4)}$$

Compare with

$$Y = mX + C \quad . \quad . \quad . \quad . \quad \text{(VIII.5)}$$

We see that the substitutions to produce a straight line graph are $Y = \log_{10} P$ and $X = \log_{10} V$, and that the gradient and intercept are $-n$ and $\log_{10} C$ respectively. Tabulating we have

Values of P . .	12·27	17·62	24·92	34·77	47·87	65·06
Values of V . .	3 390	2 406	1 732	1 264	934·6	699·0
$Y = \log_{10} P$ . .	1·0887	1·2459	1·3966	1·5412	1·6800	1·8133
$X = \log_{10} V$ . .	3·5302	3·3813	3·2386	3·1017	2·9706	2·8445

The graph is shown in Fig. 77. As the ordinate through $X = \log V = 0$ is not on the graph, we cannot measure the intercept which would give $\log_{10} C$. We proceed thus—

From the graph the gradient is $-\dfrac{PR}{RQ}$, but $-n$ is the gradient.

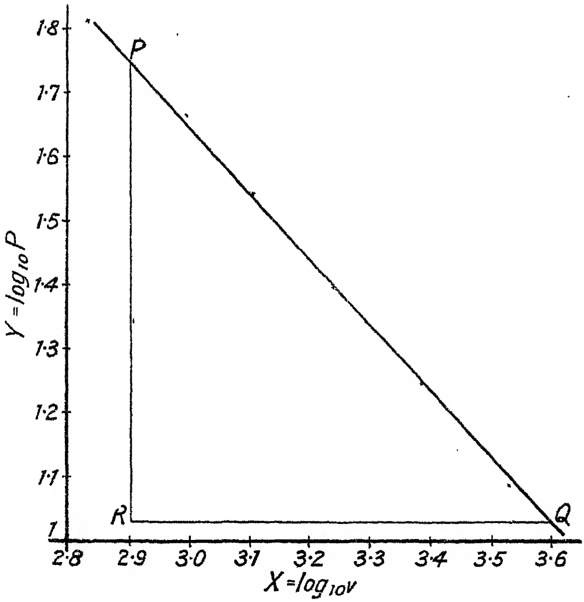

1·8
1·7
1·6
1·5
1·4
1·3
1·2
1·1
1
Y = log₁₀P
P
Q
R
2·8
2·9
3·0
3·1
3·2
3·3
3·4
3·5
3·6
X = log₁₀v

FIG. 77

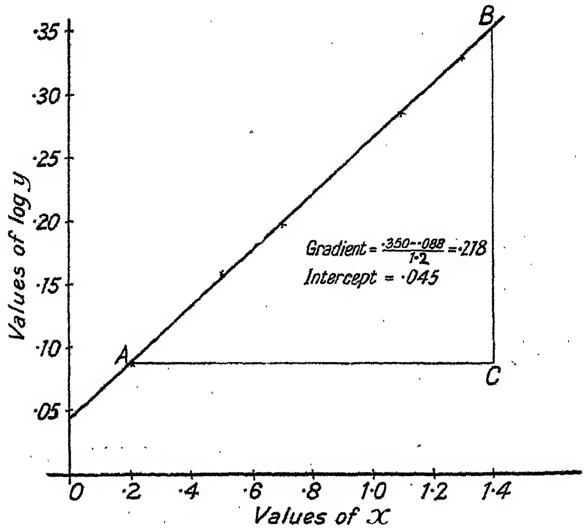

·35
·30
·25
·20
·15
·10
·05
Values of log y
B
A
C
Gradient = (·350 − ·088)/1·2 = ·218
Intercept = ·045
0
·2
·4
·6
·8
1·0
1·2
1·4
Values of x

FIG. 78

$$\therefore \qquad n = \frac{PR}{RQ} = \frac{1{\cdot}748 - 1{\cdot}026}{3{\cdot}6 - 2{\cdot}9} = 1{\cdot}03$$

From the graph, when $\log_{10} V = 3$, $\log_{10} P = 1{\cdot}642$. Substituting in

$$\log_{10} P = -n \log_{10} V + \log_{10} C$$

$$1{\cdot}642 = -1{\cdot}03 \times 3 + \log_{10} C$$

from which $\qquad \log_{10} C = 4{\cdot}732$

and $\qquad C = 53\ 950$, the required law being

$$PV^{1{\cdot}03} = 53\ 950$$

Another method of finding n and C would be to measure the values of $\log P$ and $\log V$ for each of two points on the graph (one near each end for accuracy), and to substitute the values in turn in (VIII.4). The values of n and $\log_{10} C$ would then be found by solving the resulting pair of simultaneous equations.

EXAMPLE 2

The following values of x and y follow the law $y = ae^{kx}$. Find the best values of a and k.

x	0·2	0·5	0·7	1·1	1·3
y	1·223	1·430	1·571	1·921	2·127

Taking logs, we have $\log_{10} y = 0{\cdot}4343kx + \log_{10} a$. If we plot $Y = \log_{10} y$ and $X = x$, the graph is a straight line of gradient $0{\cdot}4343k$ and intercept $\log_{10} a$.

$\log y$	0·0875	0·1553	0·1962	0·2835	0·3277

The graph is shown in Fig. 78.

The gradient of the graph is given by $\dfrac{BC}{AC} = \dfrac{0{\cdot}350 - 0{\cdot}088}{1{\cdot}4 - 0{\cdot}2} = \dfrac{0{\cdot}262}{1{\cdot}2} = 0{\cdot}218.$

Hence, $0{\cdot}4343k = 0{\cdot}218$ or $k = \dfrac{0{\cdot}218}{0{\cdot}4343} = 0{\cdot}502$, also the intercept is 0·045. Hence, $\log_{10} a = 0{\cdot}045$ or $a = 1{\cdot}11$, and the law is $y = 1{\cdot}11e^{0{\cdot}502x}$.

EXAMPLE 3

The following values of x and y are probably connected by the law $y + a\sqrt{x^2 + y^2} = b$. Test this, and if it is the correct law, find the best values of a and b.

x	1 440	1 460	1 280	805	0
y	0	413	950	1 470	1 730
$\sqrt{x^2 + y^2}$	1 440	1 518	1 594	1 676	1 730

The values in the last line are calculated from the given values of x and y. The given expression takes the form $y = -a\sqrt{x^2 + y^2} + b$, which becomes the straight line law $y = -aX + b$ if X is substituted for $\sqrt{x^2 + y^2}$. Hence, we plot the graph (Fig. 79), which shows that the points lie approximately on a straight line, and that the law is nearly correct. We cannot read off the intercept in this case, and so we take two points, A and B, on the graph and find that $X = 1\,450$ when $y = 20$ and $X = 1\,750$ when $y = 1\,875$. Substituting in $y = -a\sqrt{x^2 + y^2} + b$,

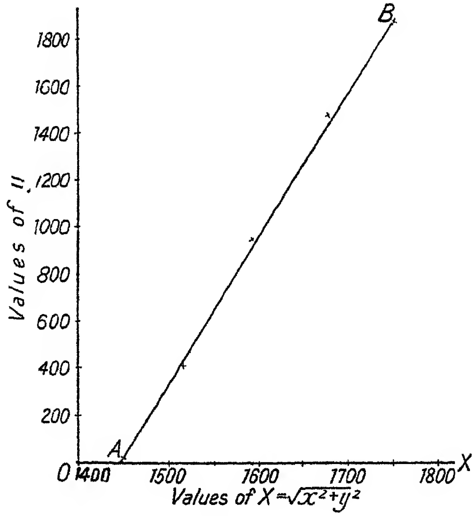


FIG. 79

we have
$$1\,875 = -1\,750a + b \quad . \quad . \quad . \quad (1)$$
$$20 = -1\,450a + b \quad . \quad . \quad . \quad (2)$$

From these, by subtracting (2) from (1)
$$1\,855 = -300a \text{ or } a = -6{\cdot}18$$
and substituting in (2)
$$20 = 6{\cdot}18 \times 1\,450 + b$$
from which
$$b = 20 - 8\,961 = -8\,941$$

Hence, $y = 6{\cdot}18\sqrt{x^2 + y^2} - 8\,941$ is the law.

It will be seen from the graph that this law is only approximately true.

103. **Laws Containing Three or More Constants whose Values are to be Determined.** If the suspected law connecting y and x contains n constants, where n is greater than two, the usual method of procedure is to draw the graph between x and y and to measure from it n pairs of values. On substitution of these in turn in the suspected law, we obtain n equations from which the n constants may be found. In some cases special methods may be applied as in the following—

(1) THE LAW $y = a + be^{nx}$. Plot the given values of x and y and draw a smooth curve through the points obtained. Take three points on the graph, one near each end and one between these so that their abscissae are in arithmetical progression. If the co-ordinates of the points are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then $x_2 = x_1 + h$ and $x_3 = x_1 + 2h$ where h is a constant (known). Substituting these co-ordinates in the equation in turn, we have

$$y_1 = a + be^{nx_1} \qquad \text{(VIII.6)}$$

$$y_2 = a + be^{nx_1 + nh} \qquad \text{(VIII.7)}$$

$$y_3 = a + be^{nx_1 + 2nh} \qquad \text{(VIII.8)}$$

From (VIII.6) and (VIII.7)

$$y_2 - y_1 = be^{nx_1}(e^{nh} - 1) \qquad \text{(VIII.9)}$$

and similarly

$$y_3 - y_2 = be^{nx_1}e^{nh}(e^{nh} - 1) \qquad \text{(VIII.10)}$$

Dividing (VIII.10) by (VIII.9)

$$\frac{y_3 - y_2}{y_2 - y_1} = e^{nh} \qquad \text{(VIII.11)}$$

From this n is calculated since all the other quantities are known. From (VIII.9)

$$b = (y_2 - y_1) \times \frac{1}{e^{nx_1}(e^{nh} - 1)}$$

and is, therefore, found by substitution; a is then found by substitution in (VIII.6), $a = y_1 - be^{nx_1}$.

EXAMPLE 1

The following values are measured from a graph which follows approximately the law $y = a + be^{nx}$. Find a, b, and n. $y = 3{\cdot}32$ when $x = 2$, $y = 7{\cdot}03$ when $x = 4$, and $y = 22{\cdot}10$ when $x = 6$.

Substituting the values in turn in $y = a + be^{nx}$ we have

$$3{\cdot}32 = a + be^{2n} \qquad (1)$$

$$7{\cdot}03 = a + be^{4n} \qquad (2)$$

$$22{\cdot}10 = a + be^{6n} \qquad (3)$$

Subtracting (1) from (2), $3{\cdot}71 = be^{2n}(e^{2n} - 1) \qquad (4)$

Subtracting (2) from (3), $15{\cdot}07 = be^{4n}(e^{2n} - 1) \qquad (5)$

Dividing (5) by (4), $\frac{15 \cdot 07}{3 \cdot 71} = e^{2n}$

i.e.
$$n = \frac{\log_{10} 15\cdot07 - \log_{10} 3\cdot71}{2 \log_{10} e}$$
$$= \frac{1\cdot1781 - 0\cdot5694}{0\cdot8686} = \frac{0\cdot6087}{0\cdot8686} = 0\cdot700$$

From above, $e^{2n} = \text{antilog}\, 0\cdot6087 = 4\cdot061$, and substituting in (4),
$$b = \frac{3\cdot71}{4\cdot061 \times 3\cdot061} = 0\cdot299$$

Now from (1),
$$a = 3\cdot32 - be^{2n}$$
$$= 3\cdot32 - 0\cdot299 \times 4\cdot061$$
$$= 2\cdot11$$

The law is
$$y = 2\cdot11 + 0\cdot299e^{0\cdot700x}.$$

If we put X for e^x in $y = a + be^{nx}$ we obtain the relation $y = a + bX^n$, which is dealt with in (2) below. If x_1, x_2, and x_3 are in arithmetical progression, then e^{x_1}, e^{x_2}, and e^{x_3} are in geometrical progression. Hence, the above method may be applied in cases where the law is $y = a + bx^n$ if the three values of x are consecutive terms of a geometrical progression.

(2) THE LAW $y = a + bx^n$. When the graph has been drawn, take three points on it (not close together) such that their abscissae x_1, x_2, and x_3 form a geometrical progression, i.e. $x_2 = rx_1$ and $x_3 = r^2x_1$ where r is a constant. If y_1, y_2, and y_3 are the corresponding ordinates, we have on substitution in $y - a = bx^n$

$$y_1 - a = bx_1^n \quad . \quad . \quad . \text{(VIII.12)}$$
$$y_2 - a = br^nx_1^n \quad . \quad . \quad . \text{(VIII.13)}$$
$$y_3 - a = br^{2n}x_1^n \quad . \quad . \quad . \text{(VIII.14)}$$

Taking (VIII.12) from (VIII.13) we have
$$y_2 - y_1 = bx_1^n(r^n - 1) \quad . \quad . \quad . \text{(VIII.15)}$$
and taking (VIII.13) from (VIII.14)
$$y_3 - y_2 = bx_1^nr^n(r^n - 1) \quad . \quad . \quad . \text{(VIII.16)}$$
Whence, dividing (VIII.16) by (VIII.15)
$$\frac{y_3 - y_2}{y_2 - y_1} = r^n \quad . \quad . \quad . \text{(VIII.17)}$$
from which n can be found. On substituting for n in (VIII.15) b can be found, and by substituting for n and b in (VIII.12) we find a.

EXAMPLE 2

If $y = a + bx^n$ and $y = 5\cdot29$ when $x = 1\cdot3$, $y = 15\cdot36$ when $x = 3\cdot25$ and $y = 52\cdot04$ when $x = 8\cdot125$, find a, b, and n.

Here $\frac{x_2}{x_1} = \frac{x_3}{x_2} = 2{\cdot}5$, i.e. the abscissae are in geometrical progression.

Substituting the values in turn in $y - a = bx^n$, we have

$$5{\cdot}29 - a = b \times 1{\cdot}3^n \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$15{\cdot}36 - a = b \times 1{\cdot}3^n \times 2{\cdot}5^n \quad . \quad . \quad . \quad . \quad (2)$$
$$52{\cdot}04 - a = b \times 1{\cdot}3^n \times 2{\cdot}5^{2n} \quad . \quad . \quad . \quad . \quad (3)$$

Take (1) from (2), and (2) from (3)

then
$$10{\cdot}07 = b \times 1{\cdot}3^n(2{\cdot}5^n - 1) \quad . \quad . \quad . \quad (4)$$
$$36{\cdot}68 = b \times 1{\cdot}3^n \times 2{\cdot}5^n(2{\cdot}5^n - 1) \quad . \quad . \quad (5)$$

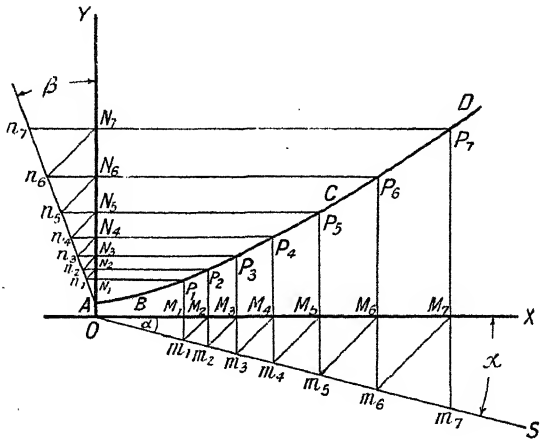


FIG. 80

Dividing (5) by (4) $\qquad \frac{36{\cdot}68}{10{\cdot}07} = 2{\cdot}5^n$

from which $\qquad n = \frac{\log 36{\cdot}68 - \log 10{\cdot}07}{\log 2{\cdot}5} = 1{\cdot}411$

From (4) $\qquad b = \frac{10{\cdot}07}{1{\cdot}3^{1{\cdot}411}(2{\cdot}5^{1{\cdot}411} - 1)} = 2{\cdot}629$

and from (1) $\qquad a = 5{\cdot}29 - 2{\cdot}629 \times 1{\cdot}3^{1{\cdot}411} = 1{\cdot}483$

The law is $\qquad y = 1{\cdot}483 + 2{\cdot}629x^{1{\cdot}411}$

104. **Graphical Method of Finding the Best Values of n and a in the Law $y = a + bx^n$.** Let BCD (Fig. 80) be the graph which lies between the points obtained by plotting observed values of x and y which are expected to follow approximately the above law. Draw OS at any convenient inclination to OX. Let P_1 be any point on BCD. Draw the ordinate through P_1 cutting OX in M_1 and OS in m_1. Draw $m_1 M_2$ at 45° to OX, cutting it in M_2. Erect the ordinate

M_2P_2 and let P_2M_2 produced cut OS in m_2. Repeat the above construction, drawing m_2M_3 at 45° to OX, erecting the ordinate M_3P_3 and producing this to cut OS in m_3. By continued repetition of the above construction we obtain points P_4, P_5, P_6, etc., on the graph. If the co-ordinates of P_r are x_r and y_r, we have

$$\frac{M_2m_2}{OM_2} = \frac{x_3 - x_2}{x_2} \quad . \quad . \quad . \text{(VIII.18)}$$

and

$$\frac{M_1m_1}{OM_1} = \frac{x_2 - x_1}{x_1} \quad . \quad . \quad . \text{(VIII.19)}$$

But if α is the angle XOS

$$\frac{M_2m_2}{OM_2} = \tan\alpha = \frac{M_1m_1}{OM_1}$$

$$\therefore \qquad \frac{x_3 - x_2}{x_2} = \frac{x_2 - x_1}{x_1} = \tan\alpha$$

or

$$\frac{x_3}{x_2} = \frac{x_2}{x_1} = 1 + \tan\alpha \quad . \quad . \quad . \text{(VIII.20)}$$

Hence, $\qquad x_1x_3 = x_2^2$

and x_1, x_2, x_3 are in geometrical progression, as also are the abscissae of any three consecutive points of the set.

Since $\qquad y - a = bx^n$

$$\left.\begin{aligned} y_1 - a &= bx_1^n \\ y_2 - a &= bx_2^n \\ y_3 - a &= bx_3^n \end{aligned}\right\} \quad . \quad . \quad . \quad . \text{(VIII.21)}$$

and therefore

$$\begin{aligned} (y_2 - a)^2 &= b^2x_2^{2n} \\ &= b^2x_1^nx_3^n \\ &= (bx_1^n)(bx_3^n) \end{aligned}$$

$$\therefore \qquad (y_2 - a)^2 = (y_1 - a)(y_3 - a) \quad . \quad . \text{(VIII.22)}$$

or the values of $y - a$ are also in geometrical progression.

Let A be the point in which the curve cuts OY. (This point is not necessarily on the plotted portion of the graph.) Then $OA = a$. Draw horizontal lines P_1N_1, P_2N_2, P_3N_3, etc., meeting OY in N_1, N_2, N_3, etc., respectively. Through these points draw lines at 45° to OY cutting P_1N_1 in n_1, P_2N_2 in n_2, etc.

Now from (VIII.22)

$$\frac{y_3 - a}{y_2 - a} = \frac{y_2 - a}{y_1 - a}$$

$$\therefore \qquad \frac{y_3 - y_2}{y_2 - a} = \frac{y_2 - y_1}{y_1 - a} \qquad . \quad . \quad . \text{(VIII.23)}$$

from which we see that if A is joined to n_1 and n_2 the triangles An_1N_1, An_2N_2 are similar, i.e. the straight line through n_1, n_2 passes through A. In the same way we can prove that all the points n_1, n_2, n_3, etc., lie on a straight line which passes through A.

As the graph is plotted from observed values, the points n_1, n_2, n_3, etc., will not lie exactly on a straight line, and the line should be drawn which passes most evenly among these points. This line cuts OY in A, so that OA is the most probable value of the constant a. When this value of a has been deducted from each of the observed values of y the law connecting $y - a$ and x, which is $(y - a) = bx^n$, may have its constants determined by using the method of Example 1, Art. 102. If the graph cuts the axis of y and the point of intersection does not coincide with A as found by the above construction, the latter is more likely to be the correct point. A value of n may be found from the figure, for if β is the angle yAn_1,

$$1 + \tan\beta = 1 + \frac{y_2 - y_1}{y_1 - a}$$

$$= \frac{y_2 - a}{y_1 - a}$$

$$= \left(\frac{x_2}{x_1}\right)^n$$

or from (VIII.20), $1 + \tan\beta = (1 + \tan\alpha)^n$ . . . (VIII.24)

By measuring the tangents of α and β and substituting, the value of n may be found.

EXAMPLE

Find the values of a, b, and n which make the law $y = a + bx^n$ fit most closely the observed values given.

x	2	4	6	8	10
y	5·0	8·6	13·9	25·3	35·6

By the construction in Fig. 81, we find that OA represents 4·1. Hence, a = 4·1.
Now, tabulating values of $y - a$, $\log x$ and $\log (y - a)$, we have

$y - a$	0·9	4·5	9·8	21·2	31·5
$\log x$	0·3010	0·6021	0·7782	0·9031	1·0000
$\log (y - a)$	$\bar{1}$·9542	0·6532	0·9912	1·3263	1·4983

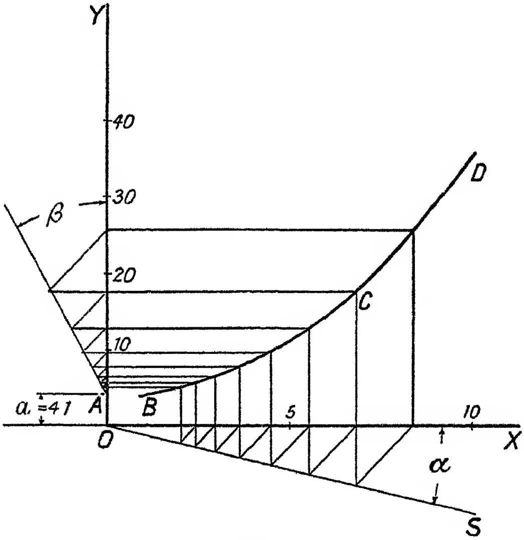


FIG. 81

Plotting values of $\log (y - 4{\cdot}1)$ against values of $\log x$, we obtain a straight line (not shown) whose slope is 2·20, and whose intercept is $-$ 0·687. Hence, n = 2·20 and b = antilog $\bar{1}$·313 = 0·206, and the law is $y = 4{\cdot}1 + 0{\cdot}206x^{2\cdot 2}$.

From Fig. 81, $\tan \alpha = 0{\cdot}22$, $\tan \beta = 0{\cdot}55$, and from (VIII.24), $n = 2{\cdot}20$.

If the suspected law is $y = b(x + a)^n$, we express x as a function of y, thus, $x = -a + \dfrac{1}{b^{\frac{1}{n}}} y^{\frac{1}{n}}$ which is of the type $y = a + bx^n$ if x and y are interchanged.

Thus, the method is the same as that in Fig. 81 if the axes of x and y are interchanged. In this case a is negative if A is on the positive part of the axis. If the suspected law is $y = a + be^{nx}$, we tabulate values of e^x and plot y against $X = e^x$.

The law is then $y = a + bX^n$, which has been dealt with above.

105. To Find the Best Values of the Constants in an Equation when the Number of Sets of Observations is Greater than the Number of Constants: Method of Least Squares.

EXAMPLE 1

Suppose we have given the three pairs of values of x and y, $x = 1, y = 5{\cdot}27$; $x = 2, y = 8{\cdot}44$; $x = 3, y = 11{\cdot}53$, which are supposed to be connected by the law $y = mx + c$, and that we require to find the values of m and c which most closely fit the given values of x and y. Substituting the values in turn in $y = mx + c$, we have

$$5{\cdot}27 = m + c$$
$$8{\cdot}44 = 2m + c$$
$$11{\cdot}53 = 3m + c$$

and rearranging these

$$m + c - 5{\cdot}27 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$2m + c - 8{\cdot}44 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$3m + c - 11{\cdot}53 = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Here there are three equations from which to determine two unknowns. In order to reduce the number of equations to two we proceed thus: Multiply through each equation by the coefficient of m in it. The relations become

$$m + c - 5{\cdot}27 = 0 \quad . \quad . \quad . \quad . \quad (4)$$
$$4m + 2c - 16{\cdot}88 = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$9m + 3c - 34{\cdot}59 = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Adding these, we obtain

$$14m + 6c - 56{\cdot}74 = 0 \quad . \quad . \quad . \quad . \quad (7)$$

Now multiply each of the equations (1), (2), and (3) through by the coefficient of c in it, and add. As the coefficient of c is one in each case, the sum is found by adding (1), (2), and (3). Thus we have

$$6m + 3c - 25{\cdot}24 = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(7) and (8) are known as the *normal equations*, and the values of m and c found from them are those which fit most closely the given values of x and y. Solving between these equations we have $2m = 56{\cdot}74 - 50{\cdot}48$, i.e. $m = 3{\cdot}13$ and $c = \dfrac{25{\cdot}24 - 18{\cdot}78}{3} = 2{\cdot}15$, and the law which most closely fits the values of x and y is $y = 3{\cdot}13x + 2{\cdot}15$.

The above method is based on the method of least squares which determines the constants in an empirical formula so that the sum of the squares of the differences between the observed and the calculated values of y is a minimum. We shall show this in the case of a suspected law $y = mx + c$ on the assumption that we are given three pairs of values (x_1, y_1), (x_2, y_2), and (x_3, y_3), of which the values of x are correct and those of y are determined by experimental methods. The values of y calculated by the formula are respectively $mx_1 + c$, $mx_2 + c$, and $mx_3 + c$, and the sum of the squares of the differences between these and the observed values is

$$E = (mx_1 + c - y_1)^2 + (mx_2 + c - y_2)^2 + (mx_3 + c - y_3)^2 \quad \text{(VIII.25)}$$

Since E is a function of m and c the conditions for a minimum value of E are

$$\frac{\partial E}{\partial m} = 0 \text{ and } \frac{\partial E}{\partial c} = 0 \quad . \quad . \quad . \quad . \quad \text{(VIII.26)}$$

By differentiation we have, therefore,

$$2x_1(mx_1 + c - y_1) + 2x_2(mx_2 + c - y_2) + 2x_3(mx_3 + c - y_3) = 0$$

and $$2(mx_1 + c - y_1) + 2(mx_2 + c - y_2) + 2(mx_3 + c - y_3) = 0$$

which reduce to

$$(x_1^2 + x_2^2 + x_3^2)m + (x_1 + x_2 + x_3)c - (x_1y_1 + x_2y_2 + x_3y_3) = 0 \quad \text{(VIII.27)}$$

and $$(x_1 + x_2 + x_3)m + 3c - (y_1 + y_2 + y_3) = 0 \quad \text{(VIII.28)}$$

These normal equations can be obtained from the relations

$$\left.\begin{aligned} mx_1 + c - y_1 &= 0 \\ mx_2 + c - y_2 &= 0 \\ mx_3 + c - y_3 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(VIII.29)}$$

by (1) multiplying through each equation by the coefficient of m in it and adding, this gives (VIII.27), and (2) by multiplying through each equation by the coefficient of c in it, unity in this case, and adding, this gives (VIII.28). This method is easily extended to the case of a law with more than two constants, as in the following example—

EXAMPLE 2

Find the parabola of the form $y = a + bx + cx^2$ which fits most closely the following observations—

$x =$	-3	-2	-1	0	1	2	3
$y =$	4·63	2·11	0·67	0·09	0·63	2·15	4·58

[The results should be correct to three places of decimals.] (U.L.)

Writing the equation in the form $a + bx + cx^2 - y = 0$, and substituting in order the pairs of values given, we have

$$a - 3b + 9c - 4{\cdot}63 = 0 \quad . \quad . \quad . \quad . \quad (1)$$
$$a - 2b + 4c - 2{\cdot}11 = 0 \quad . \quad . \quad . \quad . \quad (2)$$
$$a - b + c - 0{\cdot}67 = 0 \quad . \quad . \quad . \quad . \quad (3)$$
$$a - 0{\cdot}09 = 0 \quad . \quad . \quad . \quad . \quad (4)$$
$$a + b + c - 0{\cdot}63 = 0 \quad . \quad . \quad . \quad . \quad (5)$$
$$a + 2b + 4c - 2{\cdot}15 = 0 \quad . \quad . \quad . \quad . \quad (6)$$
$$a + 3b + 9c - 4{\cdot}58 = 0 \quad . \quad . \quad . \quad . \quad (7)$$

Each equation is now multiplied through by the coefficient of a in it, and the resulting equations added to give the first normal equation. As, however, these coefficients are all unity, we simply add the above equations. The first normal equation is therefore

$$7a + 28c - 14{\cdot}86 = 0$$

or $$a + 4c - 2{\cdot}123 = 0 \quad . \quad . \quad . \quad . \quad (8)$$

Next, multiply each of the equations (1) to (7) by the coefficient of b in it, and add. We obtain

$$a(-3-2-1+0+1+2+3)+b(3^2+2^2+1+0+1+2^2+3^2)$$
$$+c(-3^3-2^3-1^3+0+1^3+2^3+3^3)$$
$$+(13\cdot89+4\cdot22+0\cdot67-0\cdot63-4\cdot30-13\cdot74)=0$$

or

$$28b=-0\cdot11$$

and the second normal equation is

$$b=-0\cdot00393 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now multiply through each equation by the coefficient of c in it and add. We have then

$$a(3^2+2^2+1^2+0+1^2+2^2+3^2)-b(3^3+2^3+1^3+0-1^3-2^3-3^3)$$
$$+c(9^2+4^2+1^2+0+1^2+4^2+9^2)$$
$$-(41\cdot67+8\cdot44+0\cdot67+0+0\cdot63+8\cdot60+41\cdot22)=0$$

or

$$28a+196c=101\cdot23$$

Dividing through by 28, we have the third normal equation

$$a+7c=3\cdot615 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Rewriting the normal equations, we have

$$a+4c=2\cdot123;\quad b=-0\cdot00393;\quad a+7c=3\cdot615$$

From these $a=0\cdot134$, $b=-0\cdot004$, $c=0\cdot497$, correct to three places of decimals. The equation is

$$y=0\cdot134-0\cdot004x+0\cdot497x^2$$

The constants in the equation of the above example could have been obtained by plotting the smooth graph which appears to pass most nearly through the points, then measuring the co-ordinates of three points which lie on this graph, and, finally, substituting these in the equation $y=a+bx+cx^2$ in turn and solving the resulting equations for a, b, and c. This method, however, though satisfactory in most cases, does not give the values of the constants so accurately as does the method of the example. For a fuller discussion of the above method, readers should consult *The Calculus of Observations*, by Whittaker and Robinson, or some other treatise which deals particularly with errors of observation. See also Vol. II.

EXAMPLES VIII

(1) Show that for all values of n the graph of $y=x^n$ is symmetrical with the graph of $y=x^{\frac{1}{n}}$. Which line is the axis of symmetry?

(2) Show that if n is an odd number, the graph of $y=ax^n$ has a point of inflexion at the origin.

(3) Show that the graph of $y=ax^{2n}$ is concave or convex upwards everywhere, according as a is positive or negative, n being a positive integer other than unity.

(4) Show that if we draw the graph of $y=e^{kx}$ and then take as a new axis of y the ordinate $x=a$, retaining the same axis of x, the equation of the graph becomes $y=Ae^{kx}$. Find A in terms of e, k, and a.

(5) Assuming the series for $\log_e \frac{n+1}{n}$, show that for large values of n, the difference between the ordinates of the graph of $y = \log_e x$ at $x = n$ and $x = n + 1$ is very nearly the reciprocal of n. Hence, verify approximately the rule that the gradient of the graph at the point for which $x = n$ (n large) is $1/n$.

(6) Show that the graphs of $y = ae^{-kx}$ and $y = ae^{-kx} \sin (cx + b)$ touch each other at an infinite number of points, and that the projections of these points on to the axis of x are each half way between two consecutive points in which OX cuts the latter graph.

(7) Sketch the curves $r = a\theta$ (spiral of Archimedes) and $r\theta = a$ (hyperbolic spiral), when a is positive.

(8) Using the method of Art. 102, show in a table what changes must be made in the following expressions in order to obtain straight line graphs: $y = ax^n$, $y - b = a \log_e x$, $y = \frac{cx}{a(x+1)}$, $xy = a + bx$, $y = ax + b\sqrt{x} + \sqrt{y}$. Show in each case what is represented by the gradient and what by the intercept.

(9) Plot the graph of $y = 3 + \sqrt{x^2 + 1}$ from $x = 6$ to $x = 12$, and find the equation to the straight line which agrees most closely with the graph inside this range. Find the percentage errors in using this equation instead of the original one for calculating the values of y corresponding to (1) $x = 7\frac{1}{2}$, (2) $x = 0$, and (3) $x = 1$. The large percentage errors in the second and third cases illustrate the danger of making use of extrapolation when using empirical formulae.

(10) The equation $y = a + b \cdot 10^{mx}$ is thought to express the relation between the variable quantities x and y which are derived from experiment. Explain in detail a method for determining good approximations to the values of a, b, and m. (U.L.)

(11) Assuming that the following experimental values of x and y may be approximately represented by a formula of the type $y = ax + b \cdot 10^x$, find values for the constants a and b and show in a table the values of y corresponding to the given values of x as given by the formula with the ascertained values of the constants.

x	1	2	3	4
y	0·30	0·64	1·32	5·20

(U.L.)

[NOTE. Form four simultaneous equations, and obtain from these two normal equations. Then solve for a and b.]

(12) A graph has its equation of the form $y = b + ax^m$ where a, b, and m are unknown. If three points be taken on the graph whose abscissae are in geometrical progression, show how m, b, and a can be determined.

Explain how to find values of m, b, and a, when the equation represents only a near approximation to a graph derived from experiments. (U.L.)

(13) The following numbers approximately satisfy the law $y = a + bx^n$. Find the best values of a, b, and n.

x	25	35	50	75	150
y	4·19	5·31	8·82	12·70	47·40

(14) The coefficient of self-induction L of a coil and the number of turns N of wire are related by a formula of the type $L = aN^b$ where a and b are constants. The following are observed pairs of values of L and N—

N . .	25	35	50	75	150	200	250
L . .	1·09	2·21	5·72	9·60	44·3	76·0	156·0

Find the values of a and b which most nearly satisfy the whole series of observations. (U.L.)

(15) The following are the results of an experiment to find the law governing the friction of a string wrapped round a cylindrical metal bar. (θ is the angle of contact of the string with the bar, W the weight attached to one end to cause slipping.)

θ radians .	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
W oz .	2·875	4·0	5·706	8·9	12·435

Find from these numbers the equation connecting W and θ. (U.L.)

(16) A pillar of length l ft and least radius of gyration of cross-section g in., has round ends, and is loaded axially. The following table gives the safe working stress f in tons per in.² of cross-section, as calculated by a rather complicated formula, for certain values of $\frac{l}{g}$. Plot a graph showing the stress f along the vertical scale, and values of $\frac{l}{g}$ along the horizontal. Draw the straight line which passes most closely through the points, and determine the constants in the equation $f = a\frac{l}{g} + b$. Find the greatest percentage error in using this equation to calculate values of f for values of $\frac{l}{g}$ inside the given range. Compare your formula with that given below sometimes used for pillars for which $\frac{l}{g}$ lies between 100 and 140, i.e. $f = 7{\cdot}5 - \frac{l}{30g}$.

$\frac{l}{g}$.	100	110	120	130	140
f .	3·09	2·71	2·38	2·10	1·86

(17) The following numbers approximately satisfy the law $y = a + be^{nx}$. Find the best values of a, b, and n—

x .	1	2	3	4	5	6
y .	5·71	6·51	7·63	8·87	10·13	11·84

(18) Find a and b in Example 11 by plotting $\frac{y}{10^x}$ against $\frac{x}{10^x}$.

(19) Find the best values of m and c in $y = mx + c$, having given $x = 1$, $y = 4{\cdot}30$; $x = 3$, $y = 10{\cdot}66$; $x = 5$, $y = 16{\cdot}64$, $x = 7$, $y = 24{\cdot}10$. Compare the values of m and c with those obtained from a straight line graph.

(20) Find the best values of a, b, and c, if the equation $y = a + bx + cx^2$ is to fit most closely the following observations—

x .	-2	-1	0	1	2
y .	$-3{\cdot}150$	$-1{\cdot}390$	$0{\cdot}620$	$2{\cdot}880$	$5{\cdot}378$

(21) Find the best values of a and b if $y = ax + b \log_{10} x$ is the law which represents most closely the observed values given below.

x .	2	3	4	5	6
y .	3·74	5·99	7·47	8 92	9·86

(22) Sketch the graph of $y = \log_{10} x$, and show that $x \log_{10} x - 1$ has only one root. How many roots has the equation $\log_{10} x = \sin (x + c)$?

(23) A sum of £200 is invested at 5 per cent compound interest for 20 years. Find the amount at the end of the period on the following assumptions that the interest is added (1) yearly, (2) half-yearly, (3) quarterly, (4) continually at every instant of time.

(24) A sum of money is invested at simple interest, the rate being r per cent per annum. How long will it be before the total interest obtained is equal to the amount invested? Show that an equal sum of money invested for the same period at true compound interest (i.e. interest added continually) at the same rate per cent per annum would increase to e times itself.

(25) Explain how you would find the constants in the following laws from a series of more than three pairs of corresponding values of the variables found by experiment: (1) $y = a + bx^n$, (2) $y = b(x + a)^n$, (3) $y = a + bx + cx^2$, (4) $y = a + be^{nx}$.

(26) Given three pairs of values of x and y, namely (x_1, y_1), (x_2, y_2), and (x_3, y_3) which satisfy the law $y = a + bx^n$, and that $x_1x_3 = x_2^2$, find an equation involving only a and some or all of the given values of x and y. Solve this to find a.

(27) The following numbers satisfy approximately the law $y = a + bx^2$. Find (1) by modifying the relation and drawing the graph, (2) by forming and solving the normal equations, the best values of a and b.

x .	1	2	3	4	5
y	0·43	0·83	1·40	2 33	3·42

If the laws found by the two methods disagree, which is more likely to be true, and why?

(28) Plot the graph of $y = 0{\cdot}1x^2 + 2\log_{10}x + 1{\cdot}54$ from $x = 1$ to $x = 5$. Draw the straight line which most closely agrees with the curve, and find its equation.

(29) Show that if we assume the law $P = aW + b$ to connect the applied force P lb and the load raised W lb in a load-lifting machine, the law connecting the efficiency η and the load is $\eta = \dfrac{W}{a_1W + b_1}$. Find a_1 and b_1 from the following experimental values—

W .	31·25	87·25	143·25	199·25	255·25
η .	0·061	0·099	0·113	0·121	0·127

This machine has a very high velocity ratio, i.e. 205. Find the law connecting P and W, using the relation

$$\text{Efficiency} = \frac{\text{Load}}{\text{Applied force} \times \text{velocity ratio}}$$

(30) The law connecting the tensions T_1 and T_2 in the tight and slack sides respectively of a belt just on the point of slipping over the pulley surface is $\dfrac{T_1}{T_2} = e^{\mu\theta}$ where μ is the coefficient of friction and θ the angle of contact between the belt and pulley in radians. The following experimental values are given. Find the best value of μ.

$T_2 = 5$ lb for all values of θ—

θ . .	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
T_1 lb	5·57	6·14	6·84	7·61	8·47	9·34	10·32	11·45

(31) Find the best values of a, b, and c, if the parabola $y = a + bx + cx^2$ fits most closely the points defined by the following values of x and y—

x .	0	1	2	3	4
y .	2·14	6·32	16·57	33·37	56·01

Do this by forming and solving the three normal equations.

(32) Find the best values of a, b, and n if the law $y = a + be^{nx}$ fits closely the observed values of x and y—

x . .	2	4	6	8	10	12	14	16
y . .	4·000	4·280	4·631	5·015	5·445	5·880	6·370	6·935

(33) The following experimental values follow approximately the law $L = a + bN^c$. Find the best values of a, b, and c—

N .	5	7	10	15	30	40
L .	1·68	2·82	6·32	10·2	44·9	76·6

CHAPTER IX

AREA UNDER CURVE—SURFACE AND VOLUME OF SOLID OF REVOLUTION—LENGTH OF ARC—CENTROIDS —MOMENTS OF INERTIA—GRAPHICAL INTEGRATION

106. **Area under a Curve.** We have seen (Art. 47) how to find the area A of a figure like $PQNM$ (Fig. 82). If $OM = a$ and $ON = b$,

$$A = \int_a^b ydx \quad . \quad . \quad . \quad \text{(IX.1)}$$

or if $y = f(x)$,

$$A = \int_a^b f(x)dx \quad . \quad . \quad . \quad \text{(IX.2)}$$

Similarly, the area A' of the figure $PQSR$, where $OR = c$ and $OS = d$ is given by

$$A' = \int_c^d xdy \quad . \quad . \quad . \quad \text{(IX.3)}$$

Now consider the area $PQNM$ to be divided up into n strips by means of equidistant ordinates and let Δx be the distance between two adjacent ordinates. Consider the strip $CHID$ and let H be the point where the co-ordinates are x and y. Let Hd be drawn parallel to OX to meet ID in d. Then the area of the rectangle $CHdD$ is $y\Delta x$ and the area S of the stepped figure $MPaFbGcHdIeJfKgLhN$ is

$$S = \sum_{x=a}^{x=b} y\Delta x \quad . \quad . \quad . \quad \text{(IX.4)}$$

The difference between the areas A and S is the sum of the areas of the small portions PFa, FGb, etc., . . . LQh. As n, the number of strips, increases, the sum of the small areas decreases and, as n approaches the limit infinity, the value of S approaches the limit A. Thus the area A may be looked upon as being given by (1) the value of the definite integral $\int_a^b ydx$, or (2) the quantity $\underset{n\to\infty}{Lt.} \sum_{x=a}^{x=b} y\Delta x$ where $n\Delta x = b - a$. On account of the similarity of form between these two quantities, we often look upon the quantity $\int_a^b ydx$ as denoting

either the definite integral or the quantity (2), whichever point of view is most convenient Thus

$$A = \int_a^b y\,dx = \underset{n \to \infty}{Lt} \sum_{x=a}^{x=b} y\,\Delta x \qquad \text{(IX 5)}$$

The value of A may be found by evaluating the third term in (IX.5) but is generally much more easily found by evaluating the definite integral The use of (IX 5) enables us to obtain the values of quantities which can be expressed in the form $\underset{n \to \infty}{Lt} \sum_{x=a}^{x=b} f(x)\Delta x$, by

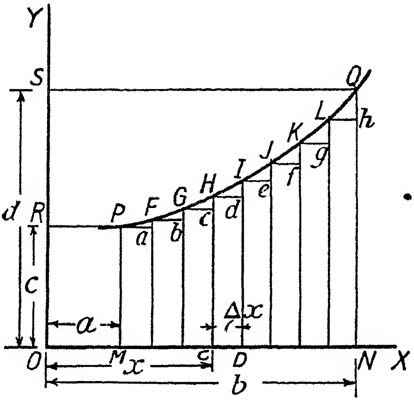


FIG 82

replacing the latter by the definite integral $\int_a^b f(x)dx$.

If throughout part of the range $x = a$ to $x = b$, $f(x)$ is negative, the quantity $\int_a^b f(x)dx$ will give the excess (positive or negative) of the area above OX over that below OX.

EXAMPLE

Prove that the area of the oval of the curve whose equation is

$$ay^2 = (x - a)(x - 5a)^2 \text{ is } \tfrac{256}{15} a^2 \qquad \text{(U L)}$$

Assuming a to be positive, it is clear that there is no part of the graph to the left of the ordinate $x = a$, for otherwise y^2 would be negative. Also the curve is symmetrical about OX. $y = 0$ when $x = a$, and when $x = 5a$ only, and so the oval referred to in the question lies between $x = a$ and $x = 5a$ We have

$$\text{Area of oval} = 2\int_a^{5a} y\,dx = \frac{2}{\sqrt{a}}\int_a^{5a} (5a - x)\sqrt{x - a}\,dx$$

Let $x - a = z^2$, so that $z = 0$ when $x = a$ and $z = 2\sqrt{a}$ when $x = 5a$. Also $dx = 2z\,dz$, and

$$\text{Area of oval} = \frac{2}{\sqrt{a}}\int_0^{2\sqrt{a}} (4a - z^2)z\;2z\,dz = \frac{4}{\sqrt{a}}\int_0^{2\sqrt{a}} (4az^2 - z^4)\,dz$$

$$= \frac{4}{\sqrt{a}}\left[\frac{4az^3}{3} - \frac{z^5}{5}\right]_0^{2\sqrt{a}} = \frac{4}{\sqrt{a}}\left(\frac{32a^{5/2}}{3} - \frac{32a^{5/2}}{5}\right)$$

$$= \tfrac{256}{15}\,a^2$$

107. Mean Value. Root Mean Square. The mean value of the ordinate y in Fig 82 over the range $x = a$ to $x = b$ is the limit as n approaches infinity of the mean value of the equidistant ordinates $y_1, y_2, y_3, \ldots, y_n$. Thus if $\bar{y}$ is the mean value

$$\bar{y} = \underset{n\to\infty}{Lt}\; \frac{y_1 + y_2 + y_3 + \ldots + y_{n-1} + y_n}{n}$$

$$= \underset{n\to\infty}{Lt}\; \frac{y_1\Delta x + y_2\Delta x + y_3\Delta x + \ldots + y_n\Delta x}{n\Delta x}$$

$$\bar{y} = \frac{\underset{n\to\infty}{Lt}\; \sum_{x=a}^{x=b} y\Delta x}{b-a}$$

or

$$\bar{y} = \frac{\int_a^b y\,dx}{b-a} \qquad \text{(IX 6)}$$

(IX.6) may be written

$$\text{Average height of graph} = \frac{\text{Area under graph}}{\text{Length of base}}$$

EXAMPLE 1

Find the mean value of $\sin(nx + \alpha)$ over the range (1) $x = -\frac{\alpha}{n}$ to $x = -\frac{\alpha}{n} + 2\pi p$, and (2) $x = -\frac{\alpha}{n}$ to $x = -\frac{\alpha}{n} + \frac{\pi}{2n}$, where n and p are positive integers

(1) $$\bar{y} = \frac{\int_{-\frac{\alpha}{n}}^{-\frac{\alpha}{n}+2\pi p} \sin(nx+\alpha)\,dx}{2\pi p} = \frac{1}{2\pi p}\left[-\frac{1}{n}\cos(nx+\alpha)\right]_{-\frac{\alpha}{n}}^{-\frac{\alpha}{n}+2\pi p}$$

$$= \frac{1}{2\pi p}\left\{-\frac{1}{n}\cos 2\pi pn + \frac{1}{n}\cos 0\right\}$$

$$= \frac{1}{2\pi pn}(1 - \cos 2\pi pn) = \frac{1}{2\pi pn}(1 - 1)$$

$\therefore \quad \bar{y} = 0$

$$(2) \quad \bar{y} = \frac{2n}{\pi}\left[-\frac{1}{n}\cos(nx+\alpha)\right]_{-\frac{\alpha}{n}}^{-\frac{\alpha}{n}+\frac{\pi}{2n}}$$

$$= \frac{2n}{\pi n}\left\{-\cos\frac{\pi}{2}+\cos 0\right\}$$

$$= \frac{2}{\pi}\{1-0\} = \frac{2}{\pi}$$

The "Root Mean Square" is a term usually applied to periodic functions only. If y is a periodic function of x, of period a, the root mean square of y is the square root of the mean value over the range $x = c$ to $x = c + a$ of the square of y, c being any constant. If R.M.S. denotes the root mean square of y, then

$$\text{R.M.S.} = \sqrt{\frac{\int_c^{c+a} y^2 dx}{a}} \quad . \qquad \text{(IX.7)}$$

EXAMPLE 2

(*a*) Find the mean value of $\sin^4 pt$ over the range $t = 0$ to $t = \frac{\pi}{p}$

(*b*) Show that the R.M.S. value of the expression

$$a \sin pt + b \cos qt + c \sin rt + d \cos st + \ldots$$

is $\sqrt{\frac{1}{2}(a^2 + b^2 + c^2 + d^2 + \ldots)}$, $p, q, r, s, \ldots$ being integers.

$$(a) \quad \sin^4 pt = (\sin^2 pt)^2 = \left(\frac{1-\cos 2pt}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2pt + \cos^2 2pt)$$

$$= \frac{1}{4}\left\{1 - 2\cos 2pt + \frac{1+\cos 4pt}{2}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} - 2\cos 2pt + \frac{1}{2}\cos 4pt\right\}$$

$\therefore \quad y_m =$ mean value of $\sin^4 pt$

$=$ mean value of $\frac{1}{4}\left\{\frac{3}{2} - 2\cos 2pt + \frac{1}{2}\cos 4pt\right\}$

$$\text{By (IX.6) } y_m = \frac{p}{\pi}\int_0^{\frac{\pi}{p}} \frac{1}{4}\left\{\frac{3}{2} - 2\cos 2pt + \frac{1}{2}\cos 4pt\right\}dt \qquad (1)$$

$$= \frac{p}{4\pi}\left[\frac{3}{2}t - \frac{1}{p}\sin 2pt + \frac{1}{8p}\sin 4pt\right]_0^{\frac{\pi}{p}}$$

$$= \frac{p}{4\pi}\left(\frac{3\pi}{2p}\right) = \frac{3}{8} \quad . \quad . \qquad (2)$$

It is evident on inspection that the terms $-2\cos 2pt$ and $\frac{1}{2}\cos 4pt$ under the integral sign in (1) contribute nothing to the result (2), because the integration of each gives the product of a constant and the area under an integral number of complete waves of a cosine curve, which area is zero.

(*b*) For all integral values of p, q, r, etc., the given expression covers a complete number of periods as t increases from 0 to 2π. We take this interval as the range of integration.

The mean square of the expression

$$\frac{1}{2\pi}\int_0^{2\pi}(a\sin pt + b\cos qt + c\sin rt + \ldots)^2 dt \quad . \quad (1)$$

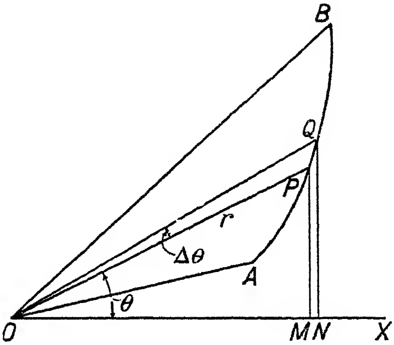


FIG. 83

On squaring the quantity in the brackets we obtain the sum of terms such as $a^2\sin^2 pt$, $b^2\cos^2 qt$, etc., together with the sum of terms like $2ab\sin pt\cos qt$, $2ac\sin pt\sin rt$, etc.

$$\text{Now } \int_0^{2\pi}\sin pt\cos qt\,dt = \int_0^{2\pi}\sin pt\sin rt\,dt$$

$$= \int_0^{2\pi}\cos qt\cos st\,dt = 0 \text{ (Art. 50)}$$

$$\text{and} \quad \int_0^{2\pi}\sin^2 pt\,dt = \int_0^{2\pi}\cos^2 qt\,dt = \int_0^{2\pi}\sin^2 rt\,dt = \pi \text{ (see Art. 49)}$$

Hence, the value of the integral in (1) is $\pi(a^2 + b^2 + c^2 + \ldots)$ and

$$\text{R.M.S.} = \text{root mean square value}$$

$$= \sqrt{\frac{1}{2\pi}\pi(a^2 + b^2 + c^2 + \ldots)}$$

$$= \sqrt{\tfrac{1}{2}(a^2 + b^2 + c^2 + \ldots)}$$

108. **Area Included by the Curve $r = f(\theta)$ and Two Radius Vectors, where (r, θ) are Polar Co-ordinates.** In Fig. 83, AB is the curve whose polar equation is $r = f(\theta)$. O is the pole and OX the initial

line. Let A be the area included between the arc AB and the radius vectors OA, OB where the angles AOX and BOX are θ_1 and θ_2 respectively.

Let P be any point (r, θ) and Q the point $(r + \Delta r, \theta + \Delta\theta)$ on the curve.

Then Area $OPQ = \frac{1}{2}OP \,.\, OQ \sin POQ = \frac{1}{2}r(r + \Delta r)\sin\Delta\theta$

As $\Delta\theta$ approaches the value zero, this area approaches the value $\frac{1}{2}r^2\,\Delta\theta$ and therefore

$$A = \underset{\Delta\theta \to 0}{Lt.} \sum_{\theta = \theta_1}^{\theta = \theta_2} \tfrac{1}{2}r^2\,\Delta\theta$$

or, by (IX.5),

$$A = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta \quad . \quad . \quad . \quad . \quad \text{(IX.8)}$$

EXAMPLE 1

Find the area bounded by the cardioid $r = a(1 + \cos\theta)$ (Fig. 91). Here the limits of integration are $\theta = 0$ to $\theta = 2\pi$. Hence,

$$A = \text{required area} = \frac{1}{2}\int_0^{2\pi} a^2(1 + \cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi a^2}{2}$$

Referring again to Fig. 83, let (x, y) and $(x + \Delta x, y + \Delta y)$ be the co-ordinates of P and Q respectively, and let PM and QN be drawn perpendicular to OX.

Then let ΔA = area OPQ

$$= \text{area } OQN - \text{area } OPM - \text{area } PMNQ$$

$$= \tfrac{1}{2}(x + \Delta x)(y + \Delta y) - \tfrac{1}{2}xy - \tfrac{1}{2}(2y + \Delta y)\,\Delta x$$

$$\therefore \quad \Delta A = \tfrac{1}{2}(x\Delta y - y\Delta x)$$

and

$$A = \text{area } OAB = \int_{\theta_1}^{\theta_2} dA$$

$$= \tfrac{1}{2}\int_{\theta = \theta_1}^{\theta = \theta_2}(x\,dy - y\,dx) \quad . \quad . \quad . \quad \text{(IX.9)}$$

The following example illustrates the use of (IX.9) in the case of an area enclosing the origin, the parametric equations to the curve being $x = f_1(t)$, $y = f_2(t)$.

EXAMPLE 2

Find the area of the ellipse given by $x = 5\cos\phi$, $y = 3\sin\phi$.

Here $dy = 3\cos\phi\,d\phi$; $dx = -5\sin\phi\,d\phi$, and, since the origin is at the centre of the ellipse, the limits are $\phi = 0$ to $\phi = 2\pi$.

$\therefore$ From (IX.9) $A = \frac{1}{2}\int_0^{2\pi} 15(\cos^2\phi + \sin^2\phi)d\phi = \frac{1}{2}\int_0^{2\pi} 15d\phi$

$\therefore \qquad A = \frac{15}{2} \times 2\pi = 15\pi$ square units

109. **Area of a Closed Curve in Terms of Rectangular Co-ordinates.** Let P_1, P_2 (Fig. 84) be the points (x, y_1), (x, y_2) respectively. Then the

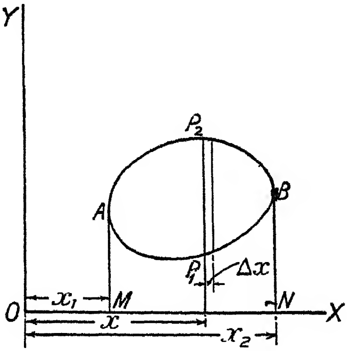


FIG. 84

area enclosed by the curve is the difference between the areas of the figures MAP_2BN and MAP_1BN, where AM and BN are vertical tangents to the curve. Hence,

$$A = \text{area of closed curve}$$

$$= \int_{x_1}^{x_2} y_2dx - \int_{x_1}^{x_2} y_1dx$$

$$= \int_{x_1}^{x_2} (y_2 - y_1)dx \qquad . \quad . \quad . \quad \text{(IX.10)}$$

where $x_1 = OM$ and $x_2 = ON$

EXAMPLE

Find the area of either of the halves into which the ellipse represented by $13x^2 + 10xy + 13y^2 = 72$ is divided by the y axis.

The equation can be written

$$y^2 + \frac{10}{13}xy + \left(\frac{5}{13}\right)^2 x^2 = \frac{72}{13} - x^2 + \frac{25}{169}x^2$$

i.e.
$$\left(y + \frac{5}{13}x\right)^2 = \frac{72}{13} - \frac{144}{169}x^2 = \frac{144}{169}(6\cdot5 - x^2)$$

whence
$$y = -\frac{5}{13}x \pm \frac{12}{13}\sqrt{(6\cdot5 - x^2)}$$

$\therefore$
$$y_2 - y_1 = \frac{24}{13}\sqrt{6\cdot5 - x^2}$$

which is zero when $x = \pm\sqrt{6\cdot5}$. Thus the limits of integration for half the ellipse are $-\sqrt{6\cdot5}$ and 0, or 0 and $\sqrt{6\cdot5}$. Taking the latter limits, we have from (IX.10)

$$A = \text{area of half ellipse}$$
$$= \int_0^{\sqrt{6\cdot5}} \frac{24}{13}\sqrt{6\cdot5 - x^2}\, dx$$

Let $x = \sqrt{6\cdot5}\sin\theta$, then $dx = \sqrt{6\cdot5}\cos\theta\, d\theta$. When $x = 0$, $\theta = 0$, and when $x = \sqrt{6\cdot5}$, $\theta = \frac{\pi}{2}$. Hence, substituting,

$$A = \int_0^{\frac{\pi}{2}} \frac{24}{13} \times 6\cdot5\cos^2\theta\, d\theta = 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$
$$= 3\pi$$

110. **Odd and Even Functions.** If $f(x)$ is any function of x such that $f(x) = f(-x)$, $f(x)$ is an *even function of* x. If $f(x) = -f(-x)$, $f(x)$ is an odd function of x. Sin x, x, x^3, $ax + bx^3 + cx^5 + dx^7$, $\sinh x$, $\tan x$ are odd functions of x because a change of the sign of x, whilst leaving unchanged the numerical value of each function, changes the sign of that value. Graphs of odd functions are symmetrical about the origin (Art. 97). Cos x, x^2, $\cosh x$, $ax^2 + bx^4 + cx^6$ are even functions of x. Graphs of even functions of x are symmetrical about the axis of y. If $f(x)$ is an odd function

$$\int_{-k}^{k} f(x)dx = 0 \quad . \quad . \qquad \text{(IX.11)}$$

If $f(x)$ is an even function

$$\int_{-k}^{k} f(x)dx = 2\int_0^k f(x)dx \quad . \qquad \text{(IX.12)}$$

because in (IX.11)

$$\int_{-k}^{0} f(x)dx = -\int_0^k f(x)dx, \text{ (odd function)}$$

whilst in (IX.12)

$$\int_{-h}^{0} f(x)dx = \int_{0}^{h} f(x)dx, \text{ (even function)}$$

The reader should verify these results by testing them in the cases of the functions given above.

As examples of (IX.11) and (IX.12) we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = 0 \text{ and } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = 2\int_{0}^{\frac{\pi}{2}} \cos x \, dx = 2 \times 1 = 2$$

111. **Volumes of Solids and Surface Areas of Solids of Revolution.** Fig. 85 shows a portion AB of a plane curve. Suppose AB to be

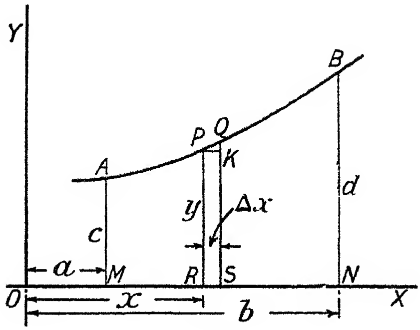


FIG. 85

rotated about OX. In making a complete revolution the area $ABNM$ generates a frustum of a solid of revolution. The ordinates MA and NB generate the end sections of the frustum and the curve AB generates the curved surface. Let P be any point on the curve and let x and y be the co-ordinates of P. Let Q be a point on the curve near P so that $RS = \Delta x$ where PR and QS are ordinates to the curve.

The figure $RPQS$ generates a solid whose volume is $\pi y^2 \Delta x$ to the first order of small quantities. Hence, the total volume is

$$V = \underset{\Delta x \to 0}{Lt.} \sum_{x=a}^{x=b} \pi y^2 \Delta x$$

where $a = OM$ and $b = ON$. By (IX.5) this gives

$$V = \int_{a}^{b} \pi y^2 \, dx \qquad . \quad . \quad . \quad \text{(IX.13)}$$

If S is the area of the curved surface generated by AB, the portion of S generated by PQ is $2\pi y\ \Delta s$ to the first order of small quantities, where Δs is the length of PQ. Hence,

$$S = \underset{\Delta s \to 0}{Lt.} \Sigma\, 2\pi y\ \Delta s$$

where the summation extends over the arc AB. By (IX.5) this is

$$S = \int_{x=a}^{x=b} 2\pi y ds \quad . \quad . \quad . \quad \text{(IX.14)}$$

Now from the triangle PQK, in which PK is parallel to OX,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

or

$$\Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

and

$$S = \underset{\Delta x \to 0}{Lt.} \Sigma\, 2\pi y \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

or

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad . \quad \text{(IX.15)}$$

Similarly

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \quad . \quad \text{(IX.16)}$$

c and d being the limits of y corresponding to $x = a$ and $x = b$ respectively.

If the area bounded by AB, the axis of y, and horizontal lines through A and B is rotated about OY the formulae corresponding to (IX.13) and (IX.14) are respectively

$$V = \int_c^d \pi x^2 dy \text{ and } S = \int_{y=c}^{y=d} 2\pi x ds$$

If the section of a body at right angles to a fixed axis is not circular but its area is a function $\phi(x)$ of x, the distance of the section from a fixed point on the axis, then the volume is given by

$$V = \int_a^b \phi(x) dx \quad . \quad . \quad . \quad \text{(IX.17)}$$

where a and b are the limiting values of x.

EXAMPLE 1

Find the volume of a segment of height h, of a sphere of radius a.

Water is being poured into a hemispherical cup of radius a at the rate of c ft^3 per sec. Find the rate at which the depth of the water is increasing when the cup is five-sixteenths full. (U.L.)

The equation of the generating circle is $x^2 + y^2 = a^2$, the centre being the origin and the x-axis being perpendicular to the plane which cuts off the segment.

$$\text{Volume of segment} = \int_{a-h}^{a} \pi y^2 dx = \pi \int_{a-h}^{a} (a^2 - x^2)dx$$
$$= \pi\left[a^2x - \frac{x^3}{3}\right]_{a-h}^{a}$$
$$= \pi[a^2(a - a + h) - \tfrac{1}{3}\{a^3 - (a-h)^3\}]$$
$$= \pi[a^2h - \tfrac{1}{3}(3a^2h - 3ah^2 + h^3)]$$
$$= \frac{\pi h}{3}[3ah - h^2]$$

Let h be the depth of the water in the cup at any instant. Then the volume V of water $= \frac{\pi}{3}[3ah^2 - h^3]$.

$$\therefore \qquad \frac{dV}{dt} = \frac{\pi}{3}[6ah - 3h^2]\frac{dh}{dt} = \pi[2ah - h^2]\frac{dh}{dt}$$

Now $\frac{dV}{dt} = c$ (given), and we have to find $\frac{dh}{dt}$ when $V = \frac{5}{16}\cdot\frac{2}{3}\pi a^3 = \frac{5}{24}\pi a^3$

i.e. when $\qquad \frac{5}{24}\pi a^3 = \frac{\pi}{3}[3ah^2 - h^3]$

i.e. „ $\qquad h^3 - 3ah^2 + \frac{5}{8}a^3 = 0$

i.e. „ $\qquad h = \frac{a}{2}$

then $\qquad \frac{dh}{dt} = \dfrac{c}{\pi\left[a^2 - \frac{a^2}{4}\right]} = \frac{4c}{3\pi a^2}$ ft per sec

EXAMPLE 2

A portion of a paraboloid of revolution is cut off by a plane perpendicular to its axis, so that the circular face is of radius R. Show that the volume of the portion is $\frac{1}{4}\pi R^4/l$, where $2l$ is the latus-rectum of the meridian section.

A cylindrical vessel of height h and radius R is full of water, and has its axis vertical. It is slowly set in rotation about its axis and attains a constant angular velocity ω. Show that the centre of the base will be uncovered if $\omega^2 > 2gh/R^2$, but that if ω^2 is less than this value the volume of water spilt is $\frac{1}{4}\pi R^4\omega^2/g$.

(U.L.)

We take the axis of the generating parabola as vertical (Fig. 86). Its equation referred to the axes OX and OY shown in the figure is $x^2 = 2ly$. $\overline{MP} = R$ and $\overline{OM} = \frac{R^2}{2l}$

$$\therefore \text{Volume of portion} = \int_0^{\frac{R^2}{2l}} \pi x^2 dy = \pi\int_0^{\frac{R^2}{2l}} 2ly\,dy = \pi l[y^2]_0^{\frac{R^2}{2l}} = \tfrac{1}{4}\pi R^4/l$$

Let a vertical plane cut the free surface of the water along the curve OQP. Consider an element of water of mass m situated at $Q(x, y)$ on the free surface. Draw $QN \perp OY$, QG normal to the curve at Q. The acceleration of m is $\overline{QN} \cdot \omega^2$ towards N, so that we can regard it as being at rest under its own weight mg acting vertically downwards, the resultant thrust on it acting along QG, and a force $m \cdot \overline{QN} \cdot \omega^2$ acting along NQ. GNQ is then a triangle of forces, and

$$\therefore \frac{\overline{GN}}{\overline{NQ}} = \frac{mg}{m\overline{QN}\omega^2} \text{ whence } \overline{GN} = \frac{g}{\omega^2}$$

$$\text{Now } \frac{dy}{dx} = \tan\psi = \tan NGQ = \frac{x}{\overline{GN}} = \frac{\omega^2}{g}x, \text{ whence } y = \frac{\omega^2 x^2}{2g}$$

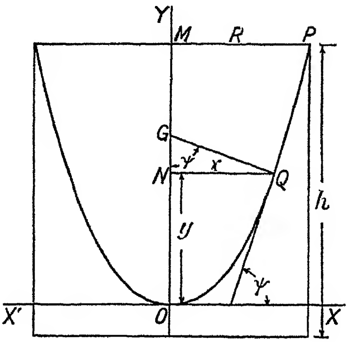


FIG. 86

If O is at the centre of the base, then $y = h$ when $x = R$,

i.e
$$h = \frac{\omega^2 R^2}{2g} \text{ or } \omega^2 = \frac{2gh}{R^2}$$

If $\omega^2 > \frac{2gh}{R^2}$, y will be $> h$ when $x = R$, and the centre of the base will be uncovered.

If $\omega^2 < \frac{2gh}{R^2}$, then the volume of the water spilt is that found in first part with $\frac{g}{\omega^2}$ substituted for l,

i.e.
$$\text{Volume spilt} = \tfrac{1}{4}\pi R^4\omega^2/g$$

EXAMPLE 3

Find the area of the curved surface of the cup formed by the revolution about its axis of the smaller part of the parabola $y^2 = 4ax$ cut off by the line $x = 3a$. (U.L.)

$$\text{The surface area required} = \int_0^{3a} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Here $2y\dfrac{dy}{dx} = 4a \quad \therefore \dfrac{dy}{dx} = \dfrac{2a}{y}$ and $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\dfrac{4a^2}{4ax}} = \sqrt{\dfrac{x+a}{x}}$

$$\therefore \text{ Surface area} = 2\pi\int_0^{3a} 2\sqrt{a}\,\sqrt{x}\cdot\frac{\sqrt{x+a}}{\sqrt{x}}\,dx$$

$$= 4\pi\sqrt{a}\int_0^{3a}\sqrt{x+a}\,dx$$

$$= 4\pi\sqrt{a}\cdot\frac{2}{3}\left[(x+a)^{\frac{3}{2}}\right]_0^{3a}$$

$$= \frac{8\pi\sqrt{a}}{3}\left[8a^{\frac{3}{2}} - a^{\frac{3}{2}}\right] = \frac{56\pi a^2}{3}$$

EXAMPLE 4

An ellipse whose semi-major and semi-minor axes are 2 ft and 1 ft respectively is rotated about its major axis. Find the surface area of the prolate spheroid so formed.

The equation of the ellipse is $\dfrac{x^2}{4} + y^2 = 1$

$$\therefore \qquad \frac{x}{2} + 2y\frac{dy}{dx} = 0, \text{ whence } \frac{dy}{dx} = -\frac{x}{4y} \text{ and}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{16y^2} = \frac{16y^2+x^2}{16y^2} = \frac{16-3x^2}{16y^2}$$

$$\therefore \text{ Surface area required} = 2\int_0^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi\int_0^2 y\cdot\frac{\sqrt{16-3x^2}}{4y}\,dx$$

$$= \sqrt{3}\,\pi\int_0^2\sqrt{\frac{16}{3}-x^2}\,dx$$

Let $x = \dfrac{4}{\sqrt{3}}\sin\theta$. $\therefore dx = \dfrac{4}{\sqrt{3}}\cos\theta\,d\theta$, and $\theta = 0, \dfrac{\pi}{3}$ when $x = 0, 2$.

$$\therefore \qquad \text{Surface area} = \sqrt{3}\,\pi\int_0^{\frac{\pi}{3}}\frac{4}{\sqrt{3}}\cos\theta\cdot\frac{4}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \frac{16\pi}{\sqrt{3}}\cdot\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{8\pi}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$= \frac{8\pi}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right] = 21{\cdot}5 \text{ ft}^2 \text{ (nearly)}$$

112. **Lengths of Curves.** Let PQ be an element of arc of length Δs of the curve $y = f(x)$, Fig. 85. The length of the arc AB is given by

$$L_{AB} = \int_{s_A}^{B} ds \quad . \quad . \quad . \quad \text{(IX.18)}$$

where s_A and s_B are the distances of A and B, measured along the curve, from some fixed point on it. If the co-ordinates of A and B are (a, c) and (b, d) respectively, then from Art. 111 we see that

$$L_{AB} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad . \quad . \quad \text{(IX.19)}$$

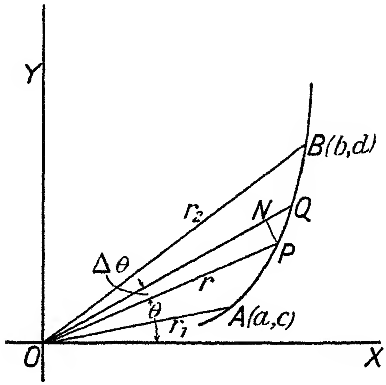


FIG. 87

or
$$L_{AB} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad . \quad . \quad \text{(IX.20)}$$

If the equation to AB is given in polar co-ordinates as $r = f(\theta)$, we have, if PN is perpendicular to OQ (Fig. 87),

$$PN = r \sin \Delta\theta = r\Delta\theta,$$

and
$$NQ = r + \Delta r - r \cos \Delta\theta = \Delta r,$$

to the first order of small quantities. (See Art. 77.)

Hence,
$$(\Delta s)^2 = r^2(\Delta\theta)^2 + (\Delta r)^2$$

$\therefore$
$$\Delta s = \sqrt{r^2(\Delta\theta)^2 + (\Delta r)^2}$$

$$= \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}\,.\,\Delta\theta \text{ or } \sqrt{r^2\left(\frac{\Delta\theta}{\Delta r}\right)^2 + 1}\,.\,\Delta r$$

Therefore,
$$L_{AB} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$= \int_{r_1}^{r_2} \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}\, dr \quad . \quad . \quad \text{(IX.21)}$$

where $\widehat{AOX} = \theta_1$, $\widehat{BOX} = \theta_2$, $OA = r_1$, $OB = r_2$

EXAMPLE 1

Show that the length of the arc of the parabola $y^2 = 4ax$ measured from $(0, 0)$ is $\int_0^x \sqrt{1 + \frac{a}{x}}\, dx$.

Integrate this by a hyperbolic substitution (or otherwise) and evaluate it between $(0, 0)$ and $(a, 2a)$. (U.L.)

As in Art. 111, Ex. 3, $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{a}{x}}$

$\therefore$ Length of arc $= \int_0^x \sqrt{1 + \frac{a}{x}}\, dx$

To find $\int_0^a \sqrt{\frac{a + x}{x}}\, dx$, let $x = a \sinh^2 \theta$, so that $dx = 2a \sinh\theta \cosh\theta\, d\theta$, and
$\sqrt{\frac{a + x}{x}} = \frac{\cosh\theta}{\sinh\theta}$

Hence,
$$\int \sqrt{\frac{a + x}{x}}\, dx = 2a \int \frac{\cosh\theta}{\sinh\theta} \sinh\theta \cosh\theta\, d\theta$$
$$= a \int (1 + \cosh 2\theta)\, d\theta$$
$$= a \left[\theta + \frac{\sinh 2\theta}{2}\right]$$

Now, when $x = 0$, $\sinh\theta = 0$, and therefore $\theta = 0$; also when $x = a$, $\sinh\theta = 1$; $\cosh\theta = \sqrt{2}$; $e^\theta = \sinh\theta + \cosh\theta = 1 + \sqrt{2}$, so that $\theta = \log_e (1 + \sqrt{2})$.

Hence,
$$\int_0^a \sqrt{\frac{a + x}{x}}\, dx = a[\log_e (1 + \sqrt{2}) + \sqrt{2}] = 2{\cdot}296a.$$

EXAMPLE 2

A cyclist's track on a straight road is the curve $y = \sin \frac{\pi x}{20}$, the direction of the axis of x being that of the road and the unit of x and y being 1 yd. Find approximately the excess registered by his cyclometer between two successive milestones, assuming the cyclometer and the milestones to be accurate. (U.L.)

Here $\frac{dy}{dx} = \frac{\pi}{20}\cos\frac{\pi x}{20}$ and $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(1+\frac{\pi^2}{400}\cos^2\frac{\pi x}{20}\right)^{\frac{1}{2}}$

$$= 1 + \frac{\pi^2}{800}\cos^2\frac{\pi x}{20} \text{ approx.}$$

(on expanding by the Binomial Theorem, $\frac{\pi^2}{400}\cos^2\frac{\pi x}{20}$ being small).

$\therefore$ Distance registered by cyclometer $= \int_0^{1\,760}\left(1+\frac{\pi^2}{800}\cos^2\frac{\pi x}{20}\right)dx$

$$= \left[x + \frac{\pi^2}{1\,600}\left(x + \frac{\sin\frac{\pi x}{10}}{\frac{\pi}{10}}\right)\right]_0^{1\,760}$$

$$= 1\,760 + \frac{\pi^2}{1\,600}(1\,760)$$

$\therefore$ Excess required $= 1{\cdot}1\pi^2 = 11$ yd (nearly)

EXAMPLE 3

Find the length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ from the pole to the point (r, θ).

Here $\frac{dr}{d\theta} = \cot\alpha \times ae^{\theta\cot\alpha} = r\cot\alpha$; also $\theta = -\infty$ when $r = 0$.

$\therefore$ Length of arc required

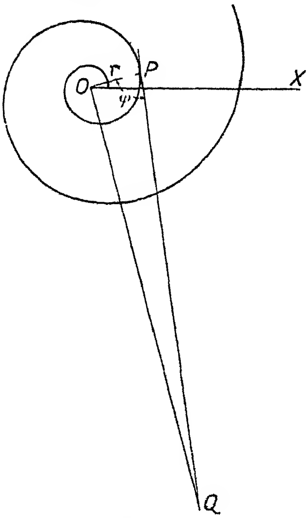


FIG. 88

$$= \int_{-\infty}^{\theta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \int_{-\infty}^{\theta}\sqrt{r^2+r^2\cot^2\alpha}\,d\theta$$

$$= \operatorname{cosec}\alpha\int_{-\infty}^{\theta} ae^{\theta\cot\alpha}\,d\theta$$

$$= \frac{a\operatorname{cosec}\alpha}{\cot\alpha}\left[e^{\theta\cot\alpha}\right]_{-\infty}^{\theta}$$

$$= \frac{a}{\cos\alpha}\left(e^{\theta\cot\alpha} - 0\right)$$

$$= r\sec\alpha$$

The curve is shown in Fig. 88. With the notation of Art. 77,

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{r}\,r\cot\alpha = \cot\alpha$$

$$\therefore\quad \phi = \alpha$$

i.e. the tangent makes a constant angle α with the radius vector. If OQ be drawn $\perp$ OP to meet the tangent at P in Q, then $PQ = r \sec \alpha =$ length of arc from pole up to P.

113. Approximate Integration (Simpson's Rule) for Areas and Volumes. Let the area under a curve be divided up into an *even* number n of strips of equal width h by ordinates $y_1, y_2, y_3, \ldots y_{n+1}$. Let the ordinates y_1, y_2, y_3 cut the curve in the points P_1, P_2, P_3. Take the foot of the ordinate y_2 as origin and assume that the equation $y = ax^2 + bx + c$ can represent the portion $P_1P_2P_3$ of the curve with sufficient accuracy.

Substituting $x = -h, 0, h$ in $y = ax^2 + bx + c$, we have

$$y_1 = a(-h)^2 + b(-h) + c = ah^2 - bh + c \quad . \qquad \text{(IX.22)}$$

$$y_2 = c \quad . \quad . \quad . \qquad \text{(IX.23)}$$

$$y_3 = ah^2 + bh + c \quad . \qquad \text{(IX.24)}$$

$$\therefore \quad y_1 + 4y_2 + y_3 = 2ah^2 + 6c \quad . \quad . \quad . \quad . \qquad \text{(IX.25)}$$

Now, by integration, the area A under the portion $P_1P_2P_3$ is given by

$$A = \int_{-h}^{h} (ax^2 + bx + c)\,dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \tfrac{2}{3}ah^3 + 2ch$$

$$= \frac{h}{3}[2ah^2 + 6c]$$

or

$$A = \frac{h}{3}[y_1 + 4y_2 + y_3] \qquad \text{(IX.26)}$$

(It is evident that the addition of a term or any number of terms of odd degree in x to the expression $ax^2 + bx + c$ will not affect the result just obtained.)

By similar reasoning, we can show that the area under the portion $P_3P_4P_5$ of the curve is $\frac{h}{3}[y_3 + 4y_4 + y_5]$; and so on, the area under the last portion $P_{n-1}P_nP_{n+1}$ being

$$\frac{h}{3}[y_{n-1} + 4y_n + y_{n+1}]$$

Hence, the whole area

$$A = \frac{h}{3}[y_1 + 4y_2 + y_3 + y_3 + 4y_4 + y_5 + \dots + y_{n-1} + 4y_n + y_{n+1}]$$

$$= \frac{h}{3}[(y_1 + y_{n+1}) + 4(y_2 + y_4 + \dots + y_n) + 2(y_3 + y_5 + \dots + y_{n-1})] \qquad \text{(IX 27)}$$

or $A = \frac{h}{3}$ [(sum of first and last ordinates) + 4 (sum of even ordinates) + 2(sum of remaining odd ordinates)] (IX 28)

If the ordinates represent the areas $A_1, A_2, A_3 \dots A_{n+1}$ of the sections of a solid made by planes at equal distances h apart, then the volume of the solid is

$$V = \frac{h}{3}[(A_1 + A_{n+1}) + 4(A_2 + A_4 + \dots + A_n) + 2(A_3 + A_5 + \dots + A_{n-1})] \qquad \text{(IX 29)}$$

EXAMPLE 1

The under-water portion of a vessel is divided by horizontal planes 1 ft apart of the following areas 472, 398, 302, 198, 116, 60, 34, 12, 4 ft² Find the volume in cubic feet between the extreme areas (U L)

We arrange the work as follows —

Area of Section	Multiplier	Result
$A_1 = 472$	1	472
$A_2 = 398$	4	1 592
$A_3 = 302$	2	604
$A_4 = 198$	4	792
$A_5 = 116$	2	232
$A_6 = 60$	4	240
$A_7 = 34$	2	68
$A_8 = 12$	4	48
$A_9 = 4$	1	4
		Sum=4 052

Volume required $= \frac{1}{3}(4\,052) = 1\,351$ ft³ (nearly)

The above shows a convenient method of arranging the working, but there is actually less working to be done if the quantities in the left-hand column are taken in sets as in (IX 29) Thus $A_1 + A_9 = 476$, $A_2 + A_4 + A_6 + A_8 = 668$, $A_3 + A_5 + A_7 = 452$

Hence, by (IX 29),

$$V = \tfrac{1}{3}(476 + 4 \times 668 + 2 \times 452)$$
$$= \tfrac{1}{3} \times 4\,052 = 1\,351 \text{ ft}^3 \text{ as before}$$

The number of ordinates given in this case is not sufficient for a very accurate determination of the volume Where greater accuracy is required a graph should be plotted between the area A of any section and the distance of this section from a fixed plane From this graph a sufficient number of equidistant ordinates should be measured, and these values should then be substituted in (IX 29) along with the appropriate value of h Simpson's Rule requires for its application that there should be an odd number of equidistant values of y or A For ordinary accuracy this number should not usually be less than eleven except in the cases mentioned below

EXAMPLE 2

A tank is discharging water through an orifice at a depth x ft below the water surface, the area of the water surface being A ft^2 The table below gives corresponding values of A and x

A	12.57	13.91	15.20	16.50	18.09	19.62	21.23	22.95	24.62	26.50	28.27
x	5	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10

The water level falls from 10 ft to 5 ft above the orifice in time T sec given by

$$0.301\,T = \int_5^{10} \frac{A}{\sqrt{x}}\,dx$$

Using some graphical or numerical method, determine the value of T

If $\dfrac{A}{\sqrt{x}}$ is plotted against x between $x = 5$ and $x = 10$, then $\displaystyle\int_5^{10} \frac{A}{\sqrt{x}}\,dx$ gives the area under the curve between these limits

A is not known as a function of x, so that this area must be found by an approximate method

The values A_1, A_2, A_3, etc, of $\dfrac{A}{\sqrt{x}}$ corresponding to the given values of x are 5.62, 5.93, 6.21, 6.47, 6.84, 7.16, 7.51, 7.87, 8.21, 8.60, 8.94 respectively, and as these values are equally spaced and their number is odd, Simpson's Rule may be applied at once

$$A_1 + A_{11} = 5.62 + 8.94 = 14.56$$
$$A_2 + A_4 + A_6 + A_8 + A_{10} = 5.93 + 6.47 + 7.16 + 7.87 + 8.60 = 36.03,$$
$$A_3 + A_5 + A_7 + A_9 = 6.21 + 6.84 + 7.51 + 8.21 = 28.77$$

Hence, by (IX.29) Area $= \frac{0.5}{3}[14.56 + 2 \times 28.77 + 4 \times 36.03]$

$= \frac{1}{6}[14.56 + 57.54 + 144.12]$

$= \frac{1}{6}[216.22] = 36.04$

$$T = \frac{36.04}{0.301} = 120 \text{ (nearly)}$$

Time required $= 120$ sec (nearly)

EXAMPLE 3

Using Simpson's Rule, find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$, taking $\frac{\pi}{12}$ as the common width of the strips. Compare your result with the exact area.

x	$\sin x$	Multiplier	Result
0	0	1	0
$\frac{\pi}{12}$	0.2588190	4	1.0352760
$\frac{\pi}{6}$	0.5000000	2	1.0000000
$\frac{\pi}{4}$	0.7071068	4	2.8284272
$\frac{\pi}{3}$	0.8660254	2	1.7320508
$\frac{5\pi}{12}$	0.9659258	4	3.8637032
$\frac{\pi}{2}$	1.0000000	2	2.0000000
$\frac{7\pi}{12}$	0.9659258	4	3.8637032
$\frac{2\pi}{3}$	0.8660254	2	1.7320508
$\frac{3\pi}{4}$	0.7071068	4	2.8284272
$\frac{5\pi}{6}$	0.5000000	2	1.0000000
$\frac{11\pi}{12}$	0.2588190	4	1.0352760
π	0	1	0
		Sum $=$	22.9189144

Area required $= \frac{1}{3} \cdot \frac{\pi}{12}(22\cdot9189144) = 2\cdot00006$

The exact area $= \int_0^{\pi} \sin x \, dx = \left(-\cos x\right)_0^{\pi} = 2$

The above calculations could have been shortened by finding the area from $x = 0$ to $x = \frac{\pi}{2}$ and doubling it. The reader is expected to be familiar with the mid-ordinate, the mean ordinate, and the

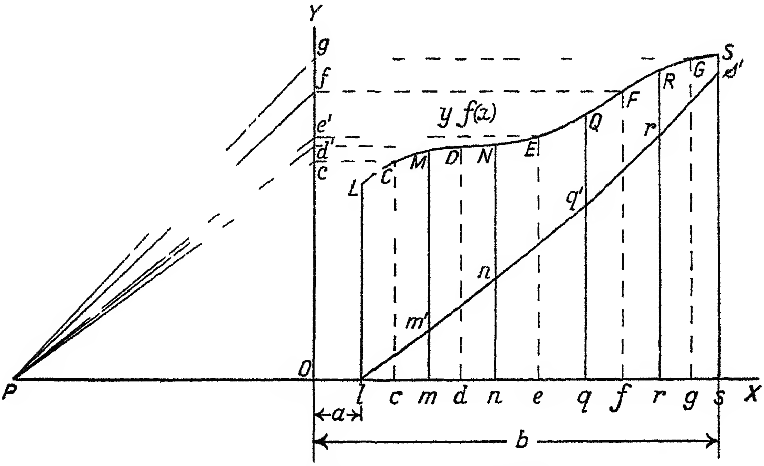


FIG 89 GRAPHICAL INTEGRATION

trapezoidal rules for finding areas. The first of these enables us to use a graphical method which, whilst not very accurate, is usually sufficiently approximate for practical purposes.

GRAPHICAL INTEGRATION. In Fig 89, LQS is the graph of a function $y = f(x)$ between $x = a$ and $x = b$. The area under the graph is divided into five strips by vertical lines through L, M, N, Q, R, and S, the feet of these ordinates being shown by corresponding small letters, and the mid-ordinates of the strips are Cc, Dd, Ee, Ff, and Gg. The points C, D, E, F, and G are projected horizontally on to OY at c', d', e', f', and g', and these points are joined to a pole P on XO produced. Through l, lm' is drawn parallel to Pc' and then $m'n'$, $n'q'$, $q'r'$, and $r's'$ are drawn in order parallel respectively to Pd', Pe', Pf', and Pg'.

By the mid-ordinate rule,

$$\begin{aligned}
\text{area of the first strip} &= Cc \times lm \\
&= Oc' \times lm \\
&= PO \times lm \times \text{gradient of } Pc' \\
&= PO \times lm \times \text{gradient of } lm' \\
&= PO \times mm'
\end{aligned}$$

$$\begin{aligned}
\text{Also, area of the second strip} &= Dd \times mn \\
&= Od' \times mn \\
&= PO \times mn \times \text{gradient of } Pd' \\
&= PO \times mn \times \text{gradient of } m'n' \\
&= PO \times (nn' - mm')
\end{aligned}$$

$$\begin{aligned}
\text{The area of the first two strips} &= PO \times nn' \\
\text{Similarly the area of } NQqn &= PO \times (qq' - nn') \text{ and} \\
\text{the area of the strips up to } Qq &= PO \times qq'
\end{aligned}$$

Thus, the area between Ll and any ordinate bounding a strip is approximately given by the ordinate of the graph $lq's'$ at that boundary measured to a suitable scale. If PO measures p units on the scale for values of x, the unit of the scale for ordinates to $lq's'$ is $\frac{1}{p}$ times that taken for values of x. On this account OP is usually made of length equal to an integral number of units on the x scale, such as 2, 5, 10. If PO is small, ss' becomes very large and either s' passes out of the figure or, if not, the lines near $r's'$ become ill-conditioned and the method becomes very inaccurate. If x and y are lengths, ordinates to the *sum* curve ls' represent areas. If ordinates represent areas and abscissae lengths, ordinates to the sum curve represent volumes. If x represent the volume of a pound of gas in cubic feet and p the pressure in lb per ft^2, the ordinate of the sum curve represents the work done in foot-pounds. For the sake of clearness, the area under the curve has been divided into only five strips. These are not enough, and ten or more strips are usually taken. The strips need not be equally wide, and greater accuracy is obtained by making the strips narrower where the curvature of the graph is greater as in the example below.

EXAMPLE 4

A variable force of F lb weight acts in the direction of motion on a moving body. The table gives values of F at distances x ft along the path of the body.

x	2	3	4	5	6	7	8	9	10	11	12
F	122	159	188	210	231	246	259	268	277	282	287

Find by graphical integration the work done by the force on the body as it moves from $x = 2$ to $x = 12$.

In Fig. 90, LM is the graph of F against x and lm' is the sum curve. The work done by F in the given range is $\int_2^{12} F\,dx$ ft-lb which is represented by mm'. To

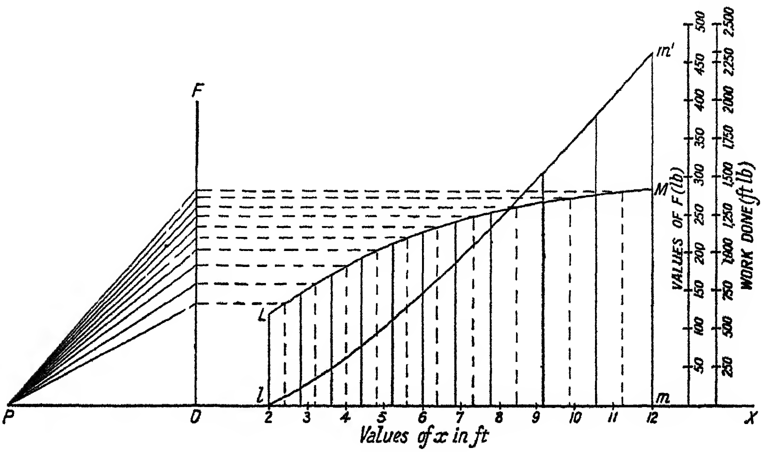


FIG. 90

find the vertical scale of the sum curve we note that $\overline{OP}$ was chosen to represent 5 ft on the x scale. The vertical scale for the sum curve must have a unit which is $\frac{1}{5}$ of that of the force scale, and readings on the scale of work done will be 5 times the corresponding numbers on the force scale. These two scales are shown on the right of the figure. Reading on the work scale we find that the work done during the displacement is 2 330 ft lb.

The above method of integration is included here for the sake of completeness. We cannot think of any example in which application of one of the ordinary rules for finding areas is not more quickly and more accurately applied. In this example the area under the graph is quickly found by the trapezoidal rule thus: Area $= 1 \times (61 + 159 + 188 + 210 + 231 + 246 + 259 + 268 + 277 + 282 + 143{\cdot}5) = 2\,325$ units.

PRISMOIDAL RULE. If l = the length of a solid, A_1 and A_3 the areas of parallel end sections, and A_2 the area of the middle section, then, using (IX.26) we have

$$\text{Volume of the solid, } V = \frac{l}{6}\,[A_1 + 4A_2 + A_3] \quad . \quad . \quad \text{(IX.30)}$$

EXAMPLE 5

A cylindrical hole of radius $\frac{1}{2}$ in. is drilled centrally through a solid sphere of radius 2 in.; find the volume of the portion remaining.

Half the length of the hole $= \sqrt{2^2 - (\frac{1}{2})^2} = \dfrac{\sqrt{15}}{2}$ in.

$\therefore$ Length of portion remaining $= \sqrt{15}$ in.

Area of each end section $= 0$

Area of middle section $= \pi\,[2^2 - (\frac{1}{2})^2] = \dfrac{15}{4}\pi$ in.^{2}

$$\therefore \text{Volume required} = \frac{\sqrt{15}}{6}\left[0 + 4\left(\frac{15}{4}\pi\right) + 0\right] = \frac{5\sqrt{15}}{2}\pi \text{ in.}^3$$
$$= 30{\cdot}4 \text{ in.}^3$$

The Prismoidal Rule gives exact results in those cases in which A is a quadratic function of x, where A is the area of the cross-section of a solid and x is the distance of that cross-section from some fixed point on the axis. In the same way the rule (IX.26) gives an exact expression for the area under an arc of the parabola $y = ax^2 + bx + c$ in terms of the end ordinates y_1 and y_3 and the middle ordinate y_2, and the distance $2h$ between the end ordinates.

114. **Centre of Gravity. Centroid.** Let Δw be the weight of an element of a body and x its distance from some fixed line (or plane); then, assuming that W, the weight of the body, acts at distance $\bar{x}$ from the line (or plane), we have $\bar{x}W = \Sigma x\,\Delta w$ since the sum of the moments of any number of forces about an axis is equal to the moment of their resultant.

If the body be continuous, then $\bar{x} = \dfrac{\int x\,dw}{\int dw}$, the integral embracing every element of the body. Referred to three mutually perpendicular axes OX, OY, OZ, the equations

$$\bar{x} = \frac{\int x dw}{\int dw},\ \bar{y} = \frac{\int y dw}{\int dw},\ \text{and } \bar{z} = \frac{\int z dw}{\int dw} \qquad . \qquad \text{(IX.31)}$$

will give the co-ordinates of the point at which the weight of the body is assumed to act. This point is called the "centre of gravity" of the body; it is also the "centre of mass" since the mass of each element is proportional to its weight.

We can apply the same method to lengths, areas, and volumes, quite apart from any notion of gravity, and the point we arrive at is called the "centroid" of the length, area, or volume. The terms

"centre of gravity" and "centre of mass" are also used in this connection, but "centroid" is a better term.

The co-ordinates of the centroid of the area $ABNM$ (Fig. 85) are given by $\bar{x}\int_a^b ydx = \int_a^b xydx$ and $\bar{y}\int_a^b ydx = \frac{1}{2}\int_a^b y^2dx$. (IX.32)

EXAMPLE 1

Obtain the area in the first quadrant bounded by the curve whose equation is $b^4y^2 = (a^2 - x^2)^3$ and the line $x = 0$. Also determine the co-ordinates of the mass centre of the same area. (U.L.)

In the first quadrant $y = \frac{1}{b^2}(a^2 - x^2)^{\frac{3}{2}}$; also $y = 0$ when $x = a$, so that the limits are $x = 0$ to $x = a$.

$$\therefore \text{Area required} = \frac{1}{b^2}\int_0^a (a^2 - x^2)^{\frac{3}{2}}\, dx$$

Let $x = a \sin\theta$. $\therefore dx = a\cos\theta\, d\theta$, also $\theta = 0, \frac{\pi}{2}$ when $x = 0, a$.

$$\therefore \text{Area} = \frac{1}{b^2}\int_0^{\frac{\pi}{2}} a^3\cos^3\theta \,.\, a\cos\theta\, d\theta = \frac{a^4}{b^2}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= \frac{a^4}{b^2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \text{ (see Art. 54)}$$

$$= \frac{3\pi a^4}{16b^2}$$

To find the mass centre we take as our element of area the strip $y\Delta x$ parallel to the y-axis (see Fig. 85); its moment about the y-axis is $xy\Delta x$ and about the x-axis $\frac{1}{2}y \,.\, y\Delta x = \frac{1}{2}y^2\Delta x$. Hence, if $\bar{x}, \bar{y}$ are the co-ordinates of the mass centre, we have

$$\bar{x} \cdot \frac{3\pi a^4}{16b^2} = \sum_{x=0}^{x=a} xy\,\Delta x = \int_0^a xydx$$

$$\bar{y} \cdot \frac{3\pi a^4}{16b^2} = \sum_{x=0}^{x=a} \tfrac{1}{2}y^2\,\Delta x = \tfrac{1}{2}\int_0^a y^2dx$$

$$\text{Now } \int_0^a xydx = \frac{1}{b^2}\int_0^a x(a^2 - x^2)^{\frac{3}{2}}\, dx = \frac{1}{2b^2}\int_0^a (a^2 - x^2)^{\frac{3}{2}}\, d(x^2) \text{ (see Art. 52)}$$

$$= \frac{1}{2b^2}\left[-\frac{2(a^2 - x^2)^{\frac{5}{2}}}{5}\right]_0^a$$

$$= -\frac{1}{5b^2}[0 - a^5] = \frac{a^5}{5b^2}$$

$$\therefore \bar{x} = \frac{a^5}{5b^2}\cdot\frac{16b^2}{3\pi a^4} = \frac{16a}{15\pi}$$

Again, $\frac{1}{2}\int_0^a y^2 dx = \frac{1}{2b^4}\int_0^a (a^2 - x^2)^3 dx = \frac{1}{2b^4}\int_0^{\frac{\pi}{2}} a^6 \cos^6\theta \cdot a\cos\theta d\theta$

(with the substitution $x = a\sin\theta$ as before)

$$= \frac{a^7}{2b^4}\int_0^{\frac{\pi}{2}} \cos^7\theta\, d\theta$$

$$= \frac{a^7}{2b^4} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \text{ (see Art 54)}$$

$$= \frac{8a^7}{35b^4}$$

$$\bar{y} = \frac{8a^7}{35b^4} \cdot \frac{16b^2}{3\pi a^4} = \frac{128a^3}{105\pi b^2}$$

EXAMPLE 2

Determine the position of the centroid of that half of the cardioid $r = a(1 + \cos\theta)$ which lies above the initial line

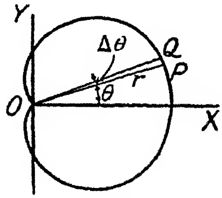


FIG 91

The curve is shown in Fig 91 The half-area $= \frac{3\pi a^2}{4}$ (see Art. 108, Ex. 1).

We take as our element of area the triangle OPQ where P and Q are the points (r, θ), $(r + \Delta r, \theta + \Delta\theta)$, on the curve Area $OPQ = \frac{1}{2}r^2\Delta\theta$, the distance of its centroid from $O = \frac{2}{3}r$ Hence, if OY be perpendicular to the initial line OX and $\bar{x}, \bar{y}$ be the co-ordinates of the centroid, then

$$\bar{x} \cdot \frac{3\pi a^2}{4} = \sum_{\theta=0}^{\theta=\pi} \tfrac{1}{2}r^2\Delta\theta \cdot \tfrac{2}{3}r\cos\theta = \tfrac{1}{3}\int_0^{\pi} r^3\cos\theta\, d\theta$$

and

$$\bar{y} \cdot \frac{3\pi a^2}{4} = \sum_{\theta=0}^{\theta=\pi} \tfrac{1}{2}r^2\Delta\theta \cdot \tfrac{2}{3}r\sin\theta = \tfrac{1}{3}\int_0^{\pi} r^3\sin\theta\, d\theta$$

Now $\int_0^{\pi} r^3\cos\theta\, d\theta = \int_0^{\pi} a^3(1 + \cos\theta)^3\cos\theta\, d\theta$

$$= a^3\int_0^{\pi}(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)d\theta$$

$$= a^3\left[0 + 6\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta + 0 + 2\int_0^{\frac{\pi}{2}}\cos^4\theta\, d\theta\right]$$

$$= a^3\left[6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{15\pi a^3}{8}$$

[For $\int_0^\pi \cos^n \theta \, d\theta = 0$ when n is an odd integer

and $\int_0^\pi \cos^n \theta \, d\theta = 2 \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$ when n is an even integer]

(Compare Art 110)

$$\therefore \quad \bar{x} = \frac{1}{3} \cdot \frac{15\pi a^3}{8} \cdot \frac{4}{3\pi a^2} = \frac{5a}{6}$$

Again $\int_0^\pi r^3 \sin\theta \, d\theta = a^3 \int_0^\pi (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d(\cos\theta)$

$$= -a^3 \left[\cos\theta + \tfrac{3}{2}\cos^2\theta + \cos^3\theta + \tfrac{1}{4}\cos^4\theta\right]_0^\pi$$

$$= -a^3 [(-2) + \tfrac{3}{2}(0) + (-2) + \tfrac{1}{4}(0)]$$

$$= 4a^3$$

$$\therefore \quad \bar{y} = \frac{1}{3} \cdot 4a^3 \cdot \frac{4}{3\pi a^2} = \frac{16a}{9\pi}$$

Following the method of Ex 2, the student should prove that the centroid of a circular sector of radius r and central angle 2α lies on the radius bisecting the central angle and is at distance $\dfrac{2r \sin\alpha}{3\alpha}$ from the centre. If the sector be a semicircle the centroid is at distance $\dfrac{4r}{3\pi}$ from the centre

EXAMPLE 3

Determine the position of the centre of gravity of the smaller part of a sphere of radius a cut off by a plane at distance $\frac{a}{2}$ from the centre, assuming (1) that the sphere is a solid of uniform density, (2) that the sphere is a thin shell

The generating curve is the circle $x^2 + y^2 = a^2$, the x-axis being perpendicular to the cutting plane From symmetry the centre of gravity in each case will lie on the x-axis, i e $\bar{y} = 0$

(1) The volume of the smaller segment $= \int_{\frac{a}{2}}^{a} \pi y^2 \, dx$

$$= \pi \int_{\frac{a}{2}}^{a} (a^2 - x^2)\,dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{\frac{a}{2}}^{a} = \frac{5}{24}\pi a^3$$

We take as our element of volume the disc $\pi y^2 \Delta x$ at distance x from the origin O. Its moment about O is $\pi x y^2 \Delta x$.

Hence $\bar{x}\cdot\frac{5}{24}\pi a^3 = \sum_{x=\frac{a}{2}}^{a}\pi x y^2\,\Delta x = \pi\int_{\frac{a}{2}}^{a} xy^2\,dx = \pi\int_{\frac{a}{2}}^{a}(a^2x - x^3)\,dx$

$$= \pi\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{\frac{a}{2}}^{a}$$

$$= \frac{9}{64}\pi a^4$$

$$\bar{x} = \frac{9}{64}a^4 \times \frac{24}{5a^3} = \frac{27a}{40}$$

(2) Since this example is illustrative, we shall find the surface area of the smaller part of the shell by actual integration

Let $PQ(=\Delta s)$ be an element of arc of the generating circle, P being the point (x, y) on the circle Then in polar co-ordinates, $x = a\cos\theta$, $y = a\sin\theta$, $\widehat{POQ} = \Delta\theta$, so that $\Delta s = a\Delta\theta$

The surface area required $= \int 2\pi y\,ds$ (between suitable limits)

$$= 2\pi\int_0^{\cos^{-1}\frac{1}{2}} a\sin\theta \cdot a\,d\theta$$

$$= 2\pi a^2\Big[-\cos\theta\Big]_0^{\frac{\pi}{3}}$$

$$= 2\pi a^2\left[\left(-\tfrac{1}{2}\right) - (-1)\right]$$

$$= \pi a^2$$

We take as our element of surface the portion $2\pi y\Delta s$ whose centre is at distance x from O Its moment about O is $2\pi xy\Delta s$

Hence $\bar{x}\cdot\pi a^2 = \Sigma 2\pi xy\Delta s = \int 2\pi xy\,ds$ (between suitable limits)

$$= \pi\int_0^{\frac{\pi}{3}} 2a\cos\theta \cdot a\sin\theta \cdot a\,d\theta$$

$$= \pi a^3\int_0^{\frac{\pi}{3}}\sin 2\theta\,d\theta$$

$$= \pi a^3\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \pi a^3\left[\left(\frac{1}{4}\right) - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{3}{4}\pi a^3$$

$$\bar{x} = \frac{3}{4}\pi a^3 \times \frac{1}{\pi a^2} = \frac{3a}{4}$$

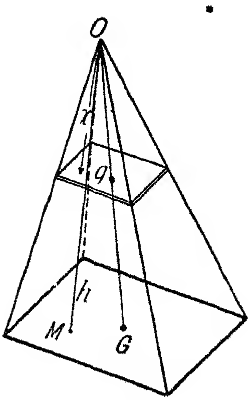


FIG 92

As in the above case, the centre of gravity of any zone of a spherical surface cut off by two parallel planes lies on the symmetrical axis and mid-way between the bounding planes

EXAMPLE 4

Determine the position of the centre of gravity of a pyramid of uniform density, the base being any plane figure

Let O be the vertex and G the centroid of the base (Fig 92) Join OG, then if a slice of thickness Δx be taken at distance x from O and parallel to the base, its centroid g will lie on OG The centre of gravity of the pyramid therefore lies on OG The area of the section at distance x from O is proportional to x^2, let it be kx^2 Then the volume of the slice is $kx^2\Delta x$ and its moment about O is $kx^2\Delta x \cdot x = kx^3\Delta x$ The volume of the pyramid is $\int_0^h kx^2\,dx$ where h = height of pyramid

Hence, if $\bar{x}$ = depth of centre of gravity below O

$$\bar{x}\int_0^h kx^2\,dx = \sum_{x=0}^{h} kx^3\Delta x = \int_0^h kx^3\,dx$$

$$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{h^4}{4}\cdot\frac{3}{h^3} = \frac{3}{4}h$$

This result also applies to the special case of a right circular cone

EXAMPLE 5

Determine the position of the centroid of the arc of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

measured from cusp to cusp

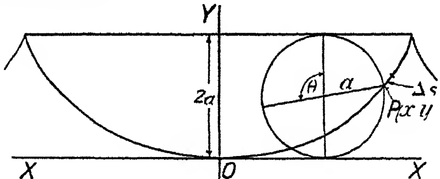


FIG 93

The curve is shown in Fig 93 The centroid is obviously on the y-axis, i e $\bar{x} = 0$ Let Δs be an element of arc of the cycloid at distance y from the x-axis Its moment about that axis is $y\Delta s$

Hence, $\bar{y}\int ds = \Sigma y\Delta s = \int y\,ds$ (between suitable limits)

Now
$$\frac{dx}{d\theta} = a(1 + \cos\theta),\quad \frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\left[(1+\cos\theta)^2 + \sin^2\theta\right]$$
$$= 2a^2(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

$$ds = 2a\cos\frac{\theta}{2}\,d\theta$$

$$\therefore \int ds = 2a\int_{-\pi}^{\pi}\cos\frac{\theta}{2}\,d\theta = 4a\left(\sin\frac{\theta}{2}\right)_{-\pi}^{\pi} = 4a\,[1-(-1)] = 8a$$

Also
$$\int y\,ds = \int_{-\pi}^{\pi} a(1-\cos\theta)\,.\,2a\cos\frac{\theta}{2}\,d\theta = 4a^2\int_{-\pi}^{\pi}\sin^2\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}\,d\theta$$

$$= 8a^2\int_{-\pi}^{\pi}\sin^2\frac{\theta}{2}\,d\left(\sin\frac{\theta}{2}\right)$$

$$= 8a^2\left[\frac{1}{3}\sin^3\frac{\theta}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{8a^2}{3}[1-(-1)]$$

$$= \frac{16a^2}{3}$$

$$\therefore \quad \bar{y} = \frac{16a^2}{3}\cdot\frac{1}{8a} = \frac{2a}{3}$$

EXAMPLE 6

A cylindrical hole of radius 1 ft is drilled centrally through a frustum of a right circular cone, radii of ends 1 ft and 3 ft, length 3 ft. If the density of the solid at any point varies as the square of the distance from the smaller end, determine the position of the mass-centre of the solid.

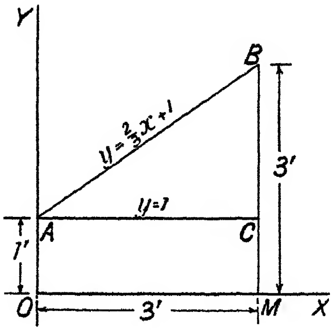


FIG. 94

We choose our axes of reference as shown in Fig. 94, OX being the axis of the cone and OA and BM the end radii. The mass-centre obviously lies on the x-axis; let $\bar{x}$ be its distance from O. The density of the solid at any point will be kx^2, where k is a constant. The conical frustum is generated by the line AB whose equation is $y = \frac{2}{3}x + 1$ and the cylinder by the line AC whose equation is $y = 1$. The volume of a slice, thickness, Δx, at distance x from O is $\pi[(\frac{2}{3}\,x + 1)^2 - 1^2]\,\Delta x = \pi\left[\frac{4x^2}{9} + \frac{4x}{3}\right]\Delta x$; its mass is $4\pi\left(\frac{x^2}{9} + \frac{x}{3}\right)kx^2\Delta x$, and its moment about O is $4k\pi\left(\frac{x^2}{9} + \frac{x}{3}\right)x^3\Delta x$

Hence, $$x \quad 4k\pi\int_0^3\left(\frac{x^4}{9}+\frac{x^3}{3}\right)dx = 4k\pi\int_0^3\left(\frac{x^5}{9}+\frac{x^4}{3}\right)dx$$

$$\therefore \quad x = \frac{\left[\frac{x^6}{54}+\frac{x^5}{15}\right]_0^3}{\left[\frac{x^5}{45}+\frac{x^4}{12}\right]_0^3} = \frac{13{\cdot}5+16{\cdot}2}{5{\cdot}4+6{\cdot}75} = 2{\cdot}44 \text{ ft}$$

The method of locating centres of gravity by the use of multiple integrals is illustrated in Vol. II.

We give below the positions of the centroids (or centres of gravity) in a few important cases not already dealt with, and leave the reader to supply the proofs.

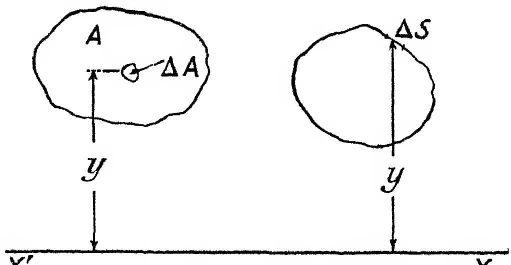


FIG. 95

The centroid of a circular arc, radius of circle r, central angle 2ϕ radians, is on the radius bisecting the central angle and at distance $\dfrac{r\sin\phi}{\phi}$ from the centre.

The centre of gravity of a hemisphere of uniform density is on the radius perpendicular to the circular base, and at $\frac{3}{8}$(radius) from the centre of the base.

The centroid of three equal particles placed at the angular points of a triangle or at the mid-points of its sides is coincident with the centroid of the triangle.

The centroid of any quadrilateral $ABCD$ is at a distance $\frac{1}{3}(h-k)$ from the diagonal BD and $\frac{1}{3}(l-m)$ from the diagonal AC, where h, k are the lengths of the perpendiculars from A and C on BD, and l, m the lengths of the perpendiculars from B and D on AC.

115. **Theorems of Pappus or Guldinus.** THEOREM 1. Let A be a closed plane area (Fig. 95) and $X'X$ a fixed straight line in its plane which does not intersect the area. Consider an element ΔA of the area at distance y from $X'X$. Its moment about $X'X = y\Delta A$; hence,

if $\bar{y}$ be the distance of the centroid of the area A from $X'X$, $\bar{y}A = \Sigma y\Delta A = \int ydA$, the integral embracing the whole area. Let the area A be rotated about $X'X$ through an angle θ radians; then the element ΔA will generate a volume $\theta y\Delta A$ and the area A will, therefore, generate a volume $\Sigma\theta y\Delta A = \theta \int ydA$ (since θ is the same for each element) $= \theta\bar{y}A$ (by above). In a complete revolution $\theta = 2\pi$, and the volume V generated is

$$V = 2\pi\bar{y}A \qquad . \quad . \quad . \quad \text{(IX.33)}$$

Hence, *if a closed plane area be rotated through any angle not greater than 2π radians about a line in its plane which does not intersect it, the volume generated is equal to the product of the area and the length of the path of its centroid.*

THEOREM 2. By reasoning similar to that used in Theorem 1 above we find that if Δs be an element of arc of a plane curve of length S at a distance y from a fixed straight line $X'X$ lying in the same plane but not intersecting the curve, then $\bar{y}S = \Sigma y\Delta s = \int yds$, where $\bar{y}$ = distance of the centroid of the curve from $X'X$; and if the curve be rotated about $X'X$ through θ radians, the surface area traced out by it (Fig. 95)

$$= \Sigma\theta y\Delta s = \theta \int yds = \theta\bar{y}S$$

In one revolution the surface area A traced out will be

$$A = 2\pi\bar{y}S \quad . \quad . \quad . \quad . \quad \text{(IX.34)}$$

Hence, *if a plane curve be rotated through any angle not greater than 2π radians about a line in its plane which does not intersect it, the surface area traced out is equal to the product of the length of the curve and the length of the path of its centroid.*

EXAMPLE 1

Prove that the volume generated by a closed plane area A revolving about a line in its plane which does not intersect it, is $2\pi\bar{y}A$, where $\bar{y}$ is the distance of the C.G. of the area from the line.

A semicircular bend of lead pipe has a mean radius of 1 ft; the internal diameter of the pipe is 4 in., and the thickness of the lead is $\frac{1}{2}$ in. Find its weight given that 1 in.^{3} of lead weighs 0·41 lb. (U.L.)

The first part of the question has been proved in Theorem 1 above. The area of a right section of the pipe

= area included between two concentric circles of radii $2\frac{1}{2}$ in. and 2 in.

$$= \pi\left[\left(\frac{5}{2}\right)^2 - 2^2\right] = \pi\cdot\frac{9}{2}\cdot\frac{1}{2} = \frac{9\pi}{4}\text{ in.}^2$$

The centroid of the section describes the semi-circumference of a circle of radius 12 in., so that the length of its path = 12π in.

$\therefore$ Volume of lead in pipe $= \frac{9\pi}{4} \times 12\pi = 27\pi^2$ in.^{3}

and the weight of the pipe $= 27\pi^2 \times 0.41 = 109.2$ lb

EXAMPLE 2

A sector of a circle of radius 5 in., the central angle being 60°, is rotated about a line through the centre parallel to the chord of the arc. Find the surface area and the volume of the solid generated in a complete revolution.

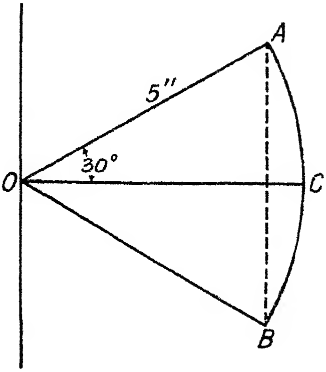


FIG. 96

Let $OACB$ (Fig. 96) be the sector, OC being the radius bisecting the central angle AOB. Then $\widehat{AOC} = \widehat{BOC} = \frac{\pi}{6}$ radians. The centroid G_1 of the arc ACB is on OC such that $OG_1 = 5\,\dfrac{\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{30}{\pi}\cdot\frac{1}{2} = \frac{15}{\pi}$ in (see end of Art. 114), the centroid G_2 of the two radii OA, OB is also on OC such that $OG_2 = \frac{5}{2}\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{4}$ in. The length of the arc $ACB = \frac{5\pi}{3}$ in. Hence, if G be the centroid of the perimeter of the sector, then

$$\left(10 + \frac{5\pi}{3}\right) OG = \frac{5\pi}{3}\cdot\frac{15}{\pi} + 10\cdot\frac{5\sqrt{3}}{4}$$

$$= 25 + \frac{25\sqrt{3}}{2}$$

$$\therefore \quad OG = \frac{25(2+\sqrt{3})}{2} \times \frac{3}{5(6+\pi)} = \frac{15(2+\sqrt{3})}{2(6+\pi)} \text{ in.}$$

$\therefore$ The surface area of the solid $= \left(10 + \frac{5\pi}{3}\right) \times 2\pi \times OG$

$$= \frac{5(6+\pi)}{3} \cdot 2\pi \cdot \frac{15(2+\sqrt{3})}{2(6+\pi)}$$

$$= 25(2+\sqrt{3})\pi = 293 \cdot 0 \text{ in.}^2$$

The area of the sector $= \frac{1}{2} \cdot 5^2 \cdot \frac{\pi}{3} = \frac{25\pi}{6}$ in.^{2}; the centroid is on OC at distance $\frac{2}{3} \cdot 5 \cdot \dfrac{\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{10}{3} \cdot \frac{6}{\pi} \cdot \frac{1}{2} = \frac{10}{\pi}$ in. from O (see Art. 114, end of Ex. 2).

$$\therefore \text{ Volume of the solid} = \frac{25\pi}{6} \cdot 2\pi \cdot \frac{10}{\pi} = \frac{250\pi}{3} = 261 \cdot 8 \text{ in.}^3$$

116. **Moments of Inertia.** If masses m_1, m_2, etc., at distances r_1, r_2, etc., from a fixed axis are rotating about that axis so that in time t each mass turns through an angle θ, then the linear velocities of the masses are $r_1\frac{d\theta}{dt}$, $r_2\frac{d\theta}{dt}$, etc.; their kinetic energies are $\frac{1}{2}m_1 r_1^2\left(\frac{d\theta}{dt}\right)^2$, $\frac{1}{2}m_2 r_2^2\left(\frac{d\theta}{dt}\right)^2$, etc.; and the sum of their kinetic energies is

$$\Sigma\tfrac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 = \tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2 \Sigma mr^2$$

the summation including every mass. The symbol I defined by the relation

$$I = \Sigma mr^2 \quad . \quad . \quad . \quad . \quad \text{(IX.35)}$$

is called the *moment of inertia*, or second moment, of the system of masses about the given axis. This quantity I is of frequent occurrence in engineering problems, and it is essential that the student should be able to find I for those bodies with which he will have to deal.

In the case of a continuous body

$$I = \int r^2 dm \quad . \quad . \quad . \quad \text{(IX.36)}$$

the integral embracing every element of the body. The term "moment of inertia" is often used when the idea of mass is not present. Thus, for an area A,

$$I = \int r^2 dA \quad . \quad . \quad . \quad \text{(IX.37)}$$

and for a volume V,

$$I = \int r^2 dV \quad . \quad . \quad . \quad \text{(IX.38)}$$

the integrals embracing the whole area or volume.

If I be the moment of inertia of a body of mass M, or a plane area A, about a given axis and k be the distance from that axis at which the whole mass or area must be concentrated in order to have the same moment of inertia I, then

$$I = Mk^2 \text{ or } I = Ak^2 \quad . \qquad \text{(IX.39)}$$

and

$$k = \sqrt{\frac{I}{M}} \text{ or } \sqrt{\frac{I}{A}} \quad . \qquad \text{(IX.40)}$$

This length k is called the "radius of gyration" of the body or area about the given axis.

EXAMPLE

A flywheel mounted on a horizontal axis is made to rotate by a descending weight W lb which is attached to one end of a string coiled round the axle. If the velocity of the weight is V ft per sec. when it has descended H ft, prove that the moment of inertia I lb-ft² of the flywheel is given by

$$I = \left(\frac{2gH}{V^2} - 1\right) Wr^2$$

where r = radius of the axle in feet. [Neglect friction.]

When the weight has descended H ft, the wheel has turned through an angle $\frac{H}{r}$ radians, and its angular velocity ω is then given by

$$\omega = \frac{d}{dt}\left(\frac{H}{r}\right) = \frac{1}{r}\frac{dH}{dt} = \frac{V}{r}$$

The kinetic energy of the system $= \frac{1}{2}\frac{I}{g}\omega^2 + \frac{1}{2}\frac{W}{g}V^2$ ft-lb

$$= \frac{V^2}{2g}\left(\frac{I}{r^2} + W\right) \text{ ft-lb}$$

The work done on the system $= W \times H$ ft-lb

Equating the kinetic energy acquired to the work done, we have

$$\frac{V^2}{2g}\left(\frac{I}{r^2} + W\right) = W \times H$$

$$\frac{I}{r^2} = \frac{2gW \times H}{V^2} - W$$

or

$$I = \left(\frac{2gH}{V^2} - 1\right) Wr^2$$

117. **Moments of Inertia about Perpendicular Axes.** Let Δm be an element of mass of a body situated at the point (x, y, z) with reference to three mutually perpendicular axes OX, OY, OZ. Then, obviously,

the moments of inertia of Δm about these axes are $(y^2 + z^2)\Delta m$, $(z^2 + x^2)\Delta m$, $(x^2 + y^2)\Delta m$ respectively, and if I_{OX}, I_{OY}, I_{OZ} denote the moments of inertia of the whole body about these axes, then

$$\left.\begin{aligned} I_{OX} = \Sigma(y^2 + z^2)\Delta m; \quad I_{OY} &= \Sigma(z^2 + x^2)\Delta m \\ I_{OZ} &= \Sigma(x^2 + y^2)\Delta m \end{aligned}\right\} \quad . \quad . \quad \text{(IX.41)}$$

It will be noted that the sum of any two of these is greater than the third.

e.g. $\quad I_{OX} + I_{OY} = \Sigma(y^2 + x^2 + 2z^2)\Delta m = I_{OZ} + 2\Sigma z^2\Delta m$

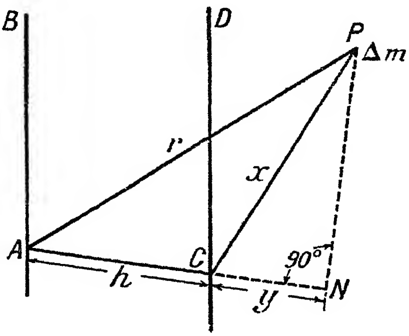


FIG. 97

Again, if r be the distance of Δm from the origin O,

$$r^2 = x^2 + y^2 + z^2$$

and we have

$$\begin{aligned} I_{OX} + I_{OY} + I_{OZ} &= \Sigma(y^2 + z^2)\Delta m + \Sigma(z^2 + x^2)\Delta m \\ &\qquad + \Sigma(x^2 + y^2)\Delta m \\ &= 2\Sigma(x^2 + y^2 + z^2)\Delta m \\ &= 2\Sigma r^2\Delta m \end{aligned}$$

$$\therefore \quad I_{OX} + I_{OY} + I_{OZ} = 2I_0 \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.42)}$$

where I_0 denotes the moment of inertia of the body about the origin O.

If the body be a lamina confined, say, to the xy plane, then

$$I_{OX} = \Sigma y^2\Delta m \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.43)}$$

$$I_{OY} = \Sigma x^2\Delta m \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.44)}$$

and $\quad I_{OZ} = \Sigma(x^2 + y^2)\Delta m = I_{OX} + I_{OY} \quad . \quad \text{(IX.45)}$

Thus, the sum of the moments of inertia of a lamina about two perpendicular axes in its plane is equal to the moment of inertia about an axis perpendicular to these two axes through their point of intersection. The moment of inertia of a plane lamina about an axis perpendicular to its plane, i.e. about a point in its plane, is called the *polar moment of inertia* about the point.

118. **Principle of Parallel Axes.** Let two parallel axes AB, CD (Fig. 97) be taken at distance h apart, the latter passing through the centre of gravity of a given body, and let an element of mass Δm of the body be situated at the point P distant r from AB and x from CD, the plane PAC being assumed perpendicular to the two axes. Then, if PN be drawn $\perp$ AC, we have, from elementary geometry,

$$r^2 = x^2 + h^2 + 2hy, \text{ where } y = CN$$

Hence, $\Sigma r^2 \Delta m = \Sigma x^2 \Delta m + \Sigma h^2 \Delta m + \Sigma 2hy \Delta m$

Now, $\Sigma r^2 \Delta m =$ moment of inertia of body about $AB = I_{AB}$

$\Sigma x^2 \Delta m =$ moment of inertia of body about $CD = I_{CD}$

$$\Sigma h^2 \Delta m = h^2 \Sigma \Delta m = Mh^2$$

where $M =$ total mass of body.

$$\Sigma 2hy \Delta m = 2h \Sigma y \Delta m = 2h\bar{y}M$$

where $\bar{y} =$ distance parallel to AC of the centre of gravity of the body from the axis CD; but this distance $= 0$, so that $\Sigma 2hy \Delta m = 0$.

Hence,
$$I_{AB} = I_{CD} + Mh^2 \quad . \quad . \quad . \quad \text{(IX.46)}$$

Thus, *the moment of inertia of a body about any axis is equal to the moment of inertia of the body about a parallel axis through its centre of gravity plus the product of the mass and the square of the distance between the two axes.*

The relations (IX.45) and (IX.46) apply also to second moments of plane areas, the axes OX, OY, AB, and CD being in the plane of the area and A replacing M.

EXAMPLE 1

Rectangle. Let $ABCD$ (Fig. 98) be a rectangle of length $AB = a$ and breadth $BC = b$. The moment of inertia about AB of an element of area $a\Delta x$ parallel to AB and at distance x from it is $ax^2\Delta x$. Hence, for the whole rectangle

$$I_{AB} = \Sigma a x^2 \Delta x = a\int_0^b x^2 dx = a\left(\frac{x^3}{3}\right)_0^b = \frac{ab^3}{3}$$

$$= A\frac{b^2}{3} \text{ (where } A = \text{whole area)}$$

Similarly,
$$I_{BC} = \frac{a^3 b}{3} = A\frac{a^2}{3}$$

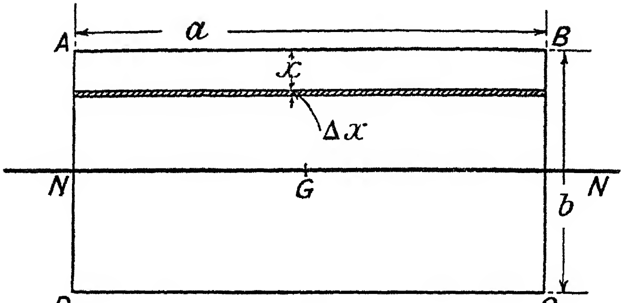


FIG. 98

If NN be an axis parallel to AB through the centroid G of the rectangle, then by the Principle of Parallel Axes

$$I_{AB} = I_{NN} + A\left(\frac{b}{2}\right)^2 \text{ whence } I_{NN} = A\frac{b^2}{3} - A\frac{b^2}{4} = A\frac{b^2}{12}$$

The moment of inertia about an axis through G perpendicular to the plane of the rectangle is (by Art. 117) equal to $A\dfrac{b^2}{12} + A\dfrac{a^2}{12} = A\dfrac{a^2+b^2}{12}$ ✓

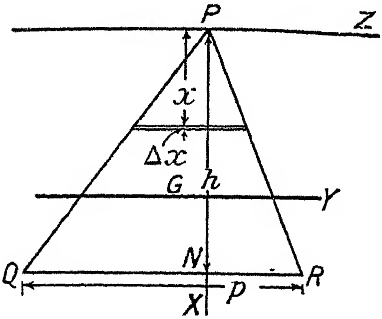


FIG. 99

Triangle. Let PQR (Fig. 99) be any triangle, the base QR being p and the height PN, h. Let PZ be drawn parallel to QR and let GY be a line parallel to PZ through G the centroid of the triangle. Take a strip of thickness Δx parallel to PZ and at distance x from it. The moment of inertia of this strip about

$PZ =$ (area of strip) $\times\, x^2 = \dfrac{xp}{h}\Delta x \,.\, x^2 = \dfrac{p}{h} x^3 \Delta x.$

For the whole triangle

$$I_{PZ} = \Sigma \frac{p}{h} x^3 \Delta x = \frac{p}{h}\int_0^h x^3 dx = \frac{p}{h} \cdot \frac{h^4}{4} = \frac{ph}{2} \cdot \frac{h^2}{2}$$

$$= A\frac{h^2}{2} \text{ (where } A = \text{area of triangle)}$$

By the Principle of Parallel Axes,

$$I_{GY} = I_{PZ} - A\left(\frac{2}{3}h\right)^2 = A\frac{h^2}{2} - A\frac{4h^2}{9} = A\frac{h^2}{18}$$

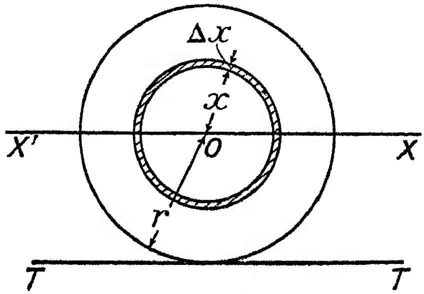


FIG. 100

and $$I_{QR} = I_{GY} + A\left(\frac{h}{3}\right)^2 = A\frac{h^2}{18} + A\frac{h^2}{9} = A\frac{h^2}{6}$$

Circle. The moment of inertia of a circular area of radius r about an axis through its centre perpendicular to its plane is $A\dfrac{r^2}{2}$, where A = area of circle

For the moment of inertia of an elementary ring of radius x and thickness Δx (Fig. 100) about O = (area of ring) $\times\, x^2 = 2\pi x \,.\, \Delta x \,.\, x^2$.

Hence, for the whole circle

$$I = \Sigma 2\pi x^3 \Delta x = 2\pi \int_0^r x^3 dx = 2\pi \frac{r^4}{4} = \pi r^2 \frac{r^2}{2} = A\frac{r^2}{2}$$

By Art. 117 we deduce that the moment of inertia of the circle about any diameter such as $X'OX$ is $A\dfrac{r^2}{4}$, and by the Principle of Parallel Axes, the moment of inertia of the circle about a tangent line TT is $A\dfrac{r^2}{4} + Ar^2 = A\dfrac{5r^2}{4}$

MOMENT OF INERTIA OF THE AREA UNDER A CURVE. Let $y = f(x)$ be the equation of the curve, the limits for x being from a to b (Fig. 82). We take as our element of area a strip $y\Delta x$ parallel to OY. The moment of inertia of this strip about $OY = x^2 y \Delta x$, and its moment of inertia about OX is $y\Delta x \dfrac{y^2}{3}$. (See above.)

For the whole area

$$I_{OY} = \Sigma x^2 y \Delta x = \int_a^b x^2 y dx \quad . \quad . \quad \text{(IX.47)}$$

and

$$I_{OX} = \Sigma \tfrac{1}{3} y^3 \Delta x = \tfrac{1}{3} \int_a^b y^3 dx \quad . \quad . \quad \text{(IX.48)}$$

MOMENT OF INERTIA OF A SOLID OF REVOLUTION. Let $y = f(x)$ be the generating curve (Fig. 85), the limits for x being from a to b; and let ρ be the density of the solid, supposed uniform. We shall find the moment of inertia of the solid about its axis of symmetry, OX, and also that about an axis perpendicular to its axis of symmetry—for this latter axis we take OY. The moment of inertia of an elementary disc of volume $\pi y^2 \Delta x$, perpendicular to OX and at distance x from O, about OX, is $\pi \rho y^2 \Delta x \frac{y^2}{2}$ (see above). Hence, for the whole solid

$$I_{OX} = \sum_{x=a}^{x=b} \tfrac{1}{2} \pi \rho y^4 \Delta x = \tfrac{1}{2} \pi \rho \int_a^b y^4 dx \quad . \quad \text{(IX.49)}$$

The moment of inertia of the elementary disc about a diameter parallel to OY

$$= \pi \rho y^2 \Delta x \, . \frac{y^2}{4}$$

and, by the Principle of Parallel Axes, its moment of inertia about OY

$$= \pi \rho y^2 \Delta x \, . \frac{y^2}{4} + \pi \rho y^2 \Delta x \, . \, x^2$$

$$= \pi \rho \left(\frac{y^4}{4} + x^2 y^2 \right) \Delta x$$

Hence, for the whole solid,

$$I_{OY} = \pi \rho \int_a^b y^2 \left(\frac{y^2}{4} + x^2 \right) dx \quad . \quad . \quad \text{(IX.50)}$$

If ρ is not uniform but is given by $\rho = \phi(x)$, then relations (IX.49) and (IX.50) become

$$I_{OX} = \tfrac{1}{2} \pi \int_a^b y^4 \phi(x) dx$$

and

$$I_{OY} = \pi \int_a^b y^2 \phi(x) \left(\frac{y^2}{4} + x^2 \right) dx$$

EXAMPLE 2

Find (1) the moment of inertia of a solid sphere of radius r and of uniform density about a diameter; (2) the moment of inertia of a solid cone of uniform density about a diameter of its base, the radius of the base being a and the height h.

(1) The equation of the generating circle is $x^2 + y^2 = r^2$.

Then, if $\rho =$ density of sphere, the moment of inertia about the x-axis

$$= \frac{\pi\rho}{2}\int_{-r}^{r} y^4 dx \text{ [by (IX.49)]}$$

$$= \frac{\pi\rho}{2}\int_{-r}^{r} (r^2 - x^2)^2 dx = \pi\rho\int_0^r (r^4 - 2r^2x^2 + x^4)dx$$

$$= \tfrac{8}{15}\pi\rho r^5$$

$$= \tfrac{4}{3}\pi\rho r^3 \cdot \tfrac{2}{5} r^2 = M \cdot \tfrac{2}{5} r^2$$

where $\quad M =$ mass of sphere

(2) If the vertex of the cone be taken as the origin and the axis of the cone as OX, then the equation of the generating line is $y = \dfrac{ax}{h}$, and, if ρ be the density,

$$I_{OY} = \pi\rho\int_0^h \frac{a^2x^2}{h^2}\left(\frac{a^2x^2}{4h^2} + x^2\right) dx \text{ [by (IX.50)]}$$

$$= \pi\rho\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)\int_0^h x^4 dx = M\left(\frac{3a^2}{20} + \frac{3h^2}{5}\right)$$

where $\quad M = \text{mass} = \frac{1}{3}\pi\rho a^2 h$

If GZ be an axis through the c.g. of the cone, and AB be a diameter of the base, both parallel to OY, then by the Principle of Parallel Axes,

$$I_{GZ} = I_{OY} - M(\tfrac{3}{4}h)^2 = M\left(\frac{3a^2}{20} + \frac{3h^2}{80}\right)$$

and

$$I_{AB} = I_{GZ} + M(\tfrac{1}{4}h)^2 = M\left(\frac{3a^2}{20} + \frac{h^2}{10}\right)$$

119. Products of Inertia. Principal Axes of a Lamina. In Art. 117 we proved that if two rectangular axes be taken in the plane of a lamina, and another axis OZ perpendicular to the plane, then $I_{OX} + I_{OY} = I_{OZ}$; so that the sum of the moments of inertia of a lamina about a pair of rectangular axes in its plane through a given point is constant for all positions of these axes. It follows that, if I_{OX}, say, is the greatest moment of inertia of the lamina for all axes in its plane through O, then I_{OY} is the least, and OX, OY are then called the *principal axes* of the lamina at the point O.

If the results obtained by multiplying each element of mass of a lamina by the product of its distances from the x and y axes be all added together, the sum, i.e. $\int xydm$, is called the *product of inertia* of

the lamina for the given axes. We shall now prove that the product of inertia for a pair of principal axes is zero.

Let the co-ordinates of an element of mass Δm at a point P of a lamina be (x, y) referred to rectangular axes OX, OY in its plane, and (x', y') referred to another pair of rectangular axes OX', OY' in the plane, angle XOX' being θ (Fig. 101).

Then $\quad x' = x \cos \theta + y \sin \theta$ and $y' = y \cos \theta - x \sin \theta$

Let $\quad A = \int y^2 dm, \; B = \int x^2 dm, \; P = \int xy dm,$

$\quad A' = \int y'^2 dm, \; B' = \int x'^2 dm, \; P' = \int x'y' dm,$

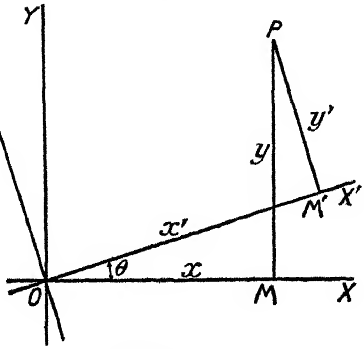


FIG. 101

the first three quantities being assumed known.

Then $\quad A' = \int (y \cos \theta - x \sin \theta)^2 dm$

$$= A \cos^2 \theta + B \sin^2 \theta - P \sin 2\theta \quad . \quad . \quad \text{(IX.51)}$$

$$B' = \int (x \cos \theta + y \sin \theta)^2 dm$$

$$= A \sin^2 \theta + B \cos^2 \theta + P \sin 2\theta \quad . \quad . \quad \text{(IX.52)}$$

$$P' = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dm$$

$$= \tfrac{1}{2}(A - B) \sin 2\theta + P \cos 2\theta \quad . \quad . \quad . \quad \text{(IX.53)}$$

Now, if OX', OY' are principal axes through O, then A' will be a maximum or a minimum, and therefore $\dfrac{d}{d\theta}(A')$ will be zero. This gives

$$-A \sin 2\theta + B \sin 2\theta - 2P \cos 2\theta = 0, \text{ i.e. } P' = 0 \quad . \quad \text{(IX.54)}$$

Also $$\tan 2\theta = \frac{2P}{B - A} \quad . \quad . \quad . \quad \text{(IX.55)}$$

which gives the directions of the principal axes.

Thus, for a pair of principal axes, the product of inertia is zero.

If OX be an axis of symmetry of the lamina, then obviously wherever O is on the axis the product of inertia $\int xy\,dm$ is zero, and therefore an axis of symmetry is a principal axis.

If OX, OY are principal axes, then $P = 0$ in (IX.51) and hence

$$A' = A\cos^2\theta + B\sin^2\theta \qquad . \qquad . \qquad \text{(IX.56)}$$

We shall use this result to find the moment of inertia of a rectangle about a diagonal. The principal axes through O, the centroid of the rectangle (Fig. 102), are OX, parallel to AB, and OY, parallel to BC. We have to find I_{BD}. Now,

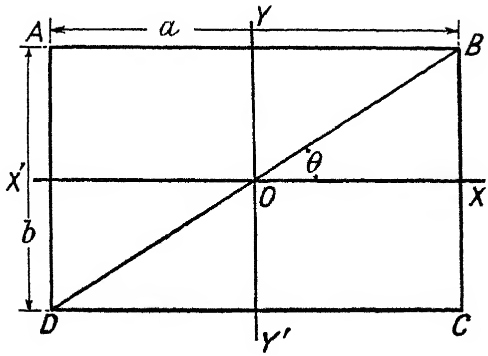


FIG. 102

$$I_{OX} = \frac{ab^3}{12} \text{ and } I_{OY} = \frac{a^3b}{12}$$

Let $\widehat{BOX} = \theta$. Then

$$I_{BD} = I_{OX}\cos^2\theta + I_{OY}\sin^2\theta$$

$$= \frac{ab^3}{12} \cdot \frac{a^2}{a^2+b^2} + \frac{a^3b}{12} \cdot \frac{b^2}{a^2+b^2} = \frac{a^3b^3}{6(a^2+b^2)}$$

If P be the product of inertia of a lamina about a pair of rectangular axes OX, OY in its plane, and P_G be the product of inertia about a pair of parallel axes through the centroid $G(\bar{x}, \bar{y})$ of the lamina, then

$$P = P_G + M\bar{x}\bar{y} \qquad . \qquad . \qquad . \qquad \text{(IX.57)}$$

where $\qquad M = \text{mass of lamina}$

For, if an element of mass Δm be situated at the point (x, y) referred to the axes OX, OY, its product of inertia about OX, OY

is $\Sigma xy\Delta m = \int xy\,dm$, and about the parallel axes through G, $\Sigma(x-\bar{x})(y-\bar{y})\Delta m = \int(x-\bar{x})(y-\bar{y})dm$, and this latter quantity

$$= \int xy\,dm - \bar{x}\int y\,dm - \bar{y}\int x\,dm + \bar{x}\bar{y}\int dm$$

$$= \int xy\,dm - \bar{x}\bar{y}M - \bar{y}\bar{x}M + \bar{x}\bar{y}M$$

$$= \int xy\,dm - \bar{x}\bar{y}M$$

i.e. $\qquad P_G = P - M\bar{x}\bar{y}$ or $P = P_G + M\bar{x}\bar{y}$

120. **The Momental Ellipse.** Let OX, OY (Fig. 103) be the principal axes of an area A through a point O in its plane, and let

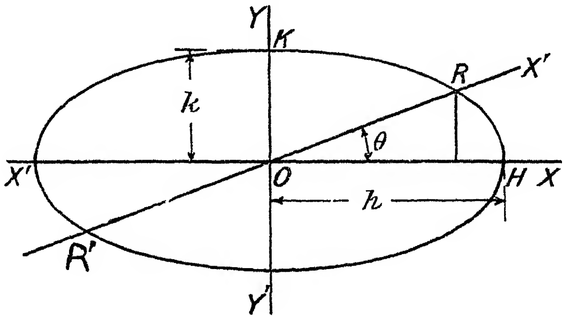


FIG. 103

OX' be another axis in the plane, $\widehat{XOX'}$ being θ. Let also the radii of gyration of the area about OX, OY, OX' be denoted by $\frac{1}{h}$, $\frac{1}{k}$, $\frac{1}{r}$ respectively, so that $I_{OX} = \frac{A}{h^2}$, $I_{OY} = \frac{A}{k^2}$, $I_{OX'} = \frac{A}{r^2}$. Mark off distances $OH = h$, $OK = k$, $OR = r$ along OX, OY, OX', and let R be the point (x, y) referred to OX, OY, so that $x = r\cos\theta$, $y = r\sin\theta$.

By (IX.56) $\qquad I_{OX'} = I_{OX}\cos^2\theta + I_{OY}\sin^2\theta$

$\therefore \qquad \frac{A}{r^2} = \frac{A}{h^2}\cos^2\theta + \frac{A}{k^2}\sin^2\theta$

$\therefore \qquad 1 = \frac{(r\cos\theta)^2}{h^2} + \frac{(r\sin\theta)^2}{k^2}$

i.e. $$\frac{x^2}{h^2} + \frac{y^2}{k^2} = 1 \qquad . \quad . \quad . \quad \text{(IX.58)}$$

Hence, R lies on an ellipse whose centre is O and whose semi-axes are h and k.

Assuming then that we know the radii of gyration of an area about a pair of principal axes, we can construct the momental ellipse on a suitable scale, its semi-axes being the reciprocals of these radii of gyration. The radius of gyration of the area about any diameter of the ellipse is equal to the reciprocal of the semi-diameter.

As an example we shall construct the momental ellipse for the section shown in Fig. 104 and find the moment of inertia of the section

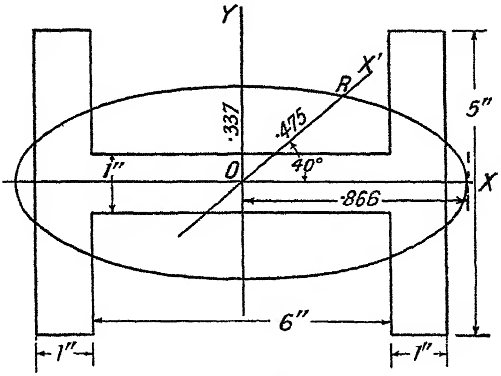


FIG. 104

about an axis OX' through the centroid inclined at 40° to the axis of symmetry OX.

The area A of the section $= 2 \times 5 \times 1 + 6 \times 1 = 16$ in.^{2}

$$I_{OX} = 6 \times \frac{1^2}{12} + 2 \times 5 \times \frac{5^2}{12} = \tfrac{1}{2} + 20\tfrac{5}{6} = 21\tfrac{1}{3}$$

$$= 16 \times \tfrac{4}{3} = A\left(\frac{2}{\sqrt{3}}\right)^2$$

$$I_{OY} = 6 \times \frac{6^2}{12} + 2\left[5 \times \frac{1^2}{12} + 5 \times \left(\frac{7}{2}\right)^2\right]$$

$$= 18 + 10\left(\tfrac{1}{12} + \tfrac{49}{4}\right) = 141\tfrac{1}{3} = 16 \times \tfrac{53}{6} = A\left(\sqrt{\frac{53}{6}}\right)^2$$

If h and k be the semi-axes of the momental ellipse,

$$h = \frac{\sqrt{3}}{2} = 0{\cdot}866'' \text{ and } k = \sqrt{\frac{6}{53}} = 0{\cdot}337''$$

Let OX' cut the ellipse at R. We find $OR = 0{\cdot}475$

$$\therefore \qquad I_{OX'} = \frac{16}{(0{\cdot}475)^2} = 70{\cdot}9 \text{ in.}^4$$

The following example shows how the moment of inertia of a body comes into use in an actual dynamical problem.

EXAMPLE

The internal and external diameters of a cylindrical shell are a and b respectively. Determine the radius of gyration about its axis.

A cylindrical shell whose external diameter is 3 ft and internal diameter 2 ft rolls down a plane inclined at 30° to the horizon. If it starts from rest, determine its speed when it has described 20 ft of the plane. (U.L.)

Let ρ = density of shell and l = its length.

The mass of an elementary shell of radius x and thickness Δx is $2\pi\rho lx\Delta x$, and its moment of inertia about the axis is $2\pi\rho lx^3\Delta x$. Hence, the moment of inertia of the whole shell about the axis

$$= \sum_{x=\frac{a}{2}}^{x=\frac{b}{2}} 2\pi\rho lx^3\Delta x = 2\pi\rho l \int_{\frac{a}{2}}^{\frac{b}{2}} x^3 dx$$

$$= \frac{\pi\rho l}{2}\left[\left(\frac{b}{2}\right)^4 - \left(\frac{a}{2}\right)^4\right]$$

$$= \pi\rho l \cdot \frac{b^2 - a^2}{4} \cdot \frac{b^2 + a^2}{8}$$

$$= M\frac{b^2 + a^2}{8}$$

where $\qquad M$ = mass of shell

$\therefore$ Radius of gyration about axis $= \sqrt{\dfrac{b^2 + a^2}{8}}$

The moment of inertia of the shell (in the second part of the question) about its axis $= M\dfrac{3^2 + 2^2}{8} = M\dfrac{13}{8}$, where M = mass of shell. Considering the central section of the shell perpendicular to its axis, let A (Fig. 105) be the point originally in contact with O, and let P be the point of contact in time t sec., the shell having then rotated through θ radians. Since there is no sliding, $x = \dfrac{3}{2}\theta$, where x = distance OP.

$$\therefore \qquad \frac{d^2x}{dt^2} = \frac{3}{2}\frac{d^2\theta}{dt^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation giving the motion of the centre of mass parallel to the plane is

$$M\frac{d^2x}{dt^2} = Mg \sin 30° - F \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

where F = the force of friction acting up the plane.

The equation for the rotational motion is $I\dfrac{d^2\theta}{dt^2} = \dfrac{3}{2}F$

i.e. $$M\frac{13}{8}\cdot\frac{d^2\theta}{dt^2} = \frac{3}{2}F \qquad (3)$$

Eliminating F between (2) and (3), and using relation (1), we have

$$M\left[\frac{3}{2}\cdot\frac{d^2x}{dt^2} + \frac{13}{8}\cdot\frac{2}{3}\cdot\frac{d^2x}{dt^2}\right] = \frac{3}{2}\cdot Mg\cdot\frac{1}{2}$$

$$\therefore \quad \frac{31}{12}\cdot\frac{d^2x}{dt^2} = \frac{3g}{4} \text{ and } \therefore \frac{d^2x}{dt^2} = \frac{9g}{31}$$

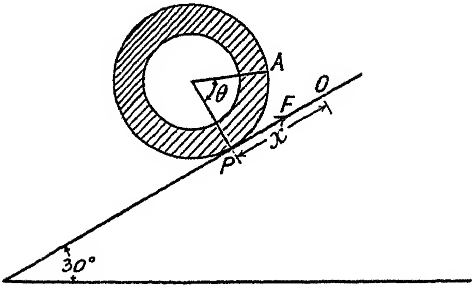


FIG. 105

The speed v when the shell has described 20 ft of the plane is given by $v = \sqrt{2\cdot\dfrac{9g}{31}\cdot 20} = 19{\cdot}34$ ft/sec.

121. **Centre of Pressure.** The "centre of pressure" of a plane area under fluid pressure is that point in the area at which the resultant pressure acts. In Fig. 106, A is a plane area immersed in a fluid and inclined at angle θ to the free horizontal surface of the fluid. YY' is the line in which the plane of the area meets the surface. Let the distances of G and C (the centroid and the centre of pressure respectively of the area A) from YY' be $\bar{x}$ and ξ; and let ΔA be an element of A distant x from YY' and at depth h below the surface $YY'VU$. Let w = weight per unit volume of the fluid. The pressure of the fluid on $\Delta A = \Delta A \times wh = wx\sin\theta\Delta A$.

$$\therefore \text{ The resultant pressure on } A = \Sigma wx\sin\theta\Delta A$$
$$= w\sin\theta\int x\,dA \qquad (IX.59)$$

the integral embracing the whole area A.

Now $\int x\,dA = \bar{x}A$, so that from (IX.59) we obtain

$$\text{Resultant pressure on } A = w\sin\theta\bar{x}A = w\bar{h}A \qquad (IX.60)$$

where $\bar{h}$ = depth of G below the surface.

Hence, *the resultant pressure on a plane area immersed in a fluid is equal to the product of the area and the pressure at the centroid.*

The moment about YY' of the pressure on ΔA

$$= wx^2 \sin\theta\Delta A$$

$\therefore$ The sum of the moments about YY' of the pressures on all the elements ΔA

$$= \Sigma wx^2 \sin\theta\Delta A = w\sin\theta\int x^2 dA = w\sin\theta \,.\, Ak^2 \quad . \quad \text{(IX.61)}$$

where k is the radius of gyration of the area A about YY'.

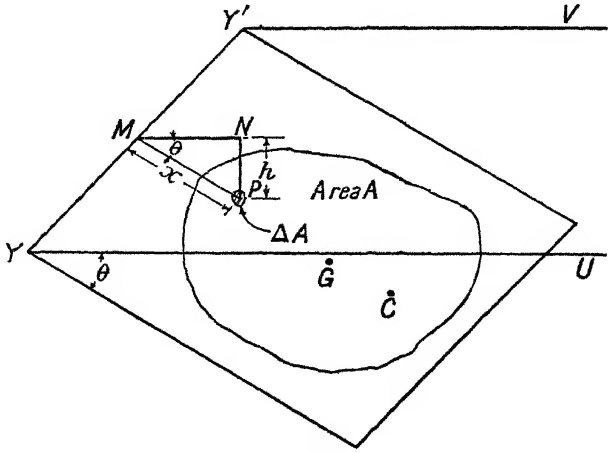


FIG. 106

The moment about YY' of the resultant pressure acting at C

$$= w\sin\theta x A\xi \quad . \quad . \quad . \quad \text{(IX.62)}$$

Equating (IX.62) and (IX.61), we have

$$w\sin\theta \,.\, xA\xi = w\sin\theta \,.\, Ak^2 \text{ or } \xi = \frac{k^2}{\bar{x}} \quad . \quad . \quad \text{(IX.63)}$$

EXAMPLE

A circular lamina, radius 3 ft, is immersed in a liquid with its plane vertical and its centre at a depth 5 ft below the surface; find the position of the centre of pressure.

With the notation above $\xi = \dfrac{k^2}{\bar{x}}$, and by (IX.46), $k^2 = \dfrac{3^2}{4} + 5^2 = \dfrac{109}{4}$

$$\therefore \qquad \xi = \frac{27{\cdot}25}{5} = 5{\cdot}45 \text{ ft}$$

Thus, the centre of pressure is at a distance 0·45 ft vertically below the centre of the lamina.

In many cases considerations of symmetry enable us to fix a line in the plane area on which the centre of pressure lies

122. **Metacentre.** A body of weight W (Fig. 107) floating freely in a fluid receives a small angular displacement θ about a horizontal axis such that the volume of the displaced fluid may be assumed to remain unaltered. The centre of gravity G of the body and the centre of buoyancy H (i.e. the centre of gravity of the displaced fluid) lie in the plane of the paper and the axis of rotation through O is perpendicular to the plane of the paper and lies in the fluid-surface plane of the body. The vertical through H', the position of the centre of buoyancy after the displacement, meets the line HG (originally vertical) at M. M is called the *metacentre* of the body and GM the *metacentric height* for the displacement in question. The weight W of the body acting vertically downwards through G, and the upthrust W of the fluid acting vertically upwards through H', form a couple of moment $W \,.\, GM \sin\theta$, which will tend to bring the body back to its equilibrium position, provided that M is above G; the equilibrium is then stable. If M is below G, the equilibrium is unstable; if M coincides with G, the equilibrium is neutral.

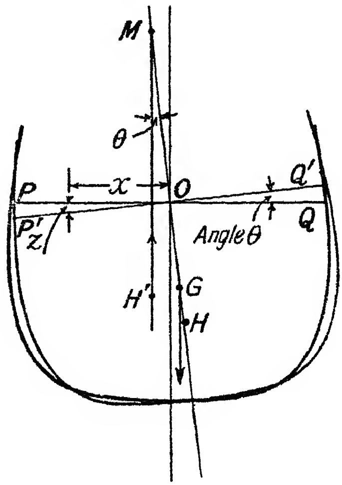


FIG. 107

We proceed to find an expression giving the position of the metacentre of a body for a small rotation in a plane of symmetry. We take the plane of the paper as a vertical plane of symmetry of the body (Fig. 107), and a perpendicular to this plane passing through O in the fluid surface plane as the axis of rotation. The angular displacement of the body through the small angle θ causes an additional thin wedge OPP' of the body to be immersed and a thin wedge OQQ' to be emersed. Let the wedge OPP', OQQ' be supposed divided up into elementary prisms by planes perpendicular to POQ, and let δA be the area of the base of one of these prisms, z its height, and

x its distance from O, A being the area of the fluid surface plane of the body. Then $z = x \tan \theta = x\theta$, since θ is small, and the force of buoyancy due to the displacement of fluid by the elementary prism is $wx\theta\Delta A$ where w = weight per unit volume of fluid, and its moment about the axis of rotation is $wx^2\theta\Delta A$. These forces of buoyancy are positive over the wedge OPP', and negative over the wedge OQQ', and their total effect is equal to that of a couple of moment

$$\Sigma wx^2\theta\Delta A = w\theta\int x^2 dA = w\theta Ak^2 \quad . \quad . \quad \text{(IX.64)}$$

where k is the radius of gyration of the fluid surface plane about the axis of rotation.

Again, if V, the volume of fluid displaced by the body, is assumed unaltered by the rotation, we can regard the effect of the rotation on the buoyancy as that due to the transfer of the force of buoyancy Vw from the point of action H to the new point of action H', the total effect being equal to that of a couple of moment

$$Vw \,.\, HM \sin\theta = Vw \,.\, HM \,.\, \theta \text{ (since } \theta \text{ is small)}. \quad \text{(IX.65)}$$

Since the couples (IX.65) and (IX.64) are equivalent, we have

$$Vw \,.\, HM \,.\, \theta = w\theta Ak^2 \text{ or } HM = \frac{Ak^2}{V} \quad . \quad \text{(IX.66)}$$

EXAMPLE

A solid cylinder, radius 1·5 ft, height 3 ft, and specific gravity 0·8, floats in water with its axis vertical. Find the metacentric height and show that the equilibrium is unstable.

Here A = area of water surface plane = $\pi(1{\cdot}5)^2$ ft²

V = volume immersed = 0·8 (volume of cylinder)

= $0{\cdot}8\pi\,(1{\cdot}5)^2\,3$ ft³

k^2 = square of radius of gyration of circle of radius 1·5 ft about a diameter $= \dfrac{(1{\cdot}5)^2}{4}$ ft²

$$\therefore \quad HM = \frac{Ak^2}{V} = \frac{\pi(1{\cdot}5)^2\dfrac{(1{\cdot}5)^2}{4}}{0{\cdot}8\pi(1{\cdot}5)^2\,3} = 0{\cdot}234 \text{ ft}$$

Height of H above bottom of cylinder $= \frac{1}{2} \times 0{\cdot}8 \times 3 = 1{\cdot}2$ ft

$\therefore \quad HG = 1{\cdot}5 - 1{\cdot}2 = 0{\cdot}3$ ft, so that

Metacentric height $= GM = HM - HG = 0{\cdot}234 - 0{\cdot}3 = -0{\cdot}066$ ft.

Since M is below G, the equilibrium is unstable.

NOTE. The reader should be familiar with the following in attempting the solutions of examples in dynamics.

QUANTITIES AND FORMULAE CONNECTED WITH THE SUBJECT OF DYNAMICS

Quantity	*Symbol*
Displacement, or distance moved, in feet .	s
Time, in seconds	t
Mass, in engineers' units	M
Weight, in lb	W
Acceleration due to gravity in feet per second per second	g
Linear velocity, in feet per second	v
Linear acceleration, in feet per second per second . .	a
Linear momentum (see below)	Mv
Kinetic energy, in foot-pounds	KE
Force, in lb weight, in the direction of motion .	F
Work done by a force in foot-pounds (see below) . .	Fs
Angular displacement or angle turned through in radians .	θ
Angular velocity, in radians per second . . .	ω
Angular acceleration in radians per second per second	α
Moment of inertia, in engineers' units . . .	I
Angular momentum (see below)	$I\omega$
Kinetic energy of rotation, in foot-pounds . . .	KE
Turning moment of a couple, in pounds-feet . . .	C
Work done by a couple, in foot-pounds (see below) . .	$C\theta$

FORMULAE

Linear Motion— $v = \frac{ds}{dt}$ $\qquad s = \int v dt$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \qquad v = \int a dt$$

$$M = \frac{W}{g}$$

Work done by a force

$= Fs$ if F is constant or $\int F ds$ if F is variable

Linear momentum $\qquad Mv$

$$KE = \tfrac{1}{2}Mv^2$$

Equation of Motion for the rectilinear motion of a particle or of a rigid body of which all points move in parallel straight lines—

$$F = Ma \quad . \quad . \quad . \quad . \qquad \text{(IX.67)}$$

Rotational Motion of a rigid body—

$$\omega = \frac{d\theta}{dt} \qquad \theta = \int \omega dt$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad \omega = \int \alpha dt$$

$$I = \Sigma \frac{Wr^2}{g}$$

where W is the weight in pounds of a small portion of the body and r is its distance in feet from the axis of rotation.

Work done by a couple

$= C\theta$ if C is constant, or $\int Cd\theta$ if C is a variable

Angular momentum $= I\omega$

$$KE = \tfrac{1}{2}I\omega^2$$

Equation of Motion for a rotating body—

$$C = I\alpha \quad . \quad . \quad . \quad . \quad . \qquad \text{(IX.68)}$$

For a rigid body in plane motion (IX.67) and (IX.68) are the equations of motion, (IX.67) referring to the motion of the whole mass assumed collected at its centre of gravity and (IX.68) to the rotational motion about the centre of gravity.

EXAMPLES IX

AREAS, MEAN VALUES, ROOT MEAN SQUARES

Find the area under each of the following graphs between the ordinates indicated—

(1) The graph of $y = 3x^2$ from $x = 0$ to $x = 4$.

(2) The graph of $y = 3x^2 + 2x - 7$ from $x = 2$ to $x = 5$.

(3) The graph of $y = \frac{3}{x}$ from $x = 1$ to $x = 2$.

(4) The graph of $y = \frac{c}{x^n}$ from $x = x_1$ to $x = x_2$.

(5) Show that the area enclosed by the axis of x and one semi-undulation of the sine curve represented by $y = A \sin nx$ is $\frac{2A}{n}$.

(6) Find the area under the catenary $y = c \cosh \frac{x}{c}$ between $x = 0$ and $x = x_1$.

(7) Find the values of k and l which will make the graph of $y = kx(l - x)$ pass through the points (0, 0), (100, 0), (50, 10). Sketch the graph. What curve is it? Find the area between the graph and the axis of x.

(8) Show that the area enclosed by the parabola $y^2 = kx$, the axis of x and the ordinate at $x = c$ is two-thirds that of the rectangle having the ordinate at $x = c$ and the abscissa c as sides.

(9) The work done in foot-pounds by a gas expanding from volume V_1 and pressure P_1 to volume V_2 and pressure P_2 is the area between $V = V_1$ and $V = V_2$ under the graph plotted with P vertical and V horizontal. P is in pounds per square foot, and V is in cubic feet. Show that the work done is $\int_{V_1}^{V_2} P dV$, and find the work done in the cases (i) $PV = c$, (ii) $PV^n = c$, where c and n are constants.

(10) Find the values of a and b which will make the graph of $y = a + bx^{1.5}$ pass through the points (1, 2) and (4, 10). Find the area under the graph between $x = 4$ and $x = 10$.

(11) Using the method indicated in Example 9, find the work done when a gas expands from a volume of 2 ft³ to one of 6 ft³ when the law connecting P and V is (i) $PV = c$, (2) $PV^{1.1} = c$. In each case $P = 7\,200$ when V is 2.

(12) The change of gradient between any two points in a horizontal beam is $\frac{A}{EI}$ where E and I are constants and A is the area of the bending moment diagram between the points. In a certain beam the bending moment at a point distant x from some fixed point in it is $\frac{1}{6}(1\,000x - 10x^3)$. Find the change of gradient between the two points for which $x = 2$ and $x = 5$ respectively.

(13) Find the turning moment T lb-in. on a hollow circular shaft, inside radius r_1 in., outside radius r_2 in., if $T = \int_{r_1}^{r_2} \frac{2\pi q}{r_2} x^3 dx$. Show that for a solid shaft of radius r in., $T = \frac{\pi}{2} qr^3$, q is a constant.

(14) A rotating vertical circular shaft, radius r in., presses on a flat footstep bearing with a total force of P lb. Assuming that the pressure is uniformly distributed over the area, find the pressure on the annulus between radii x and $x + \Delta x$. Taking μ as the coefficient of friction, show that the turning moment about the shaft axis of the frictional force on this annulus is $\frac{2\mu P x^2 \Delta x}{r^2}$. Hence, by integration, find the total frictional couple resisting rotation of the shaft.

(15) Find the frictional couple in the last example on the assumption that the pressure at any part of the bearing surface varies inversely as its distance from the axis of the shaft.

(16) Using the method of Examples 14 and 15, find the frictional couple resisting motion in a collar bearing in which the bearing surface has the form of an annulus, external radius R in., internal radius r in. Obtain two results, assuming (i) that the pressure is uniformly distributed, (ii) that the pressure varies inversely as the distance from the axis of the shaft.

(17) Show that the average value of $\sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ is $\frac{2}{\pi}$. Find the mean value between the limits (i) $x = 0$ and $x = \pi$, (ii) $x = 0$ and $x = 2\pi$

(18) Show without actual integration that if q is an integer

$$\int_0^{2\pi} \sin(qt + \alpha)\,dt = \int_0^{2\pi} \cos(qt + \alpha)\,dt = 0$$

(19) Show without actual integration that if p and q are integers, and p is not equal to q, then

$$\int_0^{2\pi} \sin(pt + \alpha)\sin(qt + \beta)\,dt = \int_0^{2\pi} \sin(pt + \alpha)\cos(qt + \beta)\,dt$$
$$\int_0^{2\pi} \cos(pt + \alpha)\cos(qt + \beta)\,dt = 0$$

(20) Find the mean values of

(i) x^2 from $x = 1$ to $x = 4$.

(ii) e^{2x} from $x = 0$ to $x = a$.

(iii) x^2e^x from $x = 1$ to $x = 3$.

(iv) $\log_e x$ from $x = 1$ to $x = a$.

(v) $x \log_e x$ from $x = 1$ to $x = a$.

(21) Evaluate the following integrals:

$$\int_0^{\frac{\pi}{2}} x \sin x\,dx, \quad \int_0^{\frac{\pi}{2}} \cos^3 x\,dx, \quad \int_0^{\pi} (2\pi - x)\cos x\,dx$$

(22) Show that the mean value of $kx(l - x)$ between $x = 0$ and $x = l$ is two-thirds of its maximum value.

(23) Find the mean values of—

(i) $\sin^2(pt + \alpha)$, (ii) $\sin^4 pt$, (iii) $\{\sin pt + 2\sin(pt + \alpha)\}^2$

over the range $t = 0$ to $t = \dfrac{\pi}{p}$ (U.L.)

(24) Prove that $\displaystyle\int_0^{\frac{\pi}{p}} \sin^2(pt + \alpha)\,dt = \frac{\pi}{2p}$ and $\displaystyle\int_0^{\frac{\pi}{p}} \sin pt \cos pt\,dt = 0$

Find the mean values of the three following expressions—

(i) $\cos(pt + \alpha)\cos(pt + \beta)$

(ii) $\sin(2pt)\cos(3pt + \alpha)$

(iii) $(2\sin 3pt + 5\sin pt)^2$ (U.L.)

(25) Show that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

(26) Find the area enclosed by the parabolas $y^2 = ax$ and $x^2 = by$, and show that if $a = b$ the area is $\frac{1}{3}a^2$.

(27) Show that if the axes of co-ordinates are oblique, the angle XOY being ω, the area A enclosed by the graph of $y = f(x)$, the axis of x, and the lines $x = a$, $x = b$ is given by $A = \sin\omega \displaystyle\int_a^b f(x)\,dx$.

(28) Find the area between the graphs of $y = x^2 - 6x + 3$ and $y = 2x - 9$.

(29) Find the area enclosed by the graph of $y = (x - 2)(x - 4)$ and the axis of x.

(30) Find the area swept out by the radius vector of the curve $r = a\theta$ (spiral of Archimedes) as θ increases from 0 to 2π.

(31) Let (r, θ) be the polar co-ordinates of a point P on a plane curve, the origin O being in the plane of the curve. Show that if PT is the tangent to the curve at P drawn on the side of OP on which θ lies and ϕ is the angle OPT, then $\dfrac{1}{r}\dfrac{dr}{d\theta} = \cot\phi$.

(32) Show that in the spiral $r = ae^{\theta \cot \alpha}$, where a and α are constants, the angle ϕ as defined in Example 31 is constant and equal to α. (For this reason the curve is known as the equiangular spiral.)

(33) Find the area between the radii $\theta = \theta_1$ and $\theta = \theta_2$ of the equiangular spiral $r = ae^{\theta \cot \alpha}$ if $\pi > \theta_2 > \theta_1$; and if $r = r_1$ when $\theta = \theta_1$, and $r = r_2$ when $\theta = \theta_2$, find the area in terms of r_1, r_2, and α.

(34) Sketch the graph of $r^2 = a^2 \cos^2 \theta$ and find its area.

(35) Integrate $\displaystyle\int \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$. Transform the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ into a polar equation by writing $x = r\cos\theta$ and $y = r\sin\theta$. Then, using the relation (IX.8), show that the area of the ellipse is πab. (Compare with Ex. 25.)

(36) Find the area enclosed by the graph of $y^2 = (x - a)(b - x)$.

(37) Show that the area enclosed by OX, OY, and any tangent to the rectangular hyperbola $xy = k$ is constant and equal to $2k$.

(38) Find the area under the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $x = 0$ to $2\pi a$. (See Art. 126.)

(39) (i) Defining the "root mean square" value of a quantity Q for an interval T as the square root of $\dfrac{1}{T}\displaystyle\int_0^T Q^2 dt$, find the "root mean square" value of an electric current i for a complete period T—

(*a*) when $i = I\sin\left(\dfrac{2\pi t}{T} + \alpha\right)$

(*b*) when $i = I_1\sin\left(\dfrac{2\pi t}{T} + \alpha_1\right) + I_2\sin\left(\dfrac{4\pi t}{T} + \alpha_2\right)$

where I, I_1, I_2, α, α_1, α_2, T are constants.

(ii) Find the mean value of the product ie for a complete period T, when

$$i = I\sin\left(\frac{2\pi t}{T} + \alpha\right) \text{ and } e = E\sin\left(\frac{2\pi t}{T} + \beta\right)$$

where I, E, T, α, β are constants. (U.L.)

(40) If the voltage in an electric circuit is V, the current I amperes, the self-induction L henries, and the resistance R ohms, then $L\dfrac{dI}{dt} + RI = V$ where t is the time in seconds. Show that if $I = I_0 \sin pt$ where I_0 and p are constants, $V = I_0\sqrt{R^2 + L^2p^2}\sin(pt + \alpha)$ where $\alpha = \tan^{-1}\dfrac{Lp}{R}$.

The power in watts P is given by $P = IV$. Show that the mean value of the power is $\frac{1}{2} I_0^2 R$.

VOLUMES, AREAS OF CURVED SURFACES

(41) Prove by integration the rules for the volumes of (i) a cone, (ii) a pyramid, (iii) a sphere. Show that the same rules are given exactly by the prismoidal formula (IX.30).

(42) Find by integration the volume of each portion into which a sphere 6 in. radius is divided by a plane which is 2 in. distant from the centre of the sphere.

(43) A paraboloid of revolution is generated by rotating the parabola $y^2 = 4ax$ about OX. Find the volume generated by that portion of the curve which lies between $x = 0$ and $x = L$. If R is the area of the cross-section at $x = L$, show that the volume is half that of a cylinder of base area R and length L.

(44) The area A of the cross-section of a solid at a distance x from the origin is a function of x. If the cross-section is everywhere perpendicular to OX, show that $\dfrac{dV}{dx} = A$ where V is the volume bounded by the cross-section of area A. If A is a function of t, show that $\dfrac{dV}{dt} = A\dfrac{dx}{dt}$.

(45) If the smaller segment of a circle of radius 5 in. cut off by a chord 8 in. long is rotated about the chord, find the volume of the solid generated.

(46) A hollow cone has a semi-vertical angle of 30° and is placed with its axis vertical and its vertex downwards. If water is running into the cone at 10 ft^3 per min., at what rate is the depth increasing (i) when the depth is x ft, (ii) when the depth is 2 ft?

(47) Find the area of the curved surface of the solid generated by rotating the portion between $x = 0$ and $x = 3$ of the graph $y = 3x^2$, (i) about OX, (ii) about OY.

(48) Find the area of the curved surface of the cup formed by the revolution about its axis of the smaller part of the parabola $y^2 = 4ax$ cut off by the line $x = 3a$. (U.L.)

(49) Find by integration the area of the surface of a sphere of radius a. Show that the area of the surface intercepted between two planes distance c apart, $c < 2a$, is $2\pi ac$.

(50) Find the volume of the solid generated by rotating about OX the portion between $x = 2$ and $x = 5$ of the graph of (i) $y = 2x^3$, (ii) $y = 2x^2 + 3$.

(51) The area intercepted by OX and the graph of $y = 3x(x - 2)$ is rotated (i) about OX, (ii) about OY. Find in each case the volume of the solid generated.

(52) A water vessel in the form of a frustum of a cone, height 5 in., diameters of ends 4 in. and 3 in. respectively, has its axis vertical and the narrow end uppermost. Find the volume of water in it if the depth is x in. If x is increasing at 0·3 in. per second, at what rate is the volume increasing?

LENGTHS OF CURVES

(53) A uniform chain hangs between two points 20 ft apart in the same horizontal line. The sag at the middle is 5 ft. Find the length of the chain on the assumption that it hangs in the following form (i) a circle, (ii) a parabola, (iii) a catenary.

(54) The chain of a suspension bridge has the form of the curve $x^2 = (b^2/h)y$ where the origin of co-ordinates is taken at the lowest point, the axis of y is vertical, b is half the span, and h is the dip of the chain. Write down an expression for the length of the chain in the form of an integral.

Show that, when h is much smaller than b, the radical under the integral sign may be expanded by the binomial theorem, and that the length of the chain is approximately $2b + 4h^2/3b$. (U.L.)

(55) For the purposes of an approximation, a half arch of the sine curve $y = a \sin x$ is to be replaced by the straight line $y = mx$. In order to find the best value of m, proceed as follows: Take the square of the difference of the ordinates of the curve and the line, and find the mean value by integration from 0 to $\frac{\pi}{2}$. Then find, in terms of a, the value of m that makes this mean value a minimum. (U.L.)

(56) Prove that, in the curve $y = c \cosh \frac{x}{c}$, the length of the arc measured from the point where $x = 0$ is $c \sinh \frac{x}{c}$.

A heavy uniform chain, 16 ft long, hangs symmetrically over two smooth pegs at the same level, so that the lowest point of the portion of the chain hanging between the pegs is 1 ft below the level of the pegs. Find the length of the two portions of the chain which hang vertically, and show that the distance between the pegs is $8 \log_e 2$ ft. (U.L.)

(57) Find the length of the cycloid in Ex. 38.

(58) Find the length of the catenary $y = 2 \cosh \frac{1}{2}x$ from $x = 0$ to $x = 4$. Find the area under the curve between these limits.

(59) Show that if c is large compared with x so that powers of $\frac{x}{c}$ higher than the second can be neglected in comparison with unity, an arc of the catenary $y = c \cosh \frac{x}{c}$ may be assumed to be an arc of a parabola. If $c = 40$, find approximately the percentage error involved in this assumption when finding the length of the arc between two points for which $x = 0$ and $x = 10$ respectively.

(60) A uniform chain of length $2l$ is stretched between two points in a horizontal line. Show that, if the sag k in the middle is small compared with l, the distance apart of the points of support is approximately $2\left(l - \frac{2k^2}{3l}\right)$.

(61) Show that the total length of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$ is $6l$. (See Fig. 114.)

(62) Given the curve whose equation is $y = c \cosh \frac{x}{c}$, find the length of arc between $x = 0$ and $x = c$. Find also—

(i) the area under the graph between the given values of x;

(ii) the volume generated by rotating the area about OX;

(iii) the volume generated by rotating the area about OY.

CENTROIDS, MOMENTS OF INERTIA, METACENTRE, CENTRE OF PRESSURE

(63) Find the position of the centroid of the area under the parabola $y = 4ax^2$ from $x = 0$ to $x = c$. Find also the centre of gravity of the solid generated by rotating the area about OX.

(64) The bending moment at a point distant x from the free end of a horizontal cantilever of length l carrying a total load of wl lb distributed uniformly is $\frac{1}{2}wx^2$.

The bending moment diagram is drawn with $\frac{1}{2}wx^2$ plotted along the vertical and x along the horizontal. Find the position of the centroid of the diagram. If $\bar{x}$ is the horizontal distance of the centroid from the free end and A the area of the diagram, the deflection is given by: Deflection $= \frac{A\bar{x}}{EI}$ where E and I are constants. Find the deflection.

(65) Show that the centre of gravity of the portion of a thin uniform spherical shell cut off by two parallel planes is halfway between the centres of the circular end sections.

(66) Find the position of the centre of gravity of (i) a semicircular area, (ii) a solid hemisphere of uniform density, (iii) a conical frustum, height h, radii of ends r_1 and r_2 ($r_2 > r_1$).

(67) Prove the rule for the distance of the centroid from the shorter of the two parallel sides of a trapezium. Parallel sides of lengths a and b, $a > b$; distance between them h.

(68) Integrate $xe^x dx$ from $x = 0$ to $x = h$.

A long vertical tapering rod of circular section has to bear a load of W at its end; the rod weighs w lb per unit volume, and the tensile stress f over every cross-section has to be constant. Determine the law giving the radius of the section at any distance y from the smaller end, and find the position of the centre of gravity of the rod when of length h. (U.L.)

(69) Find the position of the centre of gravity of the paraboloid of revolution in Ex. 43.

(70) Find the area and the centroid of the portion of a plane bounded by the parabola $y^2 = ax$, the line $x = b$, and the axis $y = 0$.

The area is revolved about the axis of y so as to form a solid ring. Find the volume of the ring. (U.L.)

(71) Find the area of the loop of the curve whose equation is $ay^2 = (x - a)(x - 5a)^2$.

Also determine the distance of the mass centre of this area from the y axis. (U.L.)

(72) Find (i) the centroid of a circular sector, radius r, angle θ; (ii) the centroid of the circular arc in (i). Find (ii) from (i) by assuming the sector to be divided up into an infinite number of equal sectors.

(73) Find the centroid of the area of the complete cardioid $r = a(1 + \cos\theta)$ and also that of the portion between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(74) Find the centroid of the area under the portion of the graph plotted between values of x and y from $x = 0$ to $x = 2\pi a$ if $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$.

(75) Find the centroid of the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

(76) Find the centroid of the area under $y = c\cosh\frac{x}{c}$ from $x = -a$ to $x = +a$.

(77) Find the centroid of a circular arc, radius r, angle α, whose density varies as the distance along the arc from one end.

(78) Using the theorems of Pappus or Guldinus, find the co-ordinates of the centroid of—

(i) a semicircular arc;
(ii) the area of a semicircle;
(iii) the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and OX.

(In (iii) assume that the volume of the ellipsoid generated is $\frac{4}{3}\pi ab^2$.)

(79) Find the position of the centroid of the smaller segment of a circle of radius 5 in. cut off by a chord 6 in. long.

(80) Find the moments of inertia of the area under $y = 3 + x^2$ from $x = 0$ to $x = 3$, (i) about OX, (ii) about OY.

(81) Find the moment of inertia of the area under $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX and about OY.

(82) Find the moment of inertia of the area of a circle, radius a, about its centre O. Using the "perpendicular axes" theorem, find the I about a diameter and, using the parallel axes theorem, find the I about a tangent. Write down the value of the radius of gyration in each case.

(83) Find I in the following cases—

(i) the area of a triangle about its base (base B, height H);
(ii) the mass M of a thin uniform triangular plate about its base;
(iii) both (i) and (ii) about an axis through the vertex parallel to the base.

(84) Find the I of the following uniform solids—

(i) a sphere about a diameter, radius r, mass M;
(ii) a circular cylinder about its axis, radius r, length l, mass M;
(iii) a circular cylinder about a diameter through the centre of gravity;
(iv) a circular cylinder about a diameter through one end.

(85) Find the I and the radius of gyration of each of the following homogeneous solids—

(i) a cone about its axis (height h, radius of base r);
(ii) a hemisphere about a diameter of the plane face, radius r.

(86) Find the moment of inertia, about an axis through the centre of gravity parallel to its length, of a rectangular solid length l, breadth b, width a. Find also the I about a parallel axis passing through the centroid of one of the faces whose area is al. Find the radius of gyration about this latter axis.

(87) Show that the momental ellipse for a square area is a circle, and that in consequence the moment of inertia of a square is the same about all axes which lie in its plane and pass through its centroid.

(88) Draw a rectangle $ABCD$ in which $AB = 4$ in. and $BC = 6$ in. Draw EF and HG parallel to AB, cutting AD in E and H respectively, and BC in F and G respectively, so that $AE = 1$ in., $EH = 4$ in., $HD = 1$ in. On EF mark off $EK = 1\frac{1}{2}$ in., $KL = 1$ in., and $LF = 1\frac{1}{2}$ in. Draw KM and LN perpendicular to EF, cutting HG in M and N respectively. Find the equation to the momental ellipse at the centroid of the area made up of the three rectangles $ABFE$, $LNMK$, $HGCD$ referred to the principal axes, taking the axis of x parallel to AB. Find the I about an axis making 30° with OX.

(89) Find the I of a cone of mass M, radius r, height h, about a diameter of the base. Find the radius of gyration.

(90) Find *ab initio* the moment of inertia of a lamina of mass M in the shape of an isosceles triangle ABC, in which $AB = AC$,

(i) about its base BC;
(ii) about the perpendicular from A to BC.

Deduce the moment of inertia about AB, and about a line through A perpendicular to the plane of the lamina. (U.L.)

(91) Two tangents, inclined at an angle of 2α to each other, are drawn to a circle of radius a. The figure bounded by the tangents and the greater arc of the

circle between their points of contact is rotated about its axis of symmetry. Find the volume of the solid thus generated, and show that, if it is of uniform density, its radius of gyration about its axis of symmetry is k given by the equation $k^2 = \frac{1}{10}a^2(3 + 2\sin\alpha - \sin^2\alpha)$. (U.L.)

(92) Find the moment of inertia about OX and deduce that about OY of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equation referred to the same axes of the momental ellipse.

(93) A rod of length l and of uniform density m has a rectangular section with sides of lengths a and b respectively. Find the moment of inertia of the rod about a diagonal of one of the ends. (U.L.)

(94) Calculate the kinetic energy of a uniform solid sphere of mass M and radius a, rolling on a plane, when the velocity of its centre of gravity is v.

Compare the time taken by a sphere to roll a given distance down an inclined plane (i) when it is solid and uniform, (ii) when it is uniform with a concentric cavity of half its radius. (See the worked example, Art. 120.) (U.L.)

(95) Find the radii of gyration of a rectangular lamina of sides a and b about the principal axes through its centre of gravity.

A uniform solid rectangular block has its edges 3 ft, 4 ft, 5 ft, and the specific gravity of its material is 0·5. Determine whether it can float in water in stable equilibrium with four of its faces vertical. (U.L.)

(96) Prove that the centre of pressure of a triangle wholly immersed in a liquid with one side parallel to the surface is at the centroid of three particles, placed at the middle points of the sides, with masses proportional to their depths below the surface.

One wall of a tank is vertical and contains a triangular trap-door which is hinged about the horizontal side BC, it has the vertex A lower than BC, and can open outwards. The vertical heights of the vertices above the bottom of the tank are a, b, b. Prove that, if water be poured into the tank to a height h so that the trap-door is entirely below the surface, the latter will remain closed provided a horizontal force is applied at A greater than $\frac{1}{6}w\Delta(2h - b - a)$, Δ being the area of the triangle, and w the weight per unit volume of the water. (U.L.)

(97) Prove the formula $HM = AK^2/V$ for the height of the metacentre (M) above the centre of buoyancy (H) in the case of a small rotational displacement in a plane of symmetry of a floating body.

Show that a hollow cylinder of height h, internal radius a, external radius b, specific gravity s, open at both ends, will float in stable equilibrium with its axis vertical, provided

$$a^2 + b^2 > 2h^2s(1 - s) \qquad \text{(U.L.)}$$

(98) Find the position of the metacentre, and the condition for the stability of the equilibrium, of a solid cylinder of specific gravity s, of radius a, and of height h, floating with its axis vertical in water. (U.L.)

(99) Prove the formula for the height of the metacentre (for a small rotation in a vertical plane of symmetry) above the centre of buoyancy in the case of a floating solid, explaining the symbols involved. A cylinder of wood 6 in. long floats in water with its axis vertical and with 4 in. of its length immersed. Find the least possible radius of the cylinder if the equilibrium is to be stable. (U.L.)

(100) Prove that in a homogeneous fluid of density ρ at rest under gravity, the fluid pressure at a depth z below the surface is $g\rho z$.

A plane area of any shape is held vertically, completely immersed in water.

The level of the surface of the water is now raised an amount h by pouring more water into the containing vessel. Prove that the centre of pressure of the *water* on the area is raised a height $h(x - y)/(h + y)$, where x, y are the original depths of the centre of pressure and the centroid of the area respectively. (U.L.)

(101) A circular plate is immersed vertically in a liquid, so that its centre is at a depth equal to the diameter.

Find the centre of liquid pressure on (*a*) one face of the whole plate, (*b*) the half of that face cut off by the vertical diameter. (U.L.)

(102) A circular area of radius a feet is immersed in water with its plane vertical. The surface of the water rises from $2a$ feet above the centre of the circle to $4a$ feet above it. Neglecting atmospheric pressure, prove that the centre of pressure rises through a distance $\frac{a}{16}$ feet. (U.L.)

(103) Prove that the force exerted by a fluid on a submerged plane area is equal to the area multiplied by the pressure at the centroid of the area.

An outlet in the vertical side of a tank is closed by a semicircular flap of radius a, hinged at its diameter, which is horizontal and uppermost. If the level of the liquid in the tank is a above the hinge, show that the smallest force that can cause the flap to open is

$$\frac{1}{24} w a^3 (3\pi + 16)$$

where w is the weight of unit volume of the liquid. (U.L.)

(104) Explain the use of the metacentre as a criterion of stability of a floating body.

A uniform hollow cone is of total height H and internal height $\tfrac{1}{2}H$, and the external and internal radii of the base are R and $\tfrac{1}{2}R$. It floats with its axis vertical and its vertex at a depth d below the surface of a liquid. Show that the equilibrium will be stable if

$$d > \frac{41 H^3}{42(H^2 + R^2)} \qquad \text{(U.L.)}$$

(105) A uniform solid in the form of a circular cylinder with hemispherical ends (outwards) has mass 300 lb, diameter 10 in., and total length 20 in. The solid rotates about its geometrical axis at 60 r.p.m. Find the kinetic energy of the solid.

CHAPTER X

ROULETTES AND GLISSETTES

123. **Displacement of a Figure in its Own Plane.** The position at any instant of a rigid plane figure moving in its own plane is known when the positions of two points of it are known. Let A and B (Fig. 108) be the positions at a given instant of time of two points in a plane figure of fixed dimensions, and let A' and B' be their respective positions at a later instant. PI and QI, which intersect at I, are the perpendicular bisectors of the straight lines AA' and BB' respectively.

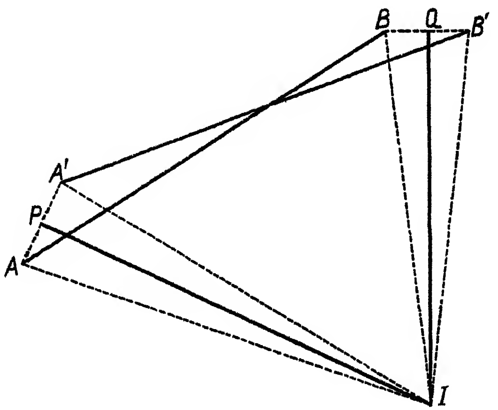


FIG. 108

Then $\overline{IA} = \overline{IA'}$ $\overline{IB} = \overline{IB'}$ and $\overline{AB} = \overline{A'B'}$, so that the triangles IAB, $IA'B'$ are congruent. Hence, AB can be made to move into the position $A'B'$ by simply rotating the figure through an angle $AIA' = BIB'$ about the point I. Rotation about I displaces the figure containing the points A and B from the first position to the second.

I is known as the "centre of rotation." When AB and $A'B'$ are parallel, the lines AA' and BB' are parallel, and so also are the lines PI and QI. In this case, I is at an infinite distance from A and B and there is no rotation, the motion being one of translation parallel to the lines AA' and BB'.

Next imagine the plane figure to be moving continuously in its own plane, and let AB, $A'B'$ (Fig. 109) be the two positions of a

straight line in it at instants of time the interval between which is small and is Δt seconds.

Assuming that AA' and BB' are short straight lines, we repeat the construction of Fig. 108 and thus obtain the centre of rotation I. Now consider what happens as we diminish the value of Δt and make it approach the value zero. As Δt diminishes, the lengths of AA' and BB' both diminish also, and tend to the value zero. The limiting positions of AA' and BB' are those of the tangents at A and B to the paths of A and B respectively, and the limiting positions of PI and QI are those of the normals at A and B respectively to the same paths. Thus, in the limit the method of finding I is that shown in Fig. 110. The curves show the paths of A and B, and I is at the intersection of the normals drawn at A and B to these paths. At the

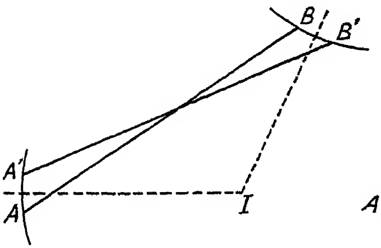


FIG. 109

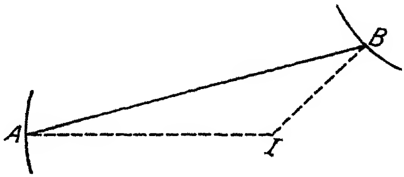


FIG. 110

instant when the line AB occupies the position shown, the figure is moving as if it were rotating about I. Being the centre of rotation at an instant of time I is known as the *instantaneous centre* of rotation. The instantaneous centre is of importance in the study of mechanisms.

If AB in Fig. 110 is a bar in a mechanism and carries with it a rigid framework, I is the instantaneous centre of rotation of the framework. If, however, AB is one bar of a deformable linkwork, each part of the linkwork which moves relatively to the other parts has its own instantaneous centre. For the application of the use of the instantaneous centre to the study of the relative motions of machine parts, the reader is referred to textbooks on the theory of machines. We shall content ourselves with two simple examples.

EXAMPLE 1

Fig. 111 shows the mechanism of the simple steam-engine. OC is the crank, CP the connecting-rod, and P the cross-head. Find the instantaneous centre of rotation of CP and show how to find the velocity of any point Q in CP.

OC is the normal to the path of C, and a vertical line through P is the normal

to the path of P. These normals produced intersect at I, which is therefore the instantaneous centre of CP

Let $\omega =$ angular velocity of OC in radians per sec.

$r =$ length of OC in ft

Then velocity of $C = \omega r$ ft per sec.

But if Ω is the instantaneous angular velocity of CP, the velocity of C is $\Omega \cdot \overline{IC}$

and $\Omega \cdot \overline{IC} = \omega r$

or $$\Omega = \omega \frac{r}{\overline{IC}} \qquad \text{(X.1)}$$

The velocity of Q is given by V_Q where

$$V_Q = \Omega \cdot \overline{IQ}$$

$$= \frac{\overline{IQ}}{\overline{IC}} \cdot \omega r$$

or $$\frac{V_Q}{V_C} = \frac{\overline{IQ}}{\overline{IC}} \qquad \text{(X.2)}$$

where $V_C =$ velocity of C. We see therefore that the velocity of any point in CP is proportional to its distance from the instantaneous centre I.

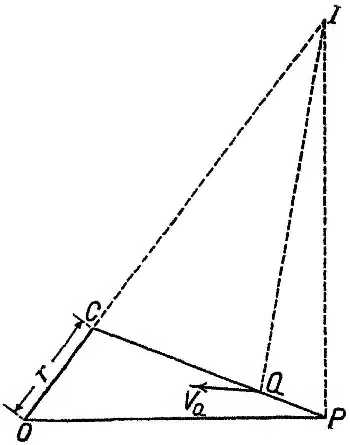


FIG. 111

If we make Q coincide with P, (X.2) becomes

$$V_P = \frac{\overline{IP}}{\overline{IC}} V_C \qquad \text{(X.3)}$$

EXAMPLE 2

A ladder ACB of length l ft has its lower end B on the ground, and overhangs the top of a vertical wall OC, as shown in Fig. 112. If B moves along the ground whilst the ladder touches the wall at C, find the position of the instantaneous centre of rotation of AB, and find in what position of the ladder the two ends will move with the same speed.

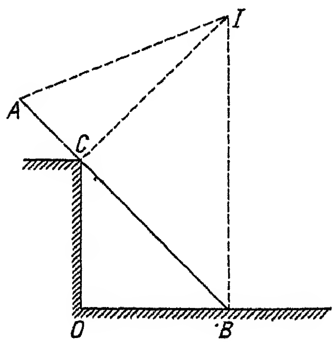


FIG. 112

At the instant when the ladder occupies the position AB the point C of the ladder has no component velocity perpendicular to the direction of the ladder. Hence, the point C is moving along AB and the instantaneous centre is on a line through C perpendicular to AB. Again, the point B of the ladder is moving along OB and the instantaneous centre is on a line through B perpendicular to OB. The instantaneous centre is the point I in which these two lines CI and BI meet.

If V_A = velocity of A, and V_B = velocity of B

$$\frac{V_A}{V_B} = \frac{\overline{IA}}{\overline{IB}}$$

and if $V_A = V_B$, we have $IA = IB$, so that the speeds of the ends will be equal when the middle point of AB touches the top of the wall.

124. **Space-centrode and Body-centrode.** As the line AB (Fig. 110) moves in the plane of the paper so also does the point I. If for every position of AB the position of I is marked on the paper and a curve is drawn through the successive positions of I, this curve is the locus in space of the instantaneous centre. The locus is known as the *space-centrode* of the rigid figure of which AB is a line. Suppose that

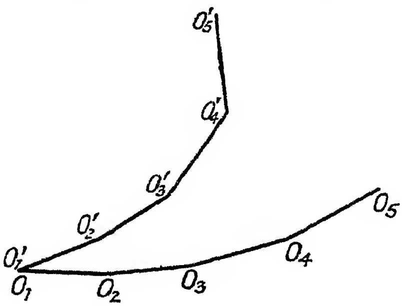


FIG. 113

a second piece of paper is carried by the line AB and that the successive positions of the instantaneous centre are drawn on this paper. The curve obtained is the locus of the instantaneous centre relative to AB and is known as the *body-centrode*.

Let O_1, O_2, O_3, O_4, etc. (Fig. 113), be consecutive positions of the instantaneous centre in space, i.e. on the space-centrode; and let O'_1, O'_2, O'_3, O'_4, etc., be the corresponding points on the body-centrode. The motion of the figure may be assumed to be made up of a number of rotations. First, a rotation about O_1 brings O'_2 into coincidence with O_2. This is followed by a rotation about O_2 until O'_3 coincides with O_3. Then a rotation about O_3 makes O_4' coincide with O_4, and so on. This is the motion which would be produced by rolling the polygon $O_1'\ O_2'\ O_3'\ O_4'$ . . . on the polygon $O_1O_2O_3O_4$ . . . The above motion is not the actual motion of the body, but, by taking the consecutive positions of the instantaneous centre closer together, we can obtain a closer approximation to the actual motion of the body. In the limit when we make the distances between each

pair of successive positions of the instantaneous centre approach the value zero, the polygons become continuous curves and the actual motion of the body is the same as that obtained by rolling the body-centrode (which carries the figure) along the space-centrode.

EXAMPLE 1

A straight line AB of fixed length moves with its ends A and B on the rectangular axes of x and y, as shown in Fig. 114. Find the position of the

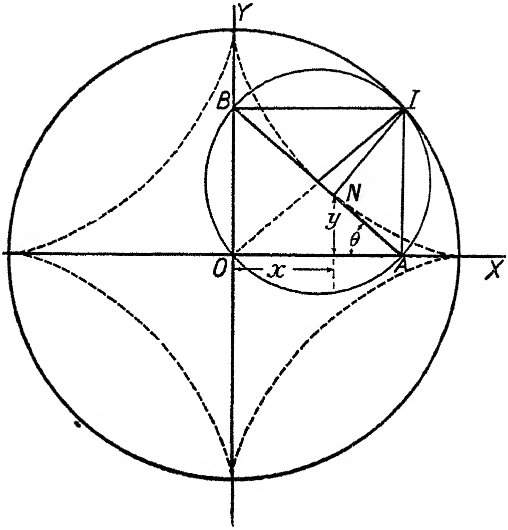


FIG. 114

instantaneous centre for the position shown, and find the space- and body-centrodes. Find also the equation of the envelope or curve which AB always touches.

Since B is moving along the vertical OY and the instantaneous centre I is situated on the normal to the path of B, I must lie on a horizontal line through B. Similarly, as A is moving along the horizontal line OX, I must lie on a vertical line through A. Drawing these lines, we see that the figure $AOBI$ is a rectangle.

Now $\overline{OI} = \overline{AB}$ = constant, hence, the locus of I for different positions of AB is a circle of centre O and radius $\overline{AB}$. This is the space-centrode.

To find the body-centrode we make use of the fact that for all positions of AB the angle AIB is a right angle. Thus, a circle on AB as diameter will always pass through I. This circle is the body-centrode. These circles are shown in the figure, and the motion of AB sliding with its ends on the two axes is exactly the same as that which would be produced if AB were carried by the smaller circle when the latter was rolling without slipping inside the larger fixed circle of twice its diameter.

Let IN be perpendicular to AB. Then N is the only point in AB which is moving along AB and has no motion perpendicular to AB. If a line moves whilst remaining in contact with a curve it is evident that the point of contact of the line and curve must be instantaneously moving along the line. Hence, N is the point of contact of the line AB with its envelope.

Let x and y be the co-ordinates of N. Then, if $\widehat{OAB} = \theta$ and $AB = l$,

$$BI = l\cos\theta,\ BN = l\cos^2\theta,\ AN = l - l\cos^2\theta = l\sin^2\theta$$

Hence, $x = BN\cos\theta = l\cos^3\theta$, and $y = AN\sin\theta = l\sin^3\theta$.

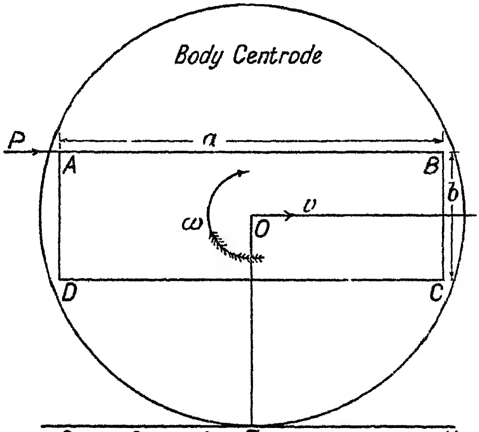


FIG. 115

Since $\sin^2\theta + \cos^2\theta = 1$, the equation to the envelope of AB is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(X.4)}$$

This envelope, shown dotted, is known as the four-cusped hypocycloid. (See Art. 127.)

If P is any point in AB such that $BP = a$, $PA = b$, it is easy to deduce that, if x, y are the co-ordinates of P, $x = a\cos\theta$ and $y = b\sin\theta$. Since $\cos^2\theta + \sin^2\theta = 1$, then $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, which is the equation of an ellipse. Hence, any point in the line AB describes an ellipse, as the ends A, B move on the axes. (The point P is not shown in the figure.)

EXAMPLE 2

A uniform rectangular lamina $ABCD$ of mass M is resting on a smooth horizontal table (Fig. 115). It receives an impulsive blow P at A in the direction AB. Find the space- and body-centrodes for its subsequent motion.

Let $AB = a$ and $BC = b$, and let O be the centre of gravity of the lamina. Let I be the moment of inertia of the lamina about O. Then, by Ex. 1, Art. 118, $I = \dfrac{M}{12}(a^2 + b^2)$.

If v is the linear velocity of the centre of gravity, and ω the angular velocity about the centre of gravity after the impulse has been applied, the equations of motion are

$$\text{Impulsive force} = \text{change of linear momentum}$$

or $$P = Mv \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and $$\text{Impulsive couple} = \text{change of angular momentum}$$

$$\tfrac{1}{2}Pb = I\omega$$

$$\therefore \qquad Pb = \frac{M}{6}(a^2 + b^2)\omega \qquad (2)$$

Hence, from (1) $$v = \frac{P}{M}$$

and from (2) $$\omega = \frac{6Pb}{M(a^2 + b^2)}$$

Draw a line OF perpendicular to DC. Let $OF = x$. Then, if we consider the line OF to move with the lamina, we see that if u is the velocity of F, then

$$u = v - \omega x$$

the first term on the right being due to the motion of O, and the second to rotation about O. If $v = \omega x$, i.e. $x = \frac{v}{\omega}$, then F is instantaneously at rest and is, in fact, the instantaneous centre. Since v and ω remain unchanged in magnitude and direction, the instantaneous centre is always at a distance $\frac{v}{\omega}$ below the line of O's motion, which is a straight line through O parallel to AB. Thus, the motion of the lamina is the same as that due to the rolling of a circle of radius $OF = \frac{v}{\omega} = \frac{a^2 + b^2}{6b}$, on the line FH parallel to AB. This circle is the body-centrode, and the line FH is the space-centrode.

125. **Roulettes.** A *roulette* is the path described by a point carried by a plane curve which rolls without slipping on a fixed curve in its own plane. A knowledge of the properties of roulettes is of importance in connection with the design of wheel teeth, as well as in the science of kinematics. In this latter connection we have seen above that the path of any point in a rigid body in plane motion is a roulette. We give below the properties of certain roulettes which are of importance to students of engineering.

EXAMPLE

A polygon $A'A_1'A_2'A_3'$ . . . (Fig. 116) rolls without slipping on a fixed polygon $AA_1A_2A_3$ . . . whose sides AA_1, A_1A_2, A_2A_3, etc., are respectively equal to the sides $A'A'_1$, $A_1'A_2'$, $A_2'A_3'$, etc., of the moving polygon. Draw the roulette described by a point P fixed relative to, and carried by the moving polygon.

The motion is similar to that described in Art. 124. The polygon first rotates about A with which A' coincides until A_1' coincides with A_1. During this motion,

P rotates about A through the circular arc PP_1. The angle subtended by PP_1 at A is equal to the angle A_1AA_1'. The polygon now rotates about A_1 until A_2' coincides with A_2. During this rotation, P moves from P_1 to P_2 as shown, the centre of the arc P_1P_2 being the point A_1. Afterwards the polygon rotates in turn about A_2, A_3, A_4, etc., and the point P traces out the circular arcs P_2P_3, P_3P_4, P_4P_5, etc., respectively. The drawing of the roulette is facilitated if the moving

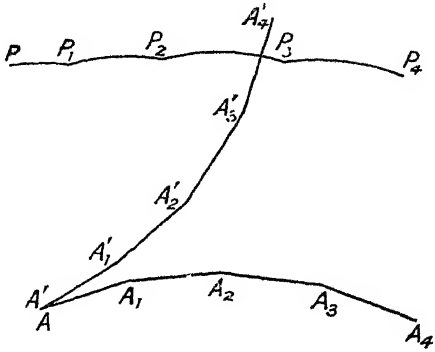


FIG. 116

polygon and the point P are drawn on tracing paper, and the polygon $A'A_1'A_2'\ldots$ rolled along the fixed polygon, the positions of P being pricked through on to the drawing paper on which the fixed polygon is drawn. The roulette may then be drawn by means of compasses. We have only shown parts of the fixed and moving polygons; $PP_1P_2P_3P_4$ is the corresponding part of the roulette.

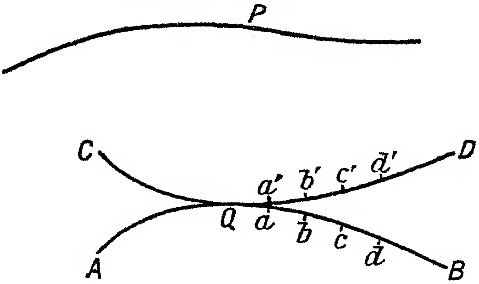


FIG. 117

In order to draw the point roulette described by a point P which is carried by a curve CD which rolls without slipping on a curve AB (Fig. 117), we adapt the method of the above example. The curve CD and the point P are drawn on tracing paper, and the curve AB is drawn on drawing paper. The curves are placed in contact at Q, as shown, and the position of P is pricked through on to the drawing paper by means of a needle. The needle is next stuck into the tracing paper and the drawing paper at Q, and the tracing paper is rotated until a point a' near Q on CD coincides with a point a on AB. The needle point is now

transferred to pass through the coincident points aa', and the tracing paper is rotated until the curves touch at aa'. The position of P is again pricked through. This procedure is repeated, the curves being made to touch at b, c, d, etc., in order, and the position of P is marked off in each case. A smooth curve through the positions of P gives the portion of the roulette due to the rolling between Q and B. A similar procedure will give the portion due to the rolling of QC on QA, and the two portions make up the complete roulette.

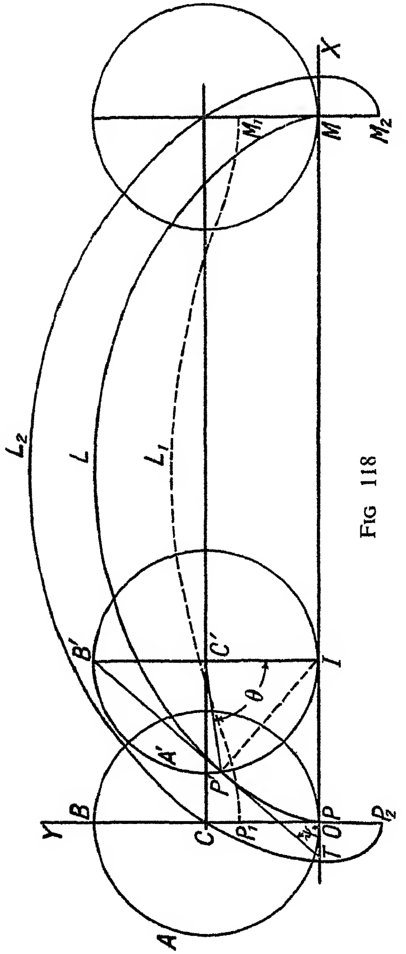


Fig 118

126. **The Cycloid.** The *cycloid* is the path of a point on the circumference of a circle which rolls without slipping along a fixed straight line. This path is a roulette and can be drawn by the method of the preceding article. In Fig. 118 OLM is the cycloid generated by a point P carried by the circle OAB, centre C, as it rolls without slipping along the straight line OX. Suppose that P originally coincides with O and that the circle OAB moves to the position $IA'B'$ in which its centre is at C'. Taking OX as axis of x, and a vertical line through O as axis of y, we have, if the angle $IC'P' = \theta$ and r is the radius of the circle,

$$OI = \text{arc } IP' = r\theta$$

and $\quad x = OI - \text{horizontal projection of } C'P'$

$$= r\theta - r\sin\theta$$

or $\quad x = r(\theta - \sin\theta) \qquad \text{(X.5)}$

Also $\quad y = C'I - \text{vertical projection of } C'P'$

$$= r - r\cos\theta$$

or $$y = r(1 - \cos\theta) \quad . \quad . \quad . \qquad \text{(X.6)}$$

(X.5) and (X.6) give x and y in terms of the parameter θ.

From (X.6) we find $\theta = \cos^{-1}\left(1 - \frac{y}{r}\right)$ and by substituting this in (X.5) we could obtain a relation independent of θ. This relation is, however, very cumbersome and we find it better to use the two relations (X.5) and (X.6), from which we can obtain any necessary properties of the curve.

To find the gradient at the point P' (x, y). By (II.45)

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$ and therefore differentiating (X.5)

and (X.6) and substituting,

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}$$

or, since $$\frac{\theta}{2} = \frac{\pi}{2} - \widehat{C'IP'}$$

$$\frac{dy}{dx} = \tan\widehat{C'IP'} \quad . \quad . \qquad \text{(X.7)}$$

If ψ = inclination of tangent to $OX = \widehat{XTP'}$, then

$$\frac{dy}{dx} = \tan\psi \quad . \quad . \quad . \qquad \text{(X.8)}$$

and therefore $$\psi = \widehat{C'IP'}$$

Hence, the tangent $P'T$ is perpendicular to IP' as is otherwise obvious, since I is the instantaneous centre of rotation.

As the angle in a semicircle is a right angle, the tangent TP' produced passes through B', the highest point of the circle. Thus we have the following simple method of drawing the tangent to a cycloid at any point P'. Draw the rolling circle through P' (there are two possible positions for the rolling circle, but there is no difficulty in

determining which is the proper one) and join P' by a straight line to the highest point of the circle. This line is the tangent at P'.

The complete locus of the point P' consists of an infinite number of portions, each of which is a copy of the portion OLM.

EXAMPLE 1

Find the area under the portion OLM of the cycloid.

Let A = area under portion OLM

Then $$A = \int_0^{\overline{OM}} y\,dx$$

But $$y = r(1 - \cos\theta)$$

and $$x = r(\theta - \sin\theta)$$

Hence, $$dx = r(1 - \cos\theta)d\theta$$

and
$$\begin{aligned} A &= \int_0^{2\pi} r^2(1 - \cos\theta)^2 d\theta \\ &= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)d\theta \\ &= 3\pi r^2 \end{aligned}$$

or the area of OLM is three times that of the rolling circle.

EXAMPLE 2

Find the length of the arc OLM.

From the triangle PQK (Fig. 85)

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Hence, $$\frac{\Delta s}{\Delta\theta} = \sqrt{\left(\frac{\Delta x}{\Delta\theta}\right)^2 + \left(\frac{\Delta y}{\Delta\theta}\right)^2}$$

and in the limit

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$\therefore \text{ length of arc} = s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

and as $$\frac{dx}{d\theta} = r(1 - \cos\theta),\ \frac{dy}{d\theta} = r\sin\theta$$

$$\begin{aligned} s &= r\int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\, d\theta \\ &= r\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = r\int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \\ &= 4r\left[-\cos\frac{\theta}{2}\right]_0^{2\pi} = 8r \end{aligned}$$

The path of any point P_1 on CP or P_2 on CP produced is known as a *trochoid.* The trochoid traced by P_2 has loops. The trochoids $P_1L_1M_1$ and $P_2L_2M_2$ which are the paths of P_1 and P_2 respectively are shown in Fig. 118. The equation to the trochoid is

$$x = r\theta - c \sin \theta \qquad \text{(X.9)}$$

$$y = r - c \cos \theta \qquad \text{(X.10)}$$

where $c = CP_1$ or CP_2, the proofs of these being left as exercises for the reader. (Exs. X, No. 22.)

127. **The Epicycloid and the Hypocycloid.** When one of two coplanar circles rolls without slipping upon the other circle, which is fixed, any point on the circumference of the rolling circle traces an *epicycloid* or a *hypocycloid,* the former if the rolling circle is outside, and the latter if it is inside of the fixed circle. Fig. 119 shows a circle PEQ, centre C, rolling on the outside of a fixed circle DQA, centre O, Q being the point of contact. Suppose that when rolling starts the line OC is horizontal and the points P and A are in contact. Let

$$\widehat{QOA} = \theta \text{ and } \widehat{QCP} = \phi$$

and let R and r be the radii of the fixed and rolling circles respectively.

Let OA produced be the axis of x and the vertical through O the axis of y.

The angle between OX and CP is $(\theta + \phi)$.

Hence, $\quad x =$ difference of horizontal projections of OC and CP

$$= OC \cos \theta - CP \cos (\theta + \phi)$$

$$= (R + r) \cos \theta - r \cos (\theta + \phi)$$

Since the arcs QP and QA are equal,

$$R\theta = r\phi \text{ and } \phi = \frac{R}{r}\theta$$

Therefore,
$$x = (R + r) \cos \theta - r \cos \left(\theta + \frac{R}{r}\theta\right)$$

or
$$x = (R + r) \cos \theta - r \cos \frac{R + r}{r}\theta \qquad \text{(X.11)}$$

Similarly, $\quad y =$ difference of vertical projections of OC and CP

$$= OC \sin \theta - CP \sin (\theta + \phi)$$

or
$$y = (R + r) \sin \theta - r \sin \frac{R + r}{r}\theta \qquad \text{(X.12)}$$

By substituting different values of θ in (X.11) and (X.12) we can find pairs of corresponding values of x and y, and from these the graph can be plotted. The cycloid, epicycloid, and hypocycloid are, however, best drawn by means of geometrical constructions which the reader should have no difficulty in discovering for himself. The path of P is made up of a number of equal portions, of which three are shown.

Fig. 120 shows the circle PEQ rolling inside the fixed circle QDA. With the same notation as before,

$$x = \text{sum of horizontal projections of } OC \text{ and } CP$$
$$= OC \cos \theta + CP \cos (\phi - \theta)$$

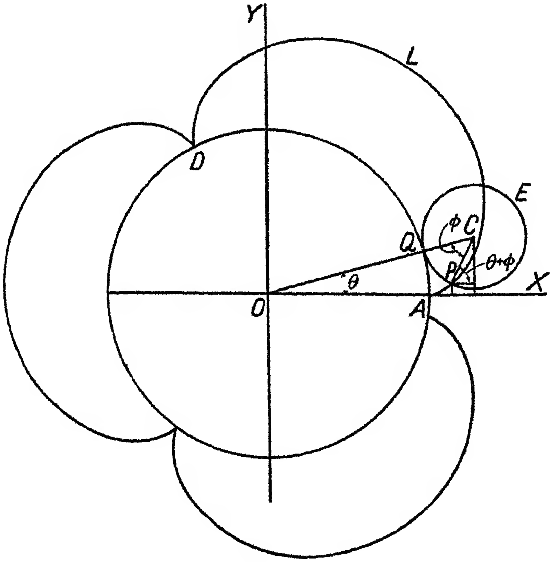


FIG. 119

and $$y = \text{difference of vertical projections of } OC \text{ and } CP$$
$$= OC \sin \theta - CP \sin (\phi - \theta)$$

from which the two equations to the hypocycloid are

$$x = (R - r) \cos \theta + r \cos \frac{R - r}{r} \theta \quad . \quad . \quad \text{(X.13)}$$

and $$y = (R - r) \sin \theta - r \sin \frac{R - r}{r} \theta \quad . \quad . \quad \text{(X.14)}$$

If r is greater than R the circle PEQ envelops the circle QDA, the curve traced out by P is then known as the *pericycloid*, and that traced out by any other point on CP or CP produced is a *peritrochoid.*

If $R = r$, the epicycloid described in Fig. 119 is a cardioid (see Ex. 1, Art. 108). If $R = 4r$ the locus given by (X.13) and (X.14) becomes the four-cusped hypocycloid of Fig. 114. If $R = 2r$ the hypocycloid becomes a diameter of the fixed circle. Thus, in Fig. 114, the points A and B are moving along the diameters OA and OB

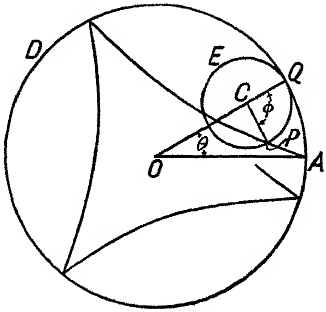


FIG. 120

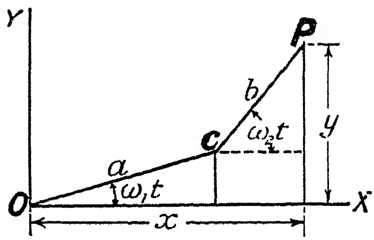


FIG. 121

respectively of the fixed circle. All other points on the circle $OAIB$ are moving along diameters of the fixed circle, for each point is moving along the perpendicular to the line joining it to I, and is therefore moving along a diameter through O. Every point on the rolling circle traces out a diameter of the fixed circle.

If r becomes infinite the pericycloid becomes an involute of the circle DQA (see Art. 85). Thus the involute of a circle is a roulette generated by a point in a straight line which rolls without slipping on the circumference of a fixed circle. (Compare this with the definition of the involute in Art. 85.)

The gradient at any point of an epicycloid or a hypocycloid is found by the method used in Art. 126 for finding the gradient of the cycloid. We can, however, make use of the properties of the instantaneous centre in order to draw the tangent to any of these curves. As Q is the instantaneous centre of rotation, the tracing point P moves along a line perpendicular to PQ, and this perpendicular is the tangent at P to the locus of P.

The curve traced out by a point lying in CP or CP produced is known as an *epitrochoid* in Fig. 119 and a *hypotrochoid* in Fig. 120.

128. **Epicyclics.** Suppose the line OC in Fig. 121 to be rotating about O with uniform angular velocity ω_1 radians per second in the anticlockwise sense, and suppose CP to be rotating about C with uniform angular velocity ω_2 radians per second. The path traced out by P is called an *epicyclic*. If the angular velocity of CP is in the same sense as that of OC, the epicyclic is *direct*; otherwise it is *retrograde*. It is easy to see that the epicycloid of Fig. 119 is an epicyclic. The inclination of CP to the horizontal is $\theta\left(\frac{R+r}{r}\right)$ in the notation of Art. 127, so that the angular velocity of CP is $\frac{R+r}{r}$ times that of OC in the same sense. Consequently, any point on CP or CP produced traces out a direct epicyclic. Thus all epitrochoids are direct epicyclics. In the same way we see that CP in Fig. 120 is rotating in the opposite sense to OC and with angular velocity $\frac{R-r}{r}$ times that of OC. Any point on CP or CP produced in this case traces out a retrograde epicyclic, and so all hypotrochoids are retrograde epicyclics. The reader will see from the following example that the ellipse is a retrograde epicyclic for which the two angular velocities are equal.

EXAMPLE 1

If a crank OC of length a rotates about O with constant angular velocity ω_1 radians per second, and a second crank CP of length b rotates about C in the same plane with constant angular velocity ω_2 radians per second, find the equations of the epicyclic described by P. If $\omega_2 = -\omega_1$, show that the epicyclic is an ellipse or a straight line according as a is different from or equal to b.

Take the axis of x along the line on which OCP is a straight line and the axis of y perpendicular to this at O. Then t sec. later the cranks will have rotated through angles $\omega_1 t$ and $\omega_2 t$ radians respectively, as shown in Fig. 121, and we have

$$x = a\cos\omega_1 t + b\cos\omega_2 t$$

$$y = a\sin\omega_1 t + b\sin\omega_2 t$$

If $\omega_2 = -\omega_1$ (Fig. 122), these equations become

$$x = a\cos\omega_1 t + b\cos\omega_1 t = (a+b)\cos\omega_1 t$$

$$y = a\sin\omega_1 t - b\sin\omega_1 t = (a-b)\sin\omega_1 t$$

Hence, since $\sin^2\omega_1 t + \cos^2\omega_1 t = 1$, we have

$$\frac{x^2}{(a+b)^2} + \frac{y^2}{(a-b)^2} = 1$$

the equation to an ellipse.

If $a = b$, the equations for x and y become

$$x = 2a\cos\omega_1 t \text{ and } y = 0$$

and the locus of P is therefore the line $y = 0$, i.e. the x-axis. (Compare this example with Exs. X, No. 37.)

The curves discussed in Arts. 126 and 127 are known as *point-roulettes*. The envelope of a line carried by the rolling curve is a *line-roulette*. The four-cusped hypocycloid in Ex. 1, Art. 124, is a line-roulette.

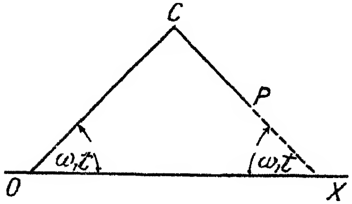


FIG. 122

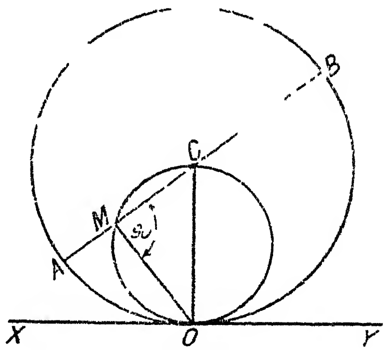


FIG. 123

EXAMPLE 2

Find the line-roulette generated by a diameter of a given circle which rolls without slipping along a given straight line.

The point in which the diameter AB (Fig. 123) touches its envelope is evidently at the foot M of the perpendicular to the diameter from the point of contact O of the circle and line, for this latter point is the instantaneous centre of rotation. Thus the point M on the roulette lies on the circumference of a circle of which OC is a diameter, C being the centre of the rolling circle. The circle OCM touches the line XY at the same point as the rolling circle, and its diameter is half that of the rolling circle. The line-roulette is therefore the cycloid generated by a point on a circle of half the diameter of the given circle rolling along the given straight line.

129. **Glissettes.** When a plane curve moves so as to touch two other fixed curves in the same plane the locus of any point which is fixed relative to the moving curve is a *glissette*. This locus is sometimes called a *point-glissette*, and the envelope of any straight line carried by the moving curve is called a *line-glissette*. The fixed and moving curves may, as particular cases, be straight lines or polygons. Either of the fixed curves may be a point which may be looked upon as a closed curve enclosing zero area. We shall give a few examples of glissettes.

EXAMPLE 1

In Ex. 1, Art. 124, we saw that when a straight line slides with its ends one on each of two fixed intersecting lines at right angles to each other, any point in it traces out an ellipse. The ellipse is then a point-glissette. The envelope of the line is the four-cusped hypocycloid which is therefore a line-glissette.

EXAMPLE 2

A plane moves parallel to itself so that two perpendicular lines in it, OP and PO', pass through the fixed points O and O'. Show that P describes a circle in

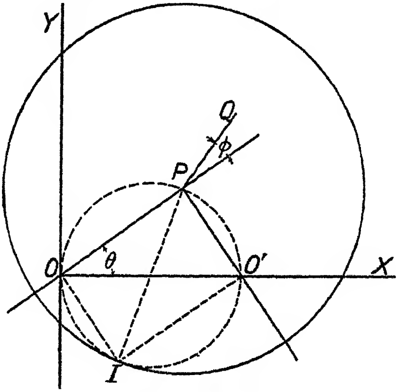


FIG. 124

space and that the glissette described by any other point is either a pericycloid or a peritrochoid.

Since in all positions of the plane $\widehat{OPO'}$ (Fig. 124) is a right angle, then the locus of P in space is a circle on OO' as diameter. At any instant the point in OP in contact with O is moving along OP, and the point in $O'P$ in contact with O' is moving along PO'. Hence, if OI and $O'I$ are drawn perpendicular to OP, $O'P$ respectively, their intersection I is the instantaneous centre of rotation of the plane. It is evident that I lies on the locus of P so that the space-centrode is the circle on OO' as diameter. Now, PI is a diameter of this circle, and, since P is a fixed point in the given plane, the body-centrode is a circle whose diameter is twice the diameter of the space-centrode. The motion of the plane is the same as that produced by the rolling without slipping of a circle (the body-centrode) on another circle (the space-centrode) of half its radius, the bigger circle enveloping the smaller. Any point in the given plane which lies on the circumference of the body-centrode will describe a pericycloid (see Art. 127), and any other point in the plane will describe a peritrochoid. We shall find the equations which give the peritrochoid described by a point Q in the plane distant c from P.

Let $\widehat{O'OP} = \theta$ and let QP make an angle ϕ with OP produced. Take the x-axis along OO' and the y-axis perpendicular to OO' through O. Then, if Q is the point (x, y) and $OO' = a$

$$x = OP \cos\theta + QP \cos(\theta + \phi) = a\cos^2\theta + c\cos(\theta + \phi) \quad . \quad . \quad (1)$$

$$y = OP \sin\theta + QP \sin(\theta + \phi) = a\cos\theta\sin\theta + c\sin(\theta + \phi) \quad . \quad (2)$$

Now, a, c, ϕ are constants, so that the equations (1) and (2) give x and y in terms of the variable θ.

The reader should compare the above example with Ex. 1, Art. 124. We saw in the latter example that when OPO' is fixed and OO' moves, any point carried by OO' traces out an ellipse relative to OPO'. In the above example, the relative motion is the same, and so any point on OO' traces out an ellipse on a plane carried by OPO'. This principle is made use of in the elliptic chuck used in a lathe for turning elliptic cylinders.

EXAMPLES X

(1) A body in plane motion moves so that two lines fixed in it always pass through two points fixed in space. Find the space- and body-centrodes.

(2) A sheet of metal moves in a plane, so that a fixed peg runs along a straight slot in the metal and a straight edge of the metal at right-angles to the slot always touches a fixed circle. Determine the locus described by the instantaneous centre (i) in space, (ii) in the body. (U.L.)

(3) A body in plane motion moves so that two perpendicular lines fixed in it always touch one each of two circles fixed in space in the plane of the body; find the centrodes.

(4) Sketch the mechanism of the simple engine and that of the oscillating cylinder engine, and show in each case how to find the instantaneous centre of rotation of the connecting-rod.

(5) The lengths of the crank and connecting-rod of a simple engine mechanism are r ft and l ft respectively. Taking rectangular axes with the centre of the crankshaft as origin and the line of stroke as axis of x, find the equation to the space-centrode for the motion of the connecting-rod.

(6) A line of length $(a + b)$ slides with an end on each of the rectangular axes OX and OY. A point in the line traces out the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Sketch graphs of the line and ellipse, and using the property of the instantaneous centre, explain a construction for drawing the normal at any point of the ellipse.

(7) Show that the normal to the cycloid passes through the point of contact of the rolling circle and the line on which it rolls, and that the tangent passes through the other end of the diameter through the point of contact.

(8) What does the hypocycloid of Fig. 120 become when $r = \frac{1}{2}R$?

(9) Find the length from cusp to cusp of a cycloid, radius of rolling circle $= a$.

(10) Find the radius of curvature at any point of the epicycloid whose parametric equations are (X.11) and (X.12).

(11) Show that the gradient at any point of the epicycloid in the last example is $\frac{dy}{dx} = \tan(\theta + \frac{1}{2}\phi)$, and that the length from cusp to cusp is $\frac{8(R + r)r}{R}$.

(12) Find the length of the arc PP' of the cycloid (Fig. 118) in terms of r and ψ. Show that if the length of the arc is s, then $\frac{ds}{d\theta} = IP'$. Hence, prove that the radius of curvature at the point P' is equal to $\frac{ds}{d\psi} = 2\frac{ds}{d\theta} = 2IP'$.

(13) Using the result of the last example, show that the evolute of the cycloid of Fig. 118 is an equal cycloid generated by a circle of radius r rolling on a straight line parallel to OX and at a distance $2r$ below it.

(14) The equation of motion of a particle moving under gravity on a smooth curve in a vertical plane is $\dfrac{d^2s}{dt^2} + g \sin \psi = 0$, where s is the distance of the particle measured along the curve from its lowest point, and ψ is the inclination of the tangent to the positive direction of OX. Show that if the cycloid of Fig. 118 is rotated through 180° about OX, the equation of motion of a particle on it is $\dfrac{d^2s}{dt^2} - \dfrac{s}{4a} g = 0$. Hence, using the result of Art. 38, show that the particle moves with simple harmonic motion of periodic time $2\pi \sqrt{\dfrac{4a}{g}}$

(15) A flat piece of metal has two edges AB and BC at right-angles to each other. The metal moves in a plane so that AB always passes through a fixed point and B moves along a fixed straight line. Find the glissette enveloped by the line BC.

(16) Show that the space-centrode of the body in Ex. 15 is a parabola, and find the equation of the body-centrode referred to rectangular axes of which B is the origin and BA the axis of x.

(17) A circle of radius a rolls, without slipping, inside a circle of radius $4a$. Show that, by a suitable choice of axes, the equation of the curve described by a point on the circumference of the rolling circle may be written

$$x = 4a \cos^3 \theta, \ y = 4a \sin^3 \theta$$

and find the equation given by eliminating θ. If this curve be revolved about the x-axis, show that the surface area of the solid formed is $192\pi a^2/5$. (U.L.)

(18) If a circle of radius a rolls outside a circle of radius $3a$, show that the locus traced out by a fixed point on the rolling circle is given by

$$x = 4a \cos \theta - a \cos 4\theta, \ y = 4a \sin \theta - a \sin 4\theta$$

the origin being the centre of the fixed circle.

If r be the distance of a point on the locus from the centre of the fixed circle, and p the perpendicular from this centre to the tangent at the point, prove that $r^2 = 9a^2 + \frac{16}{25} p^2$. (U.L.)

(19) A circle rolls on and touches the outside of a fixed circle of radius a; show that the path of a point on the circumference of the rolling circle will be given by

$$\begin{cases} x = a(1 + n) \cos \theta - na \cos \dfrac{1+n}{n} \theta \\[2ex] y = a(1 + n) \sin \theta - na \sin \dfrac{1+n}{n} \theta \end{cases}$$

or by

$$\begin{cases} x = a(1 - n') \cos \theta' + n'a \cos \dfrac{1-n'}{n'} \theta' \\[2ex] y = a(1 - n') \sin \theta' - n'a \sin \dfrac{1-n'}{n'} \theta' \end{cases}$$

according as the fixed circle is outside or inside the rolling one, the radii of the latter being na and $n'a$ respectively. If $n = 3$, find n' so that the curves may be identical and express θ' in terms of θ. (U.L.)

(20) Prove that, if the extremities A, B of a line of given length move upon two fixed straight lines OX, OY, inclined to one another at any angle, the motion

of any figure rigidly connected with AB can be produced by the rolling of a circle inside another of twice the radius. Hence, show that the path relative to OX, OY of any point rigidly connected with AB is in general an ellipse. (U.L.)

(21) Prove that any displacement of a rigid two-dimensional body in its own plane can be effected either by a motion of simple translation or by a rotation about a point in the plane.

A uniform rod of length $2a$ has an extremity A on a smooth horizontal plane, and the other extremity B on a smooth wall making an angle 120° with the plane. Initially, the rod is perpendicular to the line of intersection of the wall and the plane, with the end A close to the foot of the wall, and the rod is allowed to slip down, always remaining in the same vertical plane. Find the angular velocity of the rod when its inclination to the horizontal is 30°. (U.L.)

(22) P is a point at a distance c from the centre of a circle of radius r. Deduce the equation of the trochoid which P describes as the circle rolls without slipping on a given straight line. Prove that at any instant the normal to the trochoid at P passes through the point of contact of the circle with the straight line.

(23) Prove that in Fig. 119, if s denote the length of the arc of the epicycloid measured from A to P, then $s = \dfrac{4(R + r)r}{R}\left\{1 - \cos\dfrac{\phi}{2}\right\}$. Hence, deduce that the length of the epicycloid from cusp to cusp is $\dfrac{8(R + r)r}{R}$

(24) A straight line is tangential to a circle, centre O and radius r, at a point A, and P is the point on the line in contact with A. The straight line is made to roll without slipping on the circle. Prove that the curve described by P is given by the equations

$$x = r\cos\phi + r\phi\sin\phi, \; y = r\sin\phi - r\phi\cos\phi$$

the axes of x and y being along OA and perpendicular to OA respectively, and ϕ being the angle between OA and the radius through the point of contact of the line with the circle at any instant.

(25) Prove that the total length of the four-cusped hypocycloid or astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$ (Fig. 114) is $6l$, and that the area it includes is $\frac{3}{8}\pi l^2$.

(26) AB and AC are two rods rigidly connected together at A. If AB and AC always touch two fixed coplanar circles, determine the space-centrode and the body-centrode. Show that the locus of A is a limaçon. See (29).

(27) Two straight lines AB, AC intersect at an angle α. AB rolls without slipping on a circle, radius r, which lies in the plane of the lines. Prove that AC envelopes an involute of a concentric circle whose radius is $r\sin\alpha$.

(28) OX, OY are two rectangular axes, and P is a point on OY such that $OP = d$. A straight line moves so as always to pass through P, and a fixed point Q on the line moves along OX. Find the position of the instantaneous centre at any instant, and prove that the space-centrode is a parabola.

Show also that referred to QP as initial line and Q as origin, the equation of the body-centrode in polars is $r = d\sec^4\theta$.

(29) On a straight line OA, 3 in. long, as diameter, construct a circle. Let P be any point on the circumference and let (r, θ) be the polar co-ordinates of P referred to O as pole and OA as initial line. The polar equation of the circle is then $r = 3\cos\theta$. On OP produced take points Q and R such that $PQ = 2$ in. and $PR = 3$ in. Plot the loci of Q and R for all positions of P on the circle. The locus of Q is the curve $r = 3\cos\theta + 2$, which is called a *limaçon*, and the locus of R is the curve $r = 3(1 + \cos\theta)$, which is called a *cardioid*. Prove that if P' be

the other end of the diameter through P in any position of the line OP, then QP' and RP' are normals to the loci of Q and R respectively.

(30) A circle of radius a rolls on the outside of another circle of the same radius. Show that the polar equation of the path traced out by any point on the rolling circle is $r = 2a(1 - \cos\theta)$, the origin being the point of contact of the tracing point with the fixed circle, and the initial line the radius of this circle through the point. (U.L.)

(31) If p be the perpendicular from the origin on the tangent at a point on the hypocycloid

$$x = (a - b)\cos\theta + b\cos\frac{a-b}{b}\theta$$

$$y = (a - b)\sin\theta - b\sin\frac{a-b}{b}\theta$$

distant r from the origin, prove that $r^2 = A + Bp^2$, and determine the constants A and B in terms of a and b. (U.L.)

(32) A crank OQ rotates round O with constant angular velocity ω, and a connecting-rod QP is hinged to it at one end Q, while the other end P moves along a fixed straight line OX. If PQ meets the perpendicular to OX through O in R, prove that the angular velocity of PQ is proportional to QR, and that the velocity of P is proportional to OR. (U.L.)

(33) Prove that the equations

$$x = a\theta - a\sin\theta,\ y = a - a\cos\theta$$

represent the curve traced out by a point on the circumference of a circle rolling along a straight line.

A uniform square lamina, of side a and mass m, resting on a smooth horizontal table, receives an impulse B at one corner parallel to a side. Find the instantaneous centre of rotation, immediately after the impulse, and prove that the path of this point is a cycloid. (U.L.)

(34) Two points in a plane lamina describe fixed straight lines in the same plane. Show that the glissettes generated by all points on a certain circle carried by the lamina are straight lines.

(35) Find the glissette enveloped by a line carried by the lamina in the last example.

(36) An ellipse moving in the plane YOX is always in contact with OX and OY. Find the glissette generated by the centre of the ellipse.

(37) A crank OP rotates at a uniform angular velocity ω_1 about O. A second crank PQ rotates about P at a uniform angular velocity ω_2. The path described by Q is an epicyclic; show that it is also a roulette and discuss the cases (i) $PQ > OP$ and $\omega_1 = -\omega_2$, i.e. the angular velocities are equal but opposite, (ii) $PQ < OP$ and $\omega_1 = -\omega_2$, (iii) $PQ = OP$ and $\omega_1 = -\omega_2$, (iv) $\omega_1 = \omega_2$.

(38) $\text{Cos}\,\frac{R-r}{r}\theta = \cos\frac{R}{r}\theta \,.\, \cos\theta + \sin\frac{R}{r}\theta \,.\, \sin\theta$. By using this and the corresponding expression for $\sin\frac{R-r}{r}\theta$ and making r approach the limit infinity, obtain from (X.13) and (X.14) the equation of the involute of the circle AQD (Fig. 120) which passes through A.

(39) Show that the length of an involute of a circle of radius r measured from the point in which the involute meets the circle is $\frac{1}{2}r\theta^2$, where θ is the angle turned

through by the generating line, and l is the length of the arc over which contact has taken place. Show also that the length is $\frac{1}{2}r\theta^2$.

(40) Two pulleys are joined by a crossed belt. If one pulley drives the other and there is no belt slip, show that any point on either of the straight portions of the belt traces out an involute of a circle relative to a plane surface attached to either pulley perpendicular to its axis and carried by it in its rotation.

(41) Show that all involutes of the same circle are identical.

(42) Prove that the polar equation of a parabola of latus-rectum $2l$ is $r = \frac{1}{2}l \sec^2 \frac{1}{2}\theta$, the focus S being the pole and the initial line passing through vertex A. The parabola is made to roll without slipping on a given straight line, the vertex A being the initial point of contact. Show that if P, the point of contact at a subsequent instant, be the point (r, θ) referred to the pole and initial line given above, then the tangent to the path of S is perpendicular to SP and makes an angle ψ with the given straight line such that $\psi = \frac{1}{2}\theta$. If S be the point (x, y) referred to rectangular axes along and perpendicular to the given line, the origin being at the initial point of contact, show that $y = \frac{1}{2}l \sec \frac{1}{2}\theta = \frac{1}{2}l \sec \psi$; hence, deduce that the path of S is the catenary $y = \frac{1}{2}l \cosh \dfrac{2x}{l}$

(43) A circle of radius r and centre C rolls without slipping on a straight line. Find the equation to the curve traced out by a point P carried by the rolling circle, the distance CP being c. Distinguish between the cases $c < r$, $c = r$, $c > r$.

(44) In Fig. 118 let the origin O be shifted to the vertex of the cycloid OLM, the axes OX, OY remaining parallel to their original directions, but the positive sense of OY being now downwards. Prove that the equations of the cycloid are $x = r(\theta + \sin \theta)$ and $y = r(1 - \cos \theta)$. Prove also that $s = 4r \sin \psi$ (the "intrinsic" equation), s being the length of arc from the origin to any point P on the curve and ψ being the inclination of the tangent at P to the horizontal.

(45) A circle of radius a rolls externally on a fixed circle of radius $2a$. Show that, referred to axes through the centre of the fixed circle, the equations to the curve described by a point on the circumference of the rolling circle can be written in the form

$$x = 3a \cos \theta - a \cos 3\theta$$
$$y = 3a \sin \theta - a \sin 3\theta$$

Prove also that in this curve $p = 4a \sin \dfrac{\psi}{2}$, where p is the perpendicular from the origin on the tangent and ψ is the angle the tangent makes with the axis OX of the above co-ordinates. (U.L.)

CHAPTER XI

FIRST ORDER DIFFERENTIAL EQUATIONS

130. **Formation of Differential Equations.** Suppose we have a relation $f(x, y) = 0$ which involves n arbitrary constants A, B, C, D, etc. We can, by successive differentiation with respect to x, obtain n other relations involving x and y, and the first n derivatives of y with respect to x as well as some or all of the n arbitrary constants. From these $n + 1$ relations we can eliminate the n constants A, B, C, D, etc. The resulting relation will involve $\frac{d^n y}{dx^n}$ and will, in general, contain differential coefficients of lower orders, but will not contain any of the arbitrary constants. Such a relation is known as a *differential equation* of the nth order. As all differentiations are with respect to a single independent variable x, the equation is known as an *ordinary differential equation.*

EXAMPLE 1

Form a differential equation which does not involve a from the relation

$$y^2 = 4ax \qquad \qquad (1)$$

Differentiating with respect to x we have

$$2y\frac{dy}{dx} = 4a \qquad \qquad (2)$$

Eliminating a from this and (1) we have

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

or

$$\frac{dy}{dx} = \frac{y}{2x} \qquad \qquad (3)$$

(1) represents the family or group of parabolas whose vertices are at the origin and whose foci are situated on the axis of x. (3) represents a property common to all the curves of the family. What is this property? In Fig. 125 we have sketched several of the curves. PM is the ordinate and OM the abscissa for the point P on one of the curves, and PT is the tangent to the curve at P. The relation (3) means that

$$\text{the gradient of the tangent at } P = \frac{\text{the ordinate at } P}{2 \times \text{the abscissa at } P}$$

i.e.
$$\frac{\overline{PM}}{\overline{TM}} = \frac{y}{2x}$$

or
$$\frac{y}{\overline{TM}} = \frac{y}{2x}$$

from which
$$TM = 2x = 2 \times \overline{OM}$$

Thus, for all curves of the family, the subtangent is bisected by the origin.

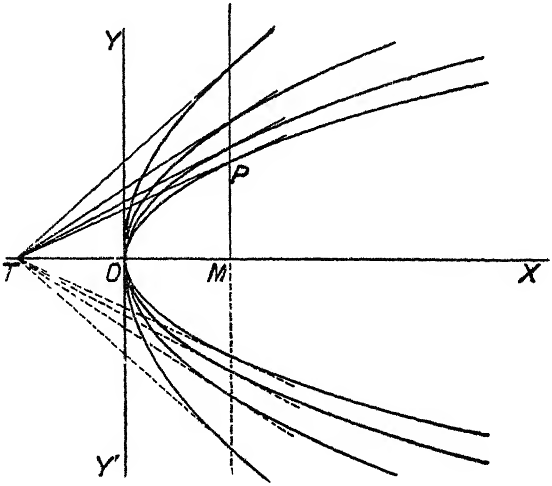


FIG. 125

EXAMPLE 2

A particle which moves with uniform acceleration f ft per sec per sec along a straight line for t sec moves through a distance s ft where s is given by

$$s = s_0 + u_0 t + \tfrac{1}{2} f t^2 \quad . \quad . \quad . \quad . \quad (1)$$

u_0 = initial velocity in feet per second and s_0 = distance of particle from origin when $t = 0$. Eliminate the constants s_0 and u_0.

Differentiating (1) with respect to t we have

$$\frac{ds}{dt} = u_0 + ft \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating again

$$\frac{d^2s}{dt^2} = f \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since (3) does not involve s_0 and u_0 it is the required relation. (3) is a second order differential equation.

EXAMPLE 3

From the relation $x^2 + y^2 + Ax + By + C = 0$, form a differential equation which does not involve either A, B, or C.

We have
$$x^2 + y^2 + Ax + By + C = 0 \quad . \quad . \quad . \quad . \quad (1)$$
Differentiate with respect to x
$$2x + 2y\frac{dy}{dx} + A + B\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad (2)$$
Differentiate again
$$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} + B\frac{d^2y}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad (3)$$
Differentiating a third time, we have
$$6\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\frac{d^3y}{dx^3} + B\frac{d^3y}{dx^3} = 0 \quad . \quad . \quad . \quad . \quad (4)$$
As (3) and (4) do not involve A or C we obtain on eliminating B from (3) and (4) the required relation
$$\frac{d^2y}{dx^2}\left(6\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\frac{d^3y}{dx^3}\right) = \frac{d^3y}{dx^3}\left\{2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right\}$$
which reduces to
$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3} = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$
a third order differential equation.

If a differential equation contains partial differential coefficients it is a *partial differential equation*. The *order* of a differential equation is that of the highest order of derivative present. The *degree* of a differential equation is that of the highest power of the highest order of derivative present.

Differential equations are not usually formed in the manner indicated in Ex. 1, 2, and 3. They often arise in investigations in geometry and in applied science, and are mathematical expressions corresponding to certain properties or laws. The equation (3) in Example 1 expresses the property stated at the end of the example. $\frac{d\theta}{dt} = k\theta$ is the expression of the compound interest law. $\frac{d^2x}{dt^2} = -n^2x$ is the expression of the relation between acceleration and displacement in simple harmonic motion, and states very clearly and tersely that the acceleration is proportional to the displacement and is always directed towards the point $x = 0$. Each of the equations (1) in the above examples is known as the *primitive* of the corresponding differential equation. The primitive is, of course, the solution of the corresponding differential equation.

We saw above that a primitive involving n arbitrary constants gives rise to a differential equation of the nth order. Conversely, the solution of a differential equation of the nth order involving no arbitrary constants is a relation between the variables involving n arbitrary constants. This is the *complete* solution. Any solution involving less than n arbitrary constants is a *particular* solution. In this chapter we shall deal only with equations of the first order and the first degree, of which the general form is

$$P\frac{dy}{dx} + Q = 0 \quad . \quad . \quad . \qquad \text{(XI.1)}$$

where P and Q are functions of x and y.

If P and Q are single-valued continuous functions of x and y, and we substitute any two values of x and y in (XI.1) we obtain a single value of $\frac{dy}{dx}$. Thus, with any point (x, y) there is associated a direction, i.e. that given by $\frac{dy}{dx}$. The relation between x and y is represented, therefore, by a family of non-intersecting curves, one of which passes through every point in the xy plane. If P and Q are not single-valued functions there will be two or more of the curves of the family passing through any given point.

First Order Differential Equations

131. **Solution of the Equation when the Variables are Separable.** We have assumed that P and Q are both functions of x and y. We shall first consider the cases in which either one or both of the variables is absent from both P and Q.

(1) *y absent.* In this case P and Q are both functions of x and

$$\frac{dy}{dx} = -\frac{Q}{P} = F(x)$$

the solution of which is

$$y = \int F(x)dx + C \quad . \quad . \quad . \qquad \text{(XI.2)}$$

where C is a constant.

If x is absent also, then $-\frac{Q}{P}$ is a constant k, say, and the solution is $y = kx + C$.

(2) *x absent.* Here (XI.1) becomes

$$-\frac{P}{Q}\frac{dy}{dx} = 1, \text{ or, writing } F(y) \text{ for } -\frac{P}{Q},$$

$$F(y)\frac{dy}{dx} = 1, \text{ the solution of which is}$$

$$\int F(y)dy = x + C \quad . \quad . \quad . \quad . \quad . \qquad \text{(XI.3)}$$

EXAMPLE 1

$$(1 - x)\frac{dy}{dx} = 1 + x$$

Here y is absent. Transposing and rearranging, we have

$$\frac{dy}{dx} = \frac{1 + x}{1 - x} = -1 + \frac{2}{1 - x}$$

Integrating directly, $y = -x - 2\log(1 - x) + c$.

EXAMPLE 2

Isochronous Governor. In the theory of governing it is shown that in an isochronous governor the balls must move on a curve whose subnormal is constant, the axis of x being the axis of rotation. Find the equation to the curve.

We saw in Art. 76 that the subnormal is equal to $y\dfrac{dy}{dx}$. Let a be the constant length of the subnormal. Then

$$y\frac{dy}{dx} = a$$

and integrating $\qquad \int y\,dy = \int a\,dx + c$

or $\qquad \frac{1}{2}y^2 = ax + c$

from which $\qquad y^2 = 2ax + 2c$

which is the equation to a parabola. Taking the origin at the lowest point of the graph, the x-axis being vertical, we have $x = 0$ when $y = 0$. Thus, $c = 0$ and the equation is

$$y^2 = 2ax$$

EXAMPLE 3

A rocket moves vertically upwards with initial velocity U. During the ascent, matter is continuously ejected vertically downwards with constant velocity u relative to the rocket, the mass of the rocket at time t being $M(1 + e^{-kt})$. If k^2 is neglected, show that the rocket will ascend to a height $\dfrac{U^2}{2g}\left(1 + \dfrac{ku}{2g}\right)$.

(U.L.)

Let v be the velocity and x the height of the rocket at time t. Since

$$e^{-kt} = 1 - kt + \frac{k^2t^2}{\underline{|2}} - \frac{k^3t^3}{\underline{|3}} + \ldots \text{ we have, neglecting } k^2\text{, etc.}$$

$$\text{Mass} = \frac{M}{g}(2 - kt)$$

The rate at which momentum is given to the ejected matter is

$$-\frac{d}{dt}(\text{mass}) \times u = \frac{kMu}{g}$$

The rocket exerts a force of this magnitude on the ejected matter, and the reaction on the rocket is therefore an upwards force of the same magnitude. The force in the direction of motion is the excess of this force over the weight of the rocket, i.e. $\frac{kMu}{g} - M(2 - kt)$. The equation of motion is, therefore, since force = mass × acceleration,

$$\frac{kMu}{g} - M(2 - kt) = \frac{M}{g}(2 - kt)\frac{dv}{dt}$$

i.e.
$$\frac{dv}{dt} = \frac{ku}{2 - kt} - g \quad . \quad . \quad . \quad . \quad (1)$$

Now $\frac{ku}{2 - kt} = \frac{ku}{2}\left(1 - \frac{kt}{2}\right)^{-1} = \frac{ku}{2}$, neglecting k^2 and higher powers of k.

Hence,
$$\frac{dv}{dt} = -\left(g - \frac{ku}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus there is a constant retardation of magnitude $g - \frac{ku}{2}$. It is proved in elementary dynamics that the maximum height attained is $\frac{(\text{initial velocity})^2}{2 \times \text{retardation}}$. If H is the greatest height attained, we have then

$$H = \frac{U^2}{2\left(g - \frac{ku}{2}\right)} = \frac{U^2}{2g}\left(1 - \frac{ku}{2g}\right)^{-1}$$

$$= \frac{U^2}{2g}\left(1 + \frac{ku}{2g}\right) \text{ neglecting } k^2\text{, etc.}$$

(NOTE. The statement in this example that k^2 may be neglected is open to objection; the term involving k^2 also involves t^2, and for large values of t it is possible for the term $\frac{k^2t^2}{\underline{|2}}$, for example, to be greater than the term kt. The result given should not be accepted without some consideration of the error involved in the approximation.)

(3) Equations of the type $f(y)\frac{dy}{dx} + \phi(x) = 0$ in which *both x and y are present* but the *variables are separable*. If one or both of the functions P and Q involves x and y it may still be possible to express $\frac{P}{Q}$ in the form $\frac{f(y)}{\phi(x)}$. In such case (XI.1) becomes

$$f(y)\frac{dy}{dx} + \phi(x) = 0 \quad . \quad . \quad (\text{XI.4})$$

the solution of which is

$$\int f(y)\, dy + \int \phi(x)\, dx = c \quad . \quad . \quad (\text{XI.5})$$

EXAMPLE 4

Solve $$x(1+y)\frac{dy}{dx}+(1-x)y=0$$

Dividing through by xy, we have

$$\left(\frac{1}{y}+1\right)\frac{dy}{dx}+\left(\frac{1}{x}-1\right)=0$$

$$\therefore \quad \log y + y + \log x - x = c$$

or $$y - x + \log xy - c = 0$$

EXAMPLE 5

Solve $$x\cos y\frac{dy}{dx}+\sin y=0$$

We put this equation in the form (XI.4) by dividing through by $x\sin y$. We have

$$\frac{\cos y}{\sin y}\frac{dy}{dx}+\frac{1}{x}=0$$

and integrating $$\log\sin y+\log x=\log C$$

where $\log C$ is an arbitrary constant,

or $$\sin y=\frac{C}{x}$$

EXAMPLE 6

Solve $$(1+x)\frac{dy}{dx}+(1-y)=0$$

Dividing through by $(1+x)(1-y)$ we have

$$\frac{1}{1-y}\frac{dy}{dx}+\frac{1}{1+x}=0$$

Integrating we have $$\int\frac{dy}{1-y}+\int\frac{dx}{1+x}=C$$

i.e. $\quad -\log_e(1-y)+\log_e(1+x)=\log_e A$, where $\log_e A=C$

$$\therefore \quad \log_e\frac{1+x}{1-y}=\log_e A$$

or $$A(1-y)=1+x$$

$$\therefore \quad y=-\frac{x}{A}+\left(1-\frac{1}{A}\right)$$

the equation to a straight line whose intercept is greater by unity than its gradient.

132. **Homogeneous Equations** are equations of the type (XI.1) in which P and Q are homogeneous functions of x and y, both functions being of the same degree. The method of solution is to substitute $y = vx$. $\frac{dy}{dx}$ becomes $\frac{d}{dx}(vx) = v + x\frac{dv}{dx}$ and $-\frac{Q}{P}$ becomes

a function of v. In this case the equation (XI.1) reduces to an equation of the type

$$v + x\frac{dv}{dx} = F(v)$$

which can be solved by separating the variables, thus

$$x\frac{dv}{dx} = F(v) - v$$

or
$$\frac{1}{F(v) - v}\frac{dv}{dx} = \frac{1}{x}$$

whence
$$\int\frac{dv}{F(v) - v} = \log_e x + C \quad . \quad . \quad . \qquad \text{(XI.6)}$$

EXAMPLE 1

Solve
$$y^2 + (3xy + x^2)\frac{dy}{dx} = 0$$

This becomes on transposition $\dfrac{dy}{dx} = -\dfrac{y^2}{3xy + x^2}$

Writing $y = vx$, we have $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, and substituting

$$v + x\frac{dv}{dx} = -\frac{v^2}{3v + 1}$$

$$\therefore \qquad x\frac{dv}{dx} = -\frac{4v^2 + v}{3v + 1}$$

Transposing, we have

$$\frac{3v + 1}{4v^2 + v}\frac{dv}{dx} = -\frac{1}{x}$$

or by partial fractions

$$\left(\frac{1}{v} - \frac{1}{4v + 1}\right)\frac{dv}{dx} = -\frac{1}{x}$$

and integrating

$$\log_e v - \tfrac{1}{4}\log_e(4v + 1) = -\log_e x + \log_e C$$

or
$$4\log_e v - \log_e(4v + 1) = -4\log_e x + 4\log_e C$$

whence
$$\frac{v^4}{4v + 1} = \frac{A}{x^4}, \text{ writing } A \text{ for } C^4.$$

Now, putting $v = \dfrac{y}{x}$ and simplifying, we have

$$y^4 = A\left(4\frac{y}{x} + 1\right) \text{ as the solution.}$$

Equations in which P and Q are linear functions of x and y can generally be made homogeneous by the substitution of $X + h$ for x and $Y + k$ for y where h and k are suitably chosen. This method is shown in the next example.

EXAMPLE 2

Solve the equation $\dfrac{dy}{dx} = \dfrac{7y - 3x - 7}{-3y + 7x + 3}$

Substitute $X + h$ for x and $Y + k$ for y. Then since $\Delta X = \Delta x$ and $\Delta Y = \Delta y$, $\dfrac{dy}{dx} = \dfrac{dY}{dX}$ and we have

$$\frac{dY}{dX} = \frac{7Y - 3X + 7k - 3h - 7}{-3Y + 7X - 3k + 7h + 3}$$

Now choose h and k so that the terms independent of X and Y disappear from both numerator and denominator.

For this we must have $7k - 3h - 7 = 0$

and $-3k + 7h + 3 = 0$

and solving for h and k we have $h = 0$, $k = 1$.

Substituting these values we have the homogeneous equation

$$\frac{dY}{dX} = \frac{7Y - 3X}{-3Y + 7X}$$

Put $Y = vX$ as before and

$$v + X\frac{dv}{dX} = \frac{7v - 3}{-3v + 7}$$

or

$$X\frac{dv}{dX} = \frac{3v^2 - 3}{-3v + 7} = \frac{3(v^2 - 1)}{7 - 3v}$$

$$\frac{7 - 3v}{v^2 - 1}\frac{dv}{dX} = \frac{3}{X}, \text{ and by partial fractions}$$

$$\left(\frac{2}{v - 1} - \frac{5}{v + 1}\right)\frac{dv}{dX} = \frac{3}{X}$$

By integration $2\log(v - 1) - 5\log(v + 1) = 3\log X + \log C$, from which

$$\frac{(v - 1)^2}{(v + 1)^5} = CX^3$$

i.e.

$$\frac{(Y - X)^2}{(Y + X)^5} = C$$

Now $x = X$ and $y = Y + 1$ or $Y = y - 1$. The solution becomes then

$$(y - x - 1)^2 = C(y + x - 1)^5$$

133 **Exact Equations** If u is a function of x and y we have by (V 5)

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad \text{(XI 7)}$$

If now we write

$$P = \frac{\partial u}{\partial x} \text{ and } Q = \frac{\partial u}{\partial y}$$

this becomes

$$du = P\,dx + Q\,dy$$

If then we have an equation of the type

$$P\,dx + Q\,dy = 0 \qquad \text{(XI 8)}$$

where we can find u, so that

$$P = \frac{\partial u}{\partial x} \text{ and } Q = \frac{\partial u}{\partial y}$$

we have $du = 0$

or $\quad u = C$ as the solution of (XI 8)

A differential equation such as (XI 8), which can be formed by direct differentiation of a primitive without any subsequent operation being performed on it, is known as an *exact equation* and the quantity $P\,dx + Q\,dy$ or du is known as a *perfect* or *complete differential*

EXAMPLE 1

Solve the differential equation $2(x + y)dx + 2(x - y)dy = 0$

If the equation is exact we have $\frac{\partial u}{\partial x} = 2(x + y)$ and $\frac{\partial u}{\partial y} = 2(x - y)$ Careful inspection will show that these are the partial differential coefficients of

$$u = x^2 + 2xy - y^2$$

Hence the solution is $u = c$, or

$$x^2 + 2xy - y^2 = c$$

EXAMPLE 2

Solve $\qquad (x + \log y)dy + y\,dx = 0$

If this is exact, we have

$$\frac{\partial u}{\partial x} = y \text{ and } \frac{\partial u}{\partial y} = x + \log y$$

and by inspection $u = y \log y - y + xy$ and the solution is $u = c$ or

$$y(\log y - 1 + x) = c$$

The reader will notice that in the last two examples we have guessed the form of the function u. We could have proceeded thus: In Ex. 1, integrating $\frac{\partial u}{\partial x} = 2(x + y)$ we have $u = x^2 + 2xy + f_1(y)$. Similarly, finding u from $\frac{\partial u}{\partial y} = 2(x - y)$ we have $u = 2xy - y^2 + f_2(x)$. Comparing these expressions for u, we see that $f_2(x)$ must be x^2 and $f_1(y)$ must be $-y^2$, so that $u = x^2 + 2xy - y^2$. In Ex. 2, $\frac{\partial u}{\partial y} = x + \log y$, from which $u = xy + y \log y - y + f_2(x)$ and $\frac{\partial u}{\partial x} = y$ from which $u = xy + f_1(y)$. By comparison, we see that $f_2(x) = 0$ and $f_1(y) = y \log y - y$, so that $u = y \log y - y + xy$ as before.

The above method of solution applies only to exact equations. Sometimes an equation can be made exact by multiplying through both sides by a factor. We have not the space to devote to the methods of doing this, for which we refer readers to treatises on differential equations. It is, however, obvious that $x\,dy - y\,dx = 0$ is made exact by division of both sides by x^2 for, doing this, we obtain

$$\frac{x\,dy - y\,dx}{x^2} = 0$$

i.e.

$$\frac{x\frac{dy}{dx} - y}{x^2} = 0$$

or

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

Hence, $\frac{y}{x} = C$ or $y = Cx$ is the solution.

A particular case of the use of integrating factors is dealt with in the next section.

Since $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$, we have from the conditions $P = \frac{\partial u}{\partial x}$ and $Q = \frac{\partial u}{\partial y}$ the relation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad . \quad . \quad . \quad (X.19)$$

This relation provides a test for exact equations. If, and only if, the values of P and Q in (XI.8) satisfy the condition (XI.9), the former is an exact equation. The reader should apply the test to each of the equations solved in this section.

If P and Q in $P\,dx + Q\,dy$ satisfy the condition (XI.9), then $P\,dx + Q\,dy$ is a perfect differential.

134. **Linear Differential Equations of the First Order.** When the dependent variable and its derivative occur in an equation only in the first degree the equation is known as a *linear equation*. The typical linear equation of the first order is

$$\frac{dy}{dx} + Py = Q \quad . \quad . \quad . \quad \text{(XI.10)}$$

where P and Q may be either constants or functions of x, or one a constant and the other a function of x. If P and Q are both constants the equation may be solved by separating the variables thus

$$\frac{dy}{dx} = -P\left(y - \frac{Q}{P}\right)$$

or

$$\frac{1}{y - \frac{Q}{P}}\frac{dy}{dx} = -P$$

from which, by integration,

$$\log_e\left(y - \frac{Q}{P}\right) = -Px + \log_e A$$

whence

$$y = Ae^{-Px} + \frac{Q}{P} \quad . \quad . \quad \text{(XI.11)}$$

The general method of solution of (XI.10) is by means of an integrating factor. If the equation is written as

$$dy + Py\,dx = Q\,dx \quad . \quad . \quad . \quad \text{(XI.12)}$$

it is clear from the form of the left-hand side that the integrating factor is a function of x alone, because dy appears in the first term and y in the second. Let $f(x)$ be the integrating factor. Multiply through by $f(x)$ in (XI.12), then

$$f(x)dy + Pf(x)y\,dx = Qf(x)\,dx \quad . \quad \text{(XI.13)}$$

We have assumed that the left-hand side is a perfect differential, and so the test (XI 9) applies Applying this test, we have

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial y}(Pf(x)y)$$
$$= Pf(x)$$

As x is the only variable involved, we may write this

$$\frac{df(x)}{dx} = Pf(x)$$

or
$$\frac{1}{f(x)}\frac{df(x)}{dx} = P \qquad \text{(XI 14)}$$

Integrating we have

$$\log f(x) = \int P\,dx$$

or
$$f(x) = e^{\int P\,dx}$$

which is an integrating factor of (XI 12) or (XI 10) We have omitted the constant of integration as we only need a particular solution of (XI 14) (The reader will see, however, that if A is a constant, $Ae^{\int P\,dx}$ is also an integrating factor)

When solving (XI 10), therefore, we multiply each side by the integrating factor $e^{\int P\,dx}$ We then have

$$e^{\int P\,dx}\frac{dy}{dx} + Pe^{\int P\,dx}\,y = Qe^{\int P\,dx}$$

or
$$\frac{d}{dx}(e^{\int P\,dx}\,y) = Qe^{\int P\,dx}$$

and, integrating,
$$e^{\int P\,dx}\,y = \int Qe^{\int P\,dx}\,dx + A$$

or
$$y = e^{-\int P\,dx}\left\{\int Qe^{\int P\,dx}\,dx + A\right\} \qquad \text{(XI 15)}$$

EXAMPLE 1

Integrate $\dfrac{dy}{dx} + x^2y = 0$

Here $P = x^2$, $\int P\,dx = \dfrac{x^3}{3}$ and the integrating factor is $e^{\int P\,dx} = e^{\frac{1}{3}x^3}$

Multiplying through by this we have

$$e^{\frac{1}{3}x^3}\frac{dy}{dx} + e^{\frac{1}{3}x^3}x^2y = 0$$

or
$$\frac{d}{dx}(e^{\frac{1}{3}x^3}y) = 0$$

whence
$$e^{\frac{1}{3}x^3}y = C$$

or
$$y = Ce^{-\frac{1}{3}x^3}$$

EXAMPLE 2

Solve $(1 + x^2)\dfrac{dy}{dx} + xy = 1$

Dividing through by $(1 + x^2)$ this becomes

$$\frac{dy}{dx} + \frac{x}{1 + x^2}y = \frac{1}{1 + x^2}$$

Here $P = \dfrac{x}{1 + x^2}$, $\int P\,dx = \frac{1}{2}\log(1 + x^2)$ and $e^{\int P\,dx} = e^{\frac{1}{2}\log(1+x^2)} = \sqrt{1 + x^2}$ which is the integrating factor. Multiplying through by this

$$\sqrt{1 + x^2}\,\frac{dy}{dx} + \frac{x}{\sqrt{1 + x^2}}\,y = \frac{1}{\sqrt{1 + x^2}}$$

$$\frac{d}{dx}\left(y\sqrt{1 + x^2}\right) = \frac{1}{\sqrt{1 + x^2}}$$

and integrating $\qquad y\sqrt{1 + x^2} = \sinh^{-1} x + C$

whence $\qquad y = \dfrac{\sinh^{-1} x}{\sqrt{1 + x^2}} + \dfrac{C}{\sqrt{1 + x^2}}$

EXAMPLE 3

Solve $\dfrac{dy}{dx} - ny = e^{\alpha x}$

Here $P = -n$, $\int P\,dx = -nx$, and the integrating factor is e^{-nx}. Multiplying through by this we have

$$\frac{d}{dx}(ye^{-nx}) = e^{(\alpha - n)x}$$

and integrating $\qquad ye^{-nx} = \dfrac{1}{\alpha - n}e^{(\alpha - n)x} + C$

and $\qquad y = \dfrac{1}{\alpha - n}e^{\alpha x} + Ce^{nx}$

135. Some Important First Order Differential Equations.

EXAMPLE 1

The Decay Function The amount of a given substance taking part in a chemical reaction is x grammes t seconds after the reaction has commenced. If $\dfrac{dx}{dt} = -kx$ where k is a constant and $x = a$ when $t = 0$, find x as a function of t.

$\dfrac{dx}{dt} = -kx$ is an equation of the type of case (2), Art 131. (It also falls under other cases.) Dividing both sides by x

$$\frac{1}{x}\frac{dx}{dt} = -k$$

hence
$$\int \frac{1}{x}\,dx = -kt + C$$
or
$$\log_e x = -kt + \log_e A, \text{ where } \log_e A = C$$
$$\therefore \qquad x = Ae^{-kt}$$
When $t = 0$, $x = a$, there
$$a = A$$
and
$$x = ae^{-kt}$$

We saw in Art. 98 that as t increases by equal increments, x decreases by fixed fractions. If then half the substance disappears in T sec from the start, three-quarters ($\frac{1}{2} + \frac{1}{4}$) of it will disappear in $2T$ sec, seven-eighths ($\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$) will disappear in $3T$ sec, etc. ($\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ to n terms) being the fraction of it which will disappear in nT sec. A simpler way of stating this is that the fraction $\left(\frac{1}{2^n}\right)$ of the substance will be left present after nT sec from the start of the reaction.

EXAMPLE 2

Tension in a Belt. If T is the tension at any part of a belt just slipping over the surface of a pulley, and θ is the angle subtended at the centre of the pulley by the piece of the belt between the points where the tension is T and the point at which the belt first makes contact with the pulley, we have the relation $\frac{dT}{d\theta} = \mu T$ where μ is the coefficient of friction and T_0 is the least tension in the belt.

This example is of the same type as the last. It becomes on transposition
$$\frac{1}{T}\frac{dT}{d\theta} = \mu$$
$$\therefore \qquad \int \frac{1}{T}\,dT = \int \mu\,d\theta + C$$
i.e.
$$\log_e T = \mu\theta + \log_e A$$
where A is a constant. Hence,
$$T = Ae^{\mu\theta}$$

Since the least tension occurs when $\theta = 0$ we have $T = T_0$ when $\theta = 0$.

Substituting these,
$$T_0 = A$$
and
$$T = T_0\,e^{\mu\theta}$$

EXAMPLE 3

Rod under Uniform Stress. A straight rod of homogeneous material of length l in. and of circular section is suspended vertically, and carries a load of W lb at its lower end (Fig. 126). If the material of the rod weighs w lb per in.^{3} and the tensile stress in the material of the rod is everywhere equal to f_0 lb per in.^{2}, find the radius of the rod at a depth x in. below the upper end.

Consider the piece of rod between two horizontal planes at distances x and $x + \Delta x$ in. respectively below the upper end (Fig. 126). Let y and $y + \Delta y$ be the radii in inches respectively of these sections. (It is clear from the conditions

of the problem that Δy is negative.) The forces acting on the element of volume are—

(1) An upward force on the upper face of magnitude $\pi y^2 f_0$ lb.
(2) A downward force on the lower face of magnitude $\pi(y + \Delta y)^2 f_0$ lb.
(3) A downward force equal to the weight of $\pi y^2 \Delta x w$ lb.

For equilibrium we have therefore

$$\pi y^2 f_0 - \pi(y + \Delta y)^2 f_0 - \pi y^2 \Delta x \, . \, w = 0$$

or $$2y \, \Delta y \, f_0 + f_0(\Delta y)^2 + wy^2 \Delta x = 0$$

Dividing by Δx we have

$$2yf_0 \frac{\Delta y}{\Delta x} + f_0 \frac{\Delta y}{\Delta x} \, . \, \Delta y + wy^2 = 0$$

and in the limit when $\Delta x \to 0$, $\Delta y \to 0$, and

$$2y f_0 \frac{dy}{dx} + wy^2 = 0$$

$$\therefore \quad 2f_0 \frac{dy}{dx} + wy = 0$$

This is the differential equation whose solution is required. It may be solved by separating the variables or by means of an integrating factor, the former being the simpler method. Dividing through by $2f_0$ and transposing

$$\frac{dy}{dx} = -\frac{w}{2f_0} y$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{w}{2f_0}$$

and integrating,

$$\log_e y = -\frac{w}{2f_0} x + \log_e A$$

whence $$y = Ae^{-\frac{w}{2f_0}x}$$

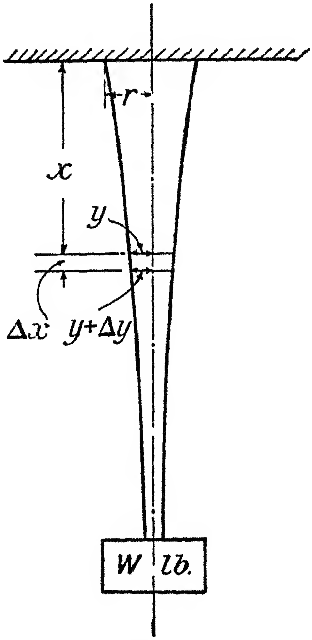


FIG. 126

If r is the radius at its upper end, then $y = r$ when $x = 0$, and substituting,

$$r = Ae^0 \quad \therefore A = r$$

and $$y = re^{-\frac{w}{2f_0}x}$$

gives the shape of the rod.

If s is the radius at the lower end, then

$$s = re^{-\frac{w}{2f_0}l}$$

Now $$\pi s^2 f_0 = W, \text{ or } s = \sqrt{\frac{W}{\pi f_0}}$$

$$\therefore \quad re^{-\frac{w}{2f_0}l} = \sqrt{\frac{W}{\pi f_0}}, \text{ or } r = \sqrt{\frac{W}{\pi f_0}} e^{\frac{w}{2f_0}l}$$

The radius y of the rod at a depth x in. below the upper end is then given by

$$y = \sqrt{\frac{W}{\pi f_0}} \cdot e^{\frac{w}{2f_0}l} \cdot e^{-\frac{w}{2f_0}x}$$

or

$$y = \sqrt{\frac{W}{\pi f_0}}\, e^{\frac{w}{2f_0}(l-x)}$$

EXAMPLE 4

Parabolic Chain Curve. A pair of chains supported at the same level at both ends carry a suspension bridge. The weight of the bridge and chains is equally distributed horizontally, i.e. each length of chain whose horizontal projection is 1 ft in length carries the same load. Show that the shape of the chain is a parabola and find an expression for the tension at any point of a chain.

Let L ft be the total span and W lb the total load. Then the load per foot on one chain is $\frac{W}{2L}$ lb. Consider the equilibrium of the piece of chain OP (Fig. 127) where O is the lowest point in the chain. Take O as origin, OM the tangent at O as the axis of X, and the vertical through O as the axis of Y. Let x, y be the co-ordinates of P. Then the load on OP is $\frac{W}{2L} \times x$ and OP is in equilibrium under the action of the three forces: (1) the tension T_0 lb along the tangent at O, (2) the tension T lb along the tangent at P, and (3) the weight $\frac{Wx}{2L}$ lb acting along a vertical line halfway between OY and MP.

Let the line of action of (3) cut OX in N. Then $\overline{ON} = \overline{NM} = \frac{x}{2}$. Since the three forces are in equilibrium they meet in a point, and the line of action of T passes through N. Thus NP is the tangent to the curve of the chain at P. Hence, we have

$$\frac{dy}{dx} = \text{gradient of tangent at } P$$

$$= \frac{y}{\frac{x}{2}}$$

i.e.
$$\frac{dy}{dx} = \frac{2y}{x}$$

Separating the variables, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$$

$\therefore$
$$\log_e y = 2\log_e x + C$$

i.e.
$$\log_e y = 2\log_e x + \log_e A \text{ where } \log_e A = C$$

This becomes
$$y = Ax^2$$

which shows that the chain hangs in the form of a parabola.

This equation can be obtained from the triangle of forces PMN thus: Since PM represents the force $\frac{Wx}{2L}$ lb and MN represents the force of T_0 lb,

we have $\dfrac{PM}{NM} = \dfrac{\dfrac{Wx}{2L}}{T_0}$. But $\dfrac{PM}{NM} = \dfrac{2y}{x}$, $\therefore$ $y = \dfrac{Wx^2}{4LT_0}$, which is the same as $y = Ax^2$ if $A = \dfrac{W}{4LT_0}$. If S is the sag in the middle, $y = S$ when $x = \dfrac{L}{2}$ and $S = \dfrac{WL}{16T_0}$ or $T_0 = \dfrac{WL}{16S}$.

From the triangle of forces PMN we have $\overline{PN}^2 = PM^2 + NM^2$, or

$$T^2 = T_0^2 + \frac{W^2x^2}{4L^2}$$

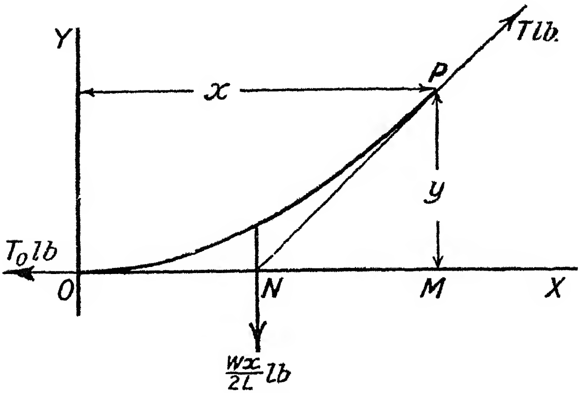


FIG. 127

The maximum tension occurs at the ends where $x = \pm\dfrac{L}{2}$ and

$$T_{max} = \sqrt{T_0^2 + \frac{W^2}{16}}$$
$$= \frac{W}{4}\sqrt{1 + \frac{L^2}{16S^2}}$$
$$= \frac{W}{16S}\sqrt{L^2 + 16S^2}$$

EXAMPLE 5

Resisted Motion. A particle falls freely under gravity from a great height, there being a resistance proportional to (1) the velocity, (2) the square of the velocity. Show that, in each case, the velocity approaches a certain limiting value, and find expressions for the distance x ft travelled by the particle in terms of the time t sec occupied by the motion.

Using engineers' units, we assume that W lb is the weight of the particle and $v = \dfrac{dx}{dt}$ its velocity in feet per second.

(1) Then, if the resistance to motion R lb varies as the velocity, we have $R = k\frac{W}{g}v$ where k is constant, and g is the acceleration due to gravity in feet per second per second. Since the weight acts vertically downwards and the resistance opposes this, the equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$W - k\frac{W}{g}v = \frac{W}{g}\frac{dv}{dt}$$

or $$\frac{dv}{dt} = g - kv \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From this $$\frac{-k}{g - kv}\frac{dv}{dt} = -k$$

and integrating, $\log(g - kv) = -kt + \log_e A$

whence $$g - kv = Ae^{-kt}$$

or $$v = \frac{1}{k}(g - Ae^{-kt}) \quad . \quad . \quad . \quad . \quad (2)$$

Now $v = 0$ when $t = 0$ if the particle starts from rest, hence,

$$0 = \frac{1}{k}(g - A) \text{ or } A = g$$

and (2) becomes $$v = \frac{g}{k}(1 - e^{-kt}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which we see that v approaches the limiting value $\frac{g}{k}$ as t approaches infinity. $v = \frac{g}{k}$ is known as the *terminal velocity* of the particle, though this velocity is never actually reached. To find the relation between x and t we write $\frac{dx}{dt}$ for v in (3) and integrate directly, thus,

$$\frac{dx}{dt} = \frac{g}{k}(1 - e^{-kt})$$

hence, $$x = \frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) + C$$

As $x = 0$ when $t = 0$, $\quad 0 = \frac{g}{k^2} + C$ and $C = -\frac{g}{k^2}$

so that $$x = \frac{g}{k}\left(t + \frac{1}{k}e^{-kt} - \frac{1}{k}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By eliminating t between (3) and (4), we can obtain the relation between x and v. The required relation can, however, be obtained much more directly by the following method—

Since $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$, (1) may be written

$$v\frac{dv}{dx} = g - kv$$

or
$$\frac{v}{\frac{g}{k} - v}\frac{dv}{dx} = k \quad . \qquad (5)$$

Let $v_0 = \frac{g}{k}$ be the terminal velocity. Then (5) becomes
$$\frac{v}{v_0 - v}\frac{dv}{dx} = \frac{g}{v_0}$$

i.e.
$$\left(-1 + \frac{v_0}{v_0 - v}\right)\frac{dv}{dx} = \frac{g}{v_0}$$

and integrating, $-v - v_0 \log_e (v_0 - v) = \frac{g}{v_0}x + C \quad . \quad . \quad . \qquad (6)$

If $v = 0$ when $x = 0$, $-v_0 \log v_0 = C$ and substituting in (6)
$$-v + v_0 \log \frac{v_0}{v_0 - v} = \frac{g}{v_0}x$$

or
$$x = \frac{v_0}{g}\left(v_0 \log \frac{v_0}{v_0 - v} - v\right)$$

is the relation between x and v.

(2) If the resistance is proportional to v^2, put $R =$ resistance in pounds $= k\frac{W}{g}v^2$. The equation of motion is
$$\frac{dv}{dt} = g - kv^2 \quad . \quad . \quad . \quad . \quad . \qquad (7)$$
$$\frac{dv}{dt} = k\left(\frac{g}{k} - v^2\right) \quad . \quad . \quad . \quad . \qquad (8)$$

or $\frac{1}{v_0^2 - v^2}\frac{dv}{dt} = k$ where $v_0 = \sqrt{\frac{g}{k}}$. By partial fractions we have
$$\frac{1}{v_0^2 - v^2} = \frac{1}{2v_0}\left(\frac{1}{v_0 - v} + \frac{1}{v_0 + v}\right)$$

Hence (8) becomes
$$\left(\frac{1}{v_0 + v} + \frac{1}{v_0 - v}\right)\frac{dv}{dt} = 2v_0k$$

and integrating,
$$\log_e \frac{v_0 + v}{v_0 - v} = 2v_0kt + \log_e A$$

from which
$$\frac{v_0 + v}{v_0 - v} = Ae^{2v_0kt}$$

If $v = 0$ when $t = 0$,
$$1 = Ae^0 = A$$

and therefore
$$\frac{v_0 + v}{v_0 - v} = e^{2v_0kt}$$

Solving this for v we have

$$v = v_0 \frac{e^{2v_0kt} - 1}{e^{2v_0kt} + 1}$$

$$= v_0 \frac{e^{v_0kt} - e^{-v_0kt}}{e^{v_0kt} + e^{-v_0kt}}$$

$$= v_0 \tanh v_0kt$$

$$v = v_0 \tanh \frac{gt}{v_0} \qquad (9)$$

As t approaches infinity the value of $\tanh \frac{gt}{v_0}$ approaches unity. Hence $v_0 = \sqrt{\frac{g}{k}}$ is the terminal velocity. It is obvious otherwise that $\sqrt{\frac{g}{k}}$ is the terminal velocity, for if $v = \sqrt{\frac{g}{k}}$ in (7), the acceleration is zero and the particle will move with uniform velocity. Actually the particle never attains its terminal velocity as it would require an infinite time to do so. To find v in terms of x we write $v \frac{dv}{dx}$ for $\frac{dv}{dt}$ in (8), obtaining

$$v \frac{dv}{dx} = k\left(\frac{g}{k} - v^2\right)$$

or

$$v \frac{dv}{dx} = k(v_0^2 - v^2)$$

This becomes

$$\frac{v}{v_0^2 - v^2} \frac{dv}{dx} = k$$

or

$$\frac{-2v}{v_0^2 - v^2} \frac{dv}{dx} = -2k$$

whence on integrating,

$$\log_e (v_0^2 - v^2) = -2kx + \log A$$

i.e.

$$v_0^2 - v^2 = Ae^{-2kx}$$

and

$$v^2 = v_0^2 - Ae^{-2kx}$$

Since $v = 0$ when $x = 0$, $\quad 0 = v_0^2 - Ae^0, \quad A = v_0^2$

and

$$v^2 = v_0^2(1 - e^{-2kx})$$

$$v^2 = v_0^2\left(1 - e^{-\frac{2g}{v_0^2}x}\right) \qquad (10)$$

which again shows that the terminal velocity is v_0 and that it is acquired in falling through an infinite distance. A relation between x and t can be obtained by writing $\frac{dx}{dt}$ for v in (9) and integrating, or by eliminating v between (9) and (10). Using this method we have

$$\tanh^2 \frac{gt}{v_0} = 1 - e^{-\frac{2g}{v_0^2}x} \qquad (11)$$

which is the required relation. Since $1 - \tanh^2 u = \operatorname{sech}^2 u$ this simplifies to

$$\operatorname{sech}\frac{gt}{v_0} = e^{-\frac{gx}{v_0^2}}$$

from which
$$\cosh\frac{gt}{v_0} = e^{\frac{gx}{v_0^2}}$$

or
$$x = \frac{v_0^2}{g}\log\cosh\frac{gt}{v_0}$$

EXAMPLE 6

Induced Current. Solve the equation $L\frac{di}{dt} + Ri = E$ where L and R are constants, for the following values of E: (1) $E = 0$; (2) $E = E_0$ (a constant); (3) $E = E_0 \sin \omega t$.

This equation gives the law of induced currents of electricity, and is of the type dealt with in Art. 134. Dividing through by L, the equation becomes

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \tag{1}$$

The integrating factor is $e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$. Multiplying through by this

$$e^{\frac{Rt}{L}}\frac{di}{dt} + \frac{R}{L}e^{\frac{Rt}{L}}i = \frac{Ee^{\frac{Rt}{L}}}{L}$$

or
$$\frac{d}{dt}\left(e^{\frac{Rt}{L}}i\right) = \frac{Ee^{\frac{Rt}{L}}}{L} \tag{2}$$

Integrating we have
$$e^{\frac{Rt}{L}}i = \frac{1}{L}\int Ee^{\frac{Rt}{L}}dt + A$$

where A is a constant. Thus

$$i = \frac{1}{L}e^{-\frac{Rt}{L}}\int Ee^{\frac{Rt}{L}}dt + Ae^{-\frac{Rt}{L}} \tag{3}$$

(1) $E = 0$. In this case the integral reduces to zero (remember that A is the constant of integration) and (3) reduces to

$$i = Ae^{-\frac{Rt}{L}} \tag{4}$$

This gives the relation between the current i amperes and the time t seconds in a circuit of resistance R ohms and inductance L henrys, t seconds after the circuit is broken. Since $i = A$ when $t = 0$, A is the value of the current at the instant when the circuit is broken.

(2) $E = E_0$. In this case we have

$$i = \frac{1}{L}e^{-\frac{Rt}{L}} \times \frac{E_0 L}{R}e^{\frac{Rt}{L}} + Ae^{-\frac{Rt}{L}}$$

or
$$i = \frac{E_0}{R} + Ae^{-\frac{Rt}{L}} \tag{5}$$

If $i = 0$ when $t = 0$ then $0 = \frac{E_0}{R} + A$ or $A = -\frac{E_0}{R}$.

Hence
$$i = \frac{E_0}{R}(1 - e^{-\frac{Rt}{L}}) \tag{6}$$

This gives the current i in a circuit such as that in (1) t seconds after the circuit is made. $e^{-\frac{Rt}{L}}$ steadily tends to the value zero as t increases and the current tends to the value $\frac{E_0}{R}$.

(3) $E = E_0 \sin \omega t$. Substituting in (3),

$$i = \frac{1}{L}e^{-\frac{Rt}{L}}\int E_0 \sin \omega t \cdot e^{\frac{Rt}{L}}\,dt + Ae^{-\frac{Rt}{L}} \tag{7}$$

By the methods of Art 53, the value of $\int \sin \omega t \cdot e^{\frac{Rt}{L}}dt$ is found to be $\frac{L}{\sqrt{R^2 + \omega^2L^2}}e^{\frac{Rt}{L}}\sin(\omega t - \alpha)$ where $\alpha = \tan^{-1}\frac{\omega L}{R}$. Substituting this in (7),

$$i = \frac{1}{L}e^{-\frac{Rt}{L}}\frac{LE_0}{\sqrt{R^2 + \omega^2L^2}}e^{\frac{Rt}{L}}\sin(\omega t - \alpha) + Ae^{-\frac{Rt}{L}}$$

or
$$i = \frac{E_0}{\sqrt{R^2 + \omega^2L^2}}\sin(\omega t - \alpha) + Ae^{-\frac{Rt}{L}} \tag{8}$$

As t increases, the value of the second term on the right decreases, until after a time it becomes negligible and

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2L^2}}\sin(\omega t - \alpha), \text{ where } \alpha = \tan^{-1}\frac{\omega L}{R} \tag{9}$$

[*Discharge of a Condenser*. A similar problem to (1) above is that of the discharge of a condenser of capacity C through a resistance R. If q is the charge in the condenser at time t the relation is

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

This corresponds to case (1) of Ex 6, and the solution is by comparison

$$q = q_0 e^{-\frac{t}{CR}}$$

where q_0 is the original charge at time $t = 0$.]

EXAMPLE 7

Stress in a Thick Cylinder When investigating the stress in the material of a thick cylinder subjected to internal pressure we obtain the relations

$$p + r\frac{dp}{dr} = q \tag{1}$$

$$p + q = 2a \tag{2}$$

where p and q are the radial stress and the circumferential stress respectively in pounds per square inch (each positive when the stress is tensile), r is the radius in inches, and a is a constant. Express p and q as functions of r, and determine the constants in these expressions for the case of a cylinder of external radius R_1 in and internal radius R_2 in under internal pressure p_0 and no external pressure.

Eliminating q from (1) and (2) we have

$$p + r\frac{dp}{dr} = 2a - p$$

$$r\frac{dp}{dr} = 2(a-p)$$

or, separating the variables, $\dfrac{dp}{a-p} = \dfrac{2\,dr}{r}$

Integrating, $\quad -\log(a-p) = 2\log_e r - \log_e b$

$-\log_e b$ being the constant of integration

$$\log_e(a-p) = \log_e b - 2\log_e r$$

i.e. $\quad a - p = \dfrac{b}{r^2}$

or $$p = a - \frac{b}{r^2} \tag{3}$$

From (2) $$q = a + \frac{b}{r^2} \tag{4}$$

a and b are arbitrary constants whose values are determined from the conditions to which the formulae are applied. Thus, in the given case $p = -p_0$ when $r = R_2$, and $p = 0$ when $r = R_1$. Substituting these in turn in (3)

$$-p_0 = a - \frac{b}{R_2^2}$$

and $$0 = a - \frac{b}{R_1^2}$$

from which a and b are found

$$a = p_0\frac{R_2^2}{R_1^2 - R_2^2} \text{ and } b = p_0\frac{R_1^2 R_2^2}{R_1^2 - R_2^2}$$

Substituting these values in (3) and (4)

$$p = \frac{p_0R_2^2}{R_1^2 - R_2^2}\left(1 - \frac{R_1^2}{r^2}\right) \text{ and } q = -\frac{p_0R_2^2}{R_1^2 - R_2^2}\left(1 + \frac{R_1^2}{r^2}\right) \qquad (5)$$

The maximum circumferential stress q_m occurs when r is least, i.e. when $r = R_2$. Hence from (5)

$$q_m = -p_0 \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2} \qquad (6)$$

EXAMPLES

Motion of a Ship It is assumed that the resistance of the water to the motion of a ship is proportional to the square of the speed of the ship. It is found that when the engines are stopped the speed of the ship is reduced from 12 knots to 3 knots in 20 minutes. Calculate the horse-power necessary to propel the ship at a constant speed of 12 knots, the mass of the ship being 5 000 tons. [1 knot is a speed of 6 080 ft per hr.] (U.L.)

$$12 \text{ knots} = \frac{12}{60} \times \frac{6\,080}{60} \frac{\text{feet}}{\text{second}} = \frac{304}{15} \text{ ft per sec}$$

$$3 \text{ knots} = \frac{76}{15} \text{ ft per sec}$$

Let R lb be the resistance to motion. Since $R \propto v^2$, we may put $R = kMv^2$ where M is the mass in pounds and k is a constant; v is the speed in feet per second. The equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$-kMv^2 = \frac{M}{g} \cdot \frac{dv}{dt}$$

t being the time in seconds. From this

$$\frac{1}{v^2}\frac{dv}{dt} = -kg$$

Integrating
$$\frac{1}{v} = kgt + C$$

Since $t = 0$ when $v = \dfrac{304}{15}$,
$$\frac{1}{v} = kgt + \frac{15}{304}$$

When $t = 1\,200$, i.e. after 20 minutes, $v = \dfrac{76}{15}$, hence, substituting,

$$\frac{15}{76} = 1\,200kg + \frac{15}{304}$$

$$kg = \frac{1}{1\,200} \times \frac{45}{304}$$

from which
$$k = \frac{45}{1\,200 \times 304 \times 32} = \frac{3}{80 \times 304 \times 32}$$

The resistance R is given by

$$R = kMv$$

and

$$\text{Power} = \frac{\text{work done per sec in ft-lb}}{550 \text{ ft-lb per sec}} = \frac{kMv^3}{550}$$

$$= \frac{3 \times 5\,000 \times 2\,240 \times 304^3}{550 \times 80 \times 304 \times 32 \times 15^3}$$

$$= 653 \text{ h p nearly}$$

EXAMPLE 9

Convective Equilibrium of the Atmosphere Find the relation between the pressure p lb per ft^2 and the height h ft above sea-level of the atmosphere which is assumed to be stationary

Let w lb be the weight of a cubic foot of air at a pressure of p lb per ft^2 It is easy to prove, and is proved in textbooks of mechanics, that $\frac{dp}{dh} = -w$ This is the differential equation with which we start, but as three variables are involved, we require some other relation

(1) Assume that the temperature is constant throughout the atmosphere In this case, if v is the volume of 1 lb of air and this air is supposed to be moved about from place to place, any expansion or contraction will follow Boyle's Law, $pv = k$ where k is a constant This, then, is our second relation

Also $$w = \frac{1}{v}$$

We have then $$\frac{dp}{dh} = -w = -\frac{1}{v} = -\frac{p}{k}$$

or $$\frac{1}{p}\frac{dp}{dh} = -\frac{1}{k}$$

and integrating, $$\log_e p = -\frac{1}{k}h + \log A$$

or $$p = Ae^{-\frac{1}{k}h}$$

Now $k = pv = \frac{p}{w}$, and if p_0 is the pressure at sea-level and w_0 lb the weight of a cubic foot of air at sea-level, we have $k = \frac{p_0}{w_0}$ Substituting this

$$p = Ae^{-\frac{w_0}{p_0}h}$$

Since $p = p_0$ when $h = 0$, $p_0 = Ae^0$ or $A = p_0$ and

$$p = p_0 e^{-\frac{w_0}{p_0}h}$$

is the required law

(2) Assume that the temperature is not constant, in which case the law connecting p and v is not $pv = k$ but $pv^n = k$ where n is some number other than unity. The assumption of constant temperature is obviously absurd, being contrary to our experience that the temperature of the atmosphere varies greatly with change of altitude. It is probable that the law connecting pressure and volume is very nearly the adiabatic law of expansion, which is in the case of air $pv^{1\cdot 4} = k$ approximately.

In this case, $\frac{dp}{dh} = -w = -\frac{1}{v}$ and $pv^n = k$. ($n = 1\cdot 4$ for air.) Substituting for v in terms of p, we have

$$\frac{dp}{dh} = -\frac{p^{\frac{1}{n}}}{k^{\frac{1}{n}}}$$

or

$$p^{-\frac{1}{n}}\frac{dp}{dh} = -\frac{1}{k^{\frac{1}{n}}}$$

and integrating,

$$\frac{n}{n-1}p^{\frac{n-1}{n}} = -\frac{h}{k^{\frac{1}{n}}} + A$$

When $h = 0$, $p = p_0$, and substituting in the above,

$$\frac{n}{n-1}p_0^{\frac{n-1}{n}} = A$$

Substituting for A, we find that

$$\frac{n}{n-1}\left(p^{\frac{n-1}{n}} - p_0^{\frac{n-1}{n}}\right) = -\frac{h}{k^{\frac{1}{n}}}$$

Solving this for h,

$$h = \frac{nk^{\frac{1}{n}}}{n-1}\left(p_0^{\frac{n-1}{n}} - p^{\frac{n-1}{n}}\right)$$

or since

$$k^{\frac{1}{n}} = p_0^{\frac{1}{n}}v_0 = \frac{p_0^{\frac{1}{n}}}{w_0}$$

$$h = \frac{n}{n-1}\frac{p_0}{w_0}\left\{1 - \left(\frac{p}{p_0}\right)^{1-\frac{1}{n}}\right\} \quad . \quad . \quad (1)$$

From this the pressure at any height may be calculated if the pressure and the density at sea-level are given. Now, for a perfect gas we have the law $pv = Rt$ where t is the absolute temperature, and

$$\left(\frac{p}{p_0}\right)^{1-\frac{1}{n}} = \frac{p}{p_0}\left(\frac{p_0}{p}\right)^{\frac{1}{n}} = \frac{pv}{p_0v_0} = \frac{Rt}{Rt_0} = \frac{t}{t_0}$$

Hence, substituting in (1) and writing $\frac{p_0}{w_0} = p_0 v_0 = Rt_0$

$$h = \frac{Rnt_0}{n-1}\left(1 - \frac{t}{t_0}\right)$$

from which
$$t = t_0 - \frac{n-1}{Rn}h$$

This relation shows that the temperature falls by equal amounts for equal increases of the height.

136. **Orthogonal Trajectories.** If two families of curves are such that each member of either family intersects at right angles every member of the other family, the members of one family are known as the *orthogonal trajectories* of the other.

Let
$$f(x, y, c) = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(XI.16)}$$

represent one family of curves, c being an arbitrary constant or parameter. We can form from (XI.16), by the method of Art. 130, a first order differential equation which is independent of c. Let this equation be

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0 \quad . \quad . \quad . \quad \text{(XI.17)}$$

At a point of intersection of a curve represented by (XI.16) and its orthogonal trajectory the gradient of the latter is

$$-1\bigg/\frac{dy}{dx}$$

where $\frac{dy}{dx}$ is the gradient of the former curve at that point. The values of x and y are the same for both curves. Hence, if we write $-1\bigg/\frac{dy}{dx}$ for $\frac{dy}{dx}$ in (XI.17) we shall obtain the differential equation of the family of orthogonal trajectories of the family represented by (XI.16). This equation is, therefore,

$$\phi\left(x, y, -1\bigg/\frac{dy}{dx}\right) = 0 \quad . \quad . \quad \text{(XI.18)}$$

EXAMPLE 1

Find the equation of the system of orthogonal trajectories of the family of curves $x^2 - y^2 = a^2$.

Differentiating both sides of the given equation with respect to x, we have

$$2x - 2y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = \frac{x}{y}$$

which is the differential equation of the given family. Changing $\frac{dy}{dx}$ into $-1\Big/\frac{dy}{dx}$, we obtain the differential equation of the family of orthogonal trajectories, i.e.

$$\frac{dy}{dx} = -\frac{y}{x} \text{ or } \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating, we have $\log y + \log x = \log A$ where A is a constant, from which $xy = A$.

Both systems of curves are rectangular hyperbolas. The reader should sketch their graphs.

EXAMPLE 2

Find the orthogonal trajectories of the system of parabolas $y^2 = 4ax$.

The parabolas of the system have all the same vertex and the same axis. We have on differentiation

$$2y\frac{dy}{dx} = 4a$$

Eliminating a between this and the given equation,

$$2xy\frac{dy}{dx} = y^2$$

$$2x\frac{dy}{dx} = y \text{ is the differential equation of the system of parabolas.}$$

Changing $\frac{dy}{dx}$ into $-1\Big/\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{2x}{y}$$

as the differential equation of the orthogonal trajectories.

Hence, $\quad y\,dy + 2x\,dx = 0$

from which $\quad 2x^2 + y^2 = c^2$

c being a constant. Thus the orthogonal trajectories form a system of ellipses of semi-axes $\frac{c}{\sqrt{2}}$ and c where c is a variable parameter.

EXAMPLES XI

(1) Form a differential equation which does not involve either a or b from the relation $(y - a)^2 = x - b$.

(2) If $y = A\sin x + B\cos x$, form a second order differential equation independent of A and B.

(3) If $y = A\sin mx + B\cos mx + C\sinh mx + D\cosh mx$, show that $\frac{d^4y}{dx^4} - m^4y = 0$.

(4) If $y = Ae^{-kt}\sin(pt + \alpha)$ show that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + (k^2 + p^2)\,y = 0$.

Solve the following differential equations

(5) $2x\dfrac{dy}{dx} \quad y \quad 0$

(6) $x\dfrac{dy}{dx} \mid y \quad 0$

(7) $y\dfrac{dy}{dx} \quad x \quad 0$

(8) $y\dfrac{dy}{dx} - x \quad 0$

(9) $\dfrac{ds}{dt} \quad at^{2} \quad bt \quad c$

(10) $\dfrac{ds}{dt} - \frac{1}{2}\omega r^{2}$ where ω is constant and $r \quad 2a\cos\omega t$

(11) $(1 \quad x)\dfrac{dy}{dx} \quad 1 \quad x^{2}$

(12) $\dfrac{1}{x}\dfrac{dy}{dx} \quad c$

(13) $\dfrac{dy}{dx} \quad y \quad xy$

(14) $\dfrac{dy}{dx} \quad \dfrac{kx \quad xy}{x^{2} - 1}$

(15) $\sqrt{a^2 - x^2}\,\dfrac{dy}{dx} \quad \sqrt{a^{2} - y^{2}}$

(16) $\dfrac{\sin x}{1 + x}\dfrac{dy}{dx} - \cos x$

(17) $r\dfrac{dp}{dr} \quad 2p - a$

Solve the homogeneous differential equations—

(18) $\dfrac{dy}{dx} - \dfrac{(x - y)}{(x + y)}$

(19) $xy^{2}\dfrac{dy}{dx} = x^{3} + y^{3}$

(20) $\dfrac{dy}{dx} = \dfrac{ax + by}{bx - cy}$

(21) $\dfrac{dy}{dx} = \dfrac{y^{2} + 3xy}{x^{2}}$

(22) $\dfrac{dy}{dx} = \dfrac{x^{2} + y^{2}}{xy}$

Solve the exact equations (23) to (26).

(23) $(x + 3y)\,dy + (x + y)\,dx = 0$

(24) (3x y) dx (6y x) dy 0

(25) (sin x x cos y x) dy (y cos x sin y y) dx 0

(26) $\frac{2y}{x \quad y} \, dy \qquad \frac{2x}{x \quad y} \, dx \quad 0$

(27) Solve $\frac{dy}{dx}$ 6y and determine the constant of integration so that y 6 when x 0

(28) If $\frac{dT}{d\theta}$ μT show that T $T_0 e^{\mu\theta}$ where T_0 is a constant

(29) Solve $\frac{dq}{dt}$ $kq - 0$ and adjust the constant of integration so that q q_0 when t 0

(30) Solve $\frac{1}{r}$ $\frac{dh}{dr}$ 0 and if $\frac{1}{2r}$ h k, k being a constant, express h as a function of r Sketch a graph showing the relation between h and r

(31) Solve (i) $\frac{dy}{dx}$ 4y 0 and (ii) $\frac{dy}{dx}$ 4y 5 Determine the constants of integration so that in each case y 6 when x 0

(32) Solve $\frac{dy}{dx}$ by a and find the value of the constant of integration, so that y 0 when x 0

(33) Solve $\frac{dy}{dx}$ y e^{x} and $\frac{dy}{dx}$ y e^{-x}

(34) The equation giving the current y in a conductor at any time t is of the form $a\frac{dy}{dt}$ + by $c \sin pt$ Find the general value of y and determine the constant of integration if y 0 when $t = 0$

Solve also $a\frac{dy}{dt} + by - ce^{-\frac{b}{a}t}$ (U L)

(35) Obtain the general solution of the equation $A\frac{dy}{dx} + By = f(x)$, where A and B are constants

A particle of mass 1 lb moves in a medium whose resistance is $\frac{1}{4}v$ lb weight, where v is the velocity, and is subject to an accelerating force constant in direction which at time t is $4t^2$ lb weight If the particle starts from rest, find its velocity after 2 seconds (U L)

(36) Solve $x\frac{dy}{dx} - 3y = x - 1$

(37) Solve $\frac{dy}{dx}$ my e^{ax}

(38) Solve $L\frac{di}{dt} + Ri - E$ where L, R, and E are constants

(39) Solve the last example for the case in which $E = 200 \cos qt$ where $q = 300$, $R = 100$, L 0 05, and find i on the assumption that $i - 0$ when t 0 What value does i approach after a long time?

(40) The subtangent of a curve is constant Find the equation to the curve

(41) Solve $\dfrac{dy}{dx} = \dfrac{4x + 3y + 6}{3x + y + 7}$ and $\dfrac{dy}{dx} = \dfrac{-12x - 6y}{8x - 2y + 4}$

(42) A particle P moves in a straight line under an attraction directed towards a fixed point O in the line, and varying inversely as the square of OP Show that its velocity v in any position P_1 is proportional to $\sqrt{\dfrac{1}{OP_1} - \dfrac{1}{OA}}$, if A is the point from which it starts with zero velocity

(43) If $\dfrac{dv}{dt} = -kv^n$, and $\dfrac{dv}{dt} = v\dfrac{dv}{ds}$, find the relation between v and s and, if $v = V$ when $s = 0$, prove that $\dfrac{1}{v^{n-2}} - \dfrac{1}{V^{n-2}} = (n-2)ks$

(44) A body falling vertically under gravity encounters the resistance of the atmosphere If the resistance varies as the velocity, the equation of motion is $\dfrac{du}{dt} = g - ku$ where k is a constant and g is the acceleration due to gravity Prove that $u = \dfrac{g}{k} - Ce^{-kt}$ Show that as t increases u approaches the value $\dfrac{g}{k}$

Now, writing $\dfrac{dx}{dt}$ for u where x is the distance fallen by the body from rest in time t, show that $x = \dfrac{gt}{k} - \dfrac{g}{k^2}(1 - e^{-kt})$

(45) Find the velocity-time and space-time relations for the body in the last example on the assumption that the resistance varies as the square on the speed Find the terminal velocity in this case

(46) If a rigid body rotates under gravity about a fixed horizontal axis, the equation of motion is

$$M(k^2 + h^2)\,\omega\frac{d\omega}{d\theta} = -Mgh\sin\theta$$

where M is the mass, k the radius of gyration about an axis parallel to the fixed axis through the centre of gravity, h the distance of the centre of gravity from the axis, θ the angular displacement of the body from its equilibrium position and $\omega = \dfrac{d\theta}{dt}$, t being the time Show that

$$\omega^2 = \frac{2hg\cos\theta}{k^2 + h^2} + C \text{ where } C \text{ is a constant}$$

Find the value of C if $\omega = \omega_0$ when $\theta = 0$

(47) If $m\dfrac{dv}{dt} = X$ and $I\dfrac{d\omega}{dt} = aX$ where m, a and I are constants and X, v, and ω functions of t show that if v_0 and ω_0 are the original values of a and ω respectively,

$$v = v_0 + \frac{I}{am}(\omega - \omega_0)$$

(48) Show that the adiabatic curve through any point in the pressure-volume diagram of a perfect gas is steeper than the isothermal through that point. Show,

further, that along a given isothermal the angle of intersection rises from zero at either end to a maximum of $\sin^{-1}\left(\frac{\gamma - 1}{\gamma + 1}\right)$, and that the points corresponding to this maximum on different isothermals are on a straight line through the origin. (γ is the ratio of the specific heats of the gas.)

A quantity of perfect gas occupying a volume v_1 at absolute temperature T is compressed adiabatically until its volume is v_2. It is then allowed to cool at constant volume until its temperature is again T. Finally, it expands isothermally until its volume is again v_1. Show that the work done on the gas is

$$RT\left[\frac{1}{\gamma - 1}\left\{\left(\frac{v_1}{v_2}\right)^{\gamma - 1} - 1\right\} + \log\frac{v_2}{v_1}\right]$$

per unit mass, where R is the gas constant. (U.L.)

[*Partial Solution.* γ is greater than unity. The adiabatic law is $pv^\gamma = C_1$, and the isothermal law is $pv = C_2$. If the law of expansion is $pv^n = C$, then $\frac{dp}{dv}$ is easily proved to be $-n\frac{p}{v}$ and the gradient of the graph is measured by $\frac{np}{v}$. Since $n > 1$ for an adiabatic curve, and $n = 1$ for an isothermal, the former is steeper at a point of intersection. The numerical values of the gradients of the adiabatic and of the isothermal are therefore $\frac{\gamma p}{v}$ and $\frac{p}{v}$ respectively, and if θ is the angle between them we have, using the formula for tan (difference of two angles),

$$\tan\theta = \frac{\gamma\frac{p}{v} - \frac{p}{v}}{1 + \gamma\left(\frac{p}{v}\right)^2}$$

or writing x for $\frac{p}{v}$

$$\tan\theta = (\gamma - 1)\frac{x}{1 + \gamma x^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is a maximum when $\frac{x}{1 + \gamma x^2}$ is a maximum, i.e. when $\frac{d}{dx}\frac{x}{1 + \gamma x^2} = 0$.

This gives $x = \sqrt{\frac{1}{\gamma}}$. Hence, the maximum difference of slope is given by

$$\tan\theta = \tfrac{1}{2}(\gamma - 1) \times \sqrt{\frac{1}{\gamma}} = \frac{\gamma - 1}{2\sqrt{\gamma}}$$

At the two ends $x = \frac{p}{v}$ is either infinite (when $v = 0$) or zero when $v = \infty$. In both cases $\tan\theta = 0$ and $\theta = 0$. Again, if

$$\tan\theta = \frac{\gamma - 1}{2\sqrt{\gamma}}$$

$$\cot\theta = \frac{2\sqrt{\gamma}}{\gamma - 1}$$

and

$$\operatorname{cosec}^2\theta = 1 + \frac{4\gamma}{(\gamma - 1)^2} = \left(\frac{\gamma + 1}{\gamma - 1}\right)^2$$

Hence,

$$\sin\theta = \frac{\gamma - 1}{\gamma + 1}$$

The angle of intersection is therefore zero at the ends and a maximum when $\frac{p}{v} \sqrt{\frac{1}{\gamma}} =$ constant. The graph of $\frac{p}{v}$ constant is a straight line through the origin, so that the points corresponding to the maximum on different isothermals lie on this straight line.

The second part is solved by finding the area enclosed between the ordinate $v = v_2$ and the isothermal and adiabatic curves through the point for which $v = v_1$ and the temperature is T.]

(49) If $mv \dfrac{dv}{dx} = a - kv^2$, where a, k, and m are constants, determine v in terms of x.

A toboggan, weighing 200 lb, descends from rest a uniform slope of 5 in 13 which is 50 yd long. If the coefficient of friction be $\frac{1}{10}$, and the air-resistance varies as the square of the velocity, and is 3 lb weight when the velocity is 10 ft-sec, prove that its velocity at the bottom is 38·6 ft-sec, and show that, however long the slope is, the velocity cannot exceed 44·14 ft-sec. (U.L.)

(50) Up to a certain height in the atmosphere it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $[n > 1]$.

If this relation continued to hold up to any height, show that the density would vanish at a finite height and that the absolute temperature would vary directly as the distance from the top of the atmosphere. (U.L.)

(51) If a volume v of a perfect gas at absolute temperature T is compressed adiabatically to absolute temperature T_0, prove that the volume becomes $v\left(\dfrac{T}{T_0}\right)^{\frac{1}{\gamma-1}}$, where γ is a constant.

A cylinder contains 5 ft³ of air at a pressure of 28 lb-in.² and at temperature 15°C. If the air is suddenly compressed to half this volume by a piston, prove that the work done is about 7 ft-tons, and find the final temperature. [Take γ to be 1·41.] (U.L.)

(52) If $\dfrac{dp}{dv} = -\dfrac{wv}{g}$ and $w = cp^{\frac{1}{\gamma}}$ where g and c are constants, show that

$$v^2 + \frac{2g\gamma p^{\frac{\gamma-1}{\gamma}}}{(\gamma-1)} = \text{constant.}$$

(53) Euler's equations of motion for a body under the action of no external forces are $A\dfrac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0$, $B\dfrac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 = 0$, and $C\dfrac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = 0$ where A, B, and C are constants. Multiply these through by ω_1, ω_2, and ω_3 respectively, and add. Integrate the resulting equation to show that $A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h^2$ where h is a constant.

(54) Find the orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$. Show that these are radii of the circles.

(55) Find the equation of the family of circles which all pass through the points $(-a, 0)$ and $(+a, 0)$ and find their orthogonal trajectories. Show that these are circles.

(56) Find the orthogonal trajectories of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1$, and of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$.

(57) A fluid contained in a vertical cylinder is rotated with the cylinder about the axis OY of the latter so that all portions of the fluid have the same constant angular velocity. The surfaces of equal pressure in the fluid are paraboloids of revolution generated by the rotation about the y-axis of the family of parabolas $y = kx^2 + c$, c being a variable parameter. OX is horizontal. Show that the surfaces generated by the family of orthogonal trajectories of the parabolas are represented by the equation $\sqrt{x^2 + z^2} = Ae^{-2ky}$, OZ being horizontal and perpendicular to OX.

(58) (i) The equation giving the horizontal oscillation of a compass needle is of the form

$$\frac{d\omega^2}{d\theta} = a \sin \theta - b\omega^2$$

Find ω in any position if it is zero when $\theta = \frac{\pi}{3}$

(ii) Solve the equation

$$\frac{dx}{dt} + 3x = e^{2t} + t^2 \qquad \text{(U.L.)}$$

(59) If $v \frac{dv}{dx} = \frac{dv}{dt} = g\left(1 - \frac{v^2}{V^2}\right)$, and when $t = 0$, $x = 0$, $v = 0$, prove that

$$e^{gx/V^2} = \cosh (gt/V)$$

A particle falls freely from rest in a slightly resisting medium in which the resistance varies as the square of the velocity. Show that so long as the velocity of the particle is small compared with the terminal velocity, the distance fallen in any time is approximately $x - x^2/6h$, where x is the distance through which it would have fallen freely in the time and h is the distance through which it would have to fall freely to acquire the terminal velocity. (U.L.)

(60) When a certain body is dropped from a great height it has a terminal velocity of 400 ft per sec. Assuming that the resistance of the air varies as the square of the speed, find the height to which this body would rise if it were projected upwards with a speed of 1 000 ft per sec. (U.L.)

(61) Find v in terms of x from the equation $v^2 \frac{dv}{dx} = a - bv^3$, where a and b are constants; given that $v = V$ when $x = c$.

A particle of mass 1 lb moves in a straight line in a medium in which the resistance varies as the square of the velocity and is acted on by a force doing work at a uniform rate of 20 ft-lb per sec. If the resistance equals 2 lb weight when $v = 1$ ft per sec, find the distance that the particle has moved from rest when $v = 2$ ft per sec. (U.L.)

(62) Solve the differential equations—

(i) $\cos x \frac{dy}{dx} + 1 = y \sin x$

(ii) $2y \frac{dy}{dx} = x - y$

(iii) $2(4x - 3y - 1) + 3(6y - 2x - 1)\frac{dy}{dx} = 0$ (U.L.)

(63) The frictional torque acting upon a flywheel is at every instant equal to $k(\omega^2 + c^2)$, where ω is the angular velocity and k and c are constants. Show that if the flywheel be set rotating with angular velocity $c \tan \lambda$, the angular velocity after the time t is $c \tan (\lambda - At)$ where $A = kc/I$ and I is the moment of inertia of the flywheel.

Is the above expression for the angular velocity at the time t valid for all values of t? (U.L.)

(64) (i) Find the value of a given that $x^3 \dfrac{dp}{dx} = a - x$, and that $p = 0$ when $x = 2$ and when $x = 6$.

(ii) Solve the equation

$$\frac{dy}{dx} = \frac{x + y - 3}{x + y + 3}$$

given that $y = 1$ when $x = 0$. (U.L.)

(65) A particle is projected vertically upwards with velocity V in a medium in which the resistance to motion is at every instant proportional to the square of the velocity. The terminal velocity, i.e. the velocity when the resistance is equal to the weight, is u. If there were no resistance the particle would ascend to height h if projected with velocity u and to height H if projected with velocity V. Show that the actual height to which it ascends is

$$h \log_e \left(1 + \frac{H}{h}\right) \qquad \text{(U.L.)}$$

(66) A flywheel, of moment of inertia I about its axis of rotation, is acted upon by a couple whose moment at time t is $G_0 + G \sin pt$, where G_0 and G are constant, and is subjected to a resisting couple of moment $k\omega$, where ω is the angular speed at any instant and k is constant. Find the value of ω at any time t, and show that its mean value over a period $\dfrac{2\pi}{p}$ tends to a limit $\dfrac{G_0}{k}$ as t increases indefinitely.

(U.L.)

SECOND ORDER DIFFERENTIAL EQUATIONS

137. Equations of the Type $\dfrac{d^2y}{dx^2} = f(x)$. The equation

$$\frac{d^2y}{dx^2} = f(x) \quad . \qquad \text{(XII.1)}$$

can be solved by two successive integrations, thus

$$\frac{dy}{dx} = \int f(x)\,dx + C \quad . \quad . \qquad \text{(XII.2)}$$

and
$$y = \iint f(x)\,dx\,dx + Cx + D \quad . \quad . \qquad \text{(XII.3)}$$

where C and D are constants of integration. Equations have been solved by direct integration in Art. 131. As a further example we give the following—

EXAMPLE

Show that at a point of inflexion $\dfrac{d^2y}{dx^2} = 0$.

The bending moment at any point of a loaded beam being $EI\dfrac{d^2y}{dx^2}$ (approximately), find a formula for this if the beam is supported at its ends and the loading is as indicated in the diagram (Fig. 128). The total load is W, and in addition there is a fixing moment M applied at each end, keeping the beam horizontal at those points. Determine M and show that the points of inflexion are about $0{\cdot}6l$ from the centre of the beam. (U.L.)

Let x be the distance of a point P in the beam from the left-hand support, and let y be the deflection of P. Each support exerts an upward force $\dfrac{W}{2}$ on the beam. The load diagram to the left of P is made up of a rectangle of area xl and a triangle of area $\frac{1}{2}x^2$. The total area of the load diagram is $3l^2$, so that the loads represented by xl and $\frac{1}{2}x^2$ are $\dfrac{Wx}{3l}$ and $\dfrac{Wx^2}{6l^2}$ respectively. The moments of these about P are $\dfrac{Wx^2}{6l}$ and $\dfrac{Wx^3}{18l^2}$, their sum being $\dfrac{W}{18l^2}(x^3 + 3lx^2)$. Hence, the bending moment at P is $M + \dfrac{W}{18l^2}(x^3 + 3lx^2) - \dfrac{Wx}{2}$. We have, therefore

$$EI\frac{d^2y}{dx^2} = M + \frac{W}{18l^2}(x^3 + 3lx^2 - 9l^2x) \quad . \quad . \quad . \quad (1)$$

Integrating,
$$EI\frac{dy}{dx} = Mx + \frac{W}{18l^2}\left(\frac{x^4}{4} + lx^3 - \frac{9}{2}l^2x^2\right) \quad . \quad (2)$$

the constant of integration being zero, because $\frac{dy}{dx} = 0$ when $x = 0$. From symmetry $\frac{dy}{dx} = 0$ when $x = l$.

$$\therefore \quad 0 = Ml + \frac{Wl^2}{18}\left(\frac{1}{4} + 1 - \frac{9}{2}\right)$$

or
$$M = \frac{13}{72} Wl$$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$ (Art. 65). Hence, substituting for M and $\frac{d^2y}{dx^2}$ in (1) we obtain the equation

$$\frac{W}{18l^2}(x^3 + 3lx^2 - 9l^2x) + \frac{13}{72} Wl = 0$$

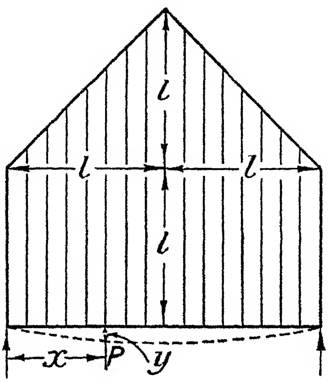


FIG. 128

or
$$\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^2 - 9\left(\frac{x}{l}\right) + \frac{13}{4} = 0 \quad . \tag{3}$$

Solving this for $\frac{x}{l}$ by one of the methods of Arts. 61 and 62, we find that $\frac{x}{l} = 0{\cdot}4$ to the nearest tenth. The points of inflexion are therefore at distances of $0{\cdot}4l$ from each end, or $0{\cdot}6l$ from the centre. If it is required to find the deflection, integration of (2) gives

$$EIy = \frac{Mx^2}{2} + \frac{W}{18l^2}\left(\frac{x^5}{20} + \frac{lx^4}{4} - \frac{3}{2}l^2x^3\right);$$

the constant of integration is again zero for $y = 0$ when $x = 0$. Hence,

$$y = \frac{W}{EI}\left\{\frac{x^5}{360l^2} + \frac{x^4}{72l} - \frac{x^3}{12} + \frac{13lx^2}{144}\right\}$$

138 **Equations of the Type $\dfrac{d^2y}{dx^2} = f(y)$** Consider the equation

$$\frac{d^2y}{dx^2} = f(y) \tag{XII 4}$$

Multiplying both sides by $2\dfrac{dy}{dx}$

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}$$

Integrating with respect to x, we have, since

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}\,\frac{d^2y}{dx^2}$$

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\frac{dy}{dx}\,dx + C \tag{XII 5}$$

or

$$\frac{dy}{dx} = \sqrt{2\int f(y)\,dy + C}$$

This is a first order equation with the variables separable and

$$\frac{dy}{\sqrt{2\int f(y)\,dy + C}} = \pm\, dx$$

the solution of which is

$$\int \frac{dy}{\sqrt{2\int f(y)\,dy + C}} = \pm\, x + D$$

It is convenient to write p for $\dfrac{dy}{dx}$ If

$$p = \frac{dy}{dx} \text{ then } \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy}$$

Then, substituting in (XII 4), we have

$$p\,\frac{dp}{dy} = f(y) \tag{XII 6}$$

which is a first order equation By the method of separating the variables

$$p\,dp = f(y)\,dy$$

and integrating,

$$\tfrac{1}{2}p^2 = \int f(y)\,dy + C'$$

If we now replace p by $\frac{dy}{dx}$ and write C for $2C$ we obtain the first order equation (XII 5) and solve as before. This latter method of solution is particularly useful in solving equations arising in dynamical problems. If x and t are distance and time in rectilinear motion $\frac{dx}{dt}$ is velocity and $\frac{d^2x}{dt^2}$ is acceleration. Alternative expressions for acceleration are, therefore, $\frac{d^2x}{dt^2}$ and $v\frac{dv}{dx}$, the latter of which may also be written $\frac{d}{dx}(\frac{1}{2}v^2)$.

EXAMPLE 1

Solve $\frac{d^2y}{dx^2} = \frac{\mu}{y^3}$ where μ is a constant.

Here, writing $p\frac{dp}{dy}$ for $\frac{d^2y}{dx^2}$

$$p\frac{dp}{dy} = \frac{\mu}{y^3}$$

and
$$p\,dp = \frac{\mu}{y^3}\,dy$$

$$\tfrac{1}{2}p^2 = -\frac{\mu}{2y^2} + C, \text{ where } C \text{ is a constant}$$

From this
$$\frac{dy}{dx} = \pm\sqrt{2C - \frac{\mu}{y^2}}$$

or
$$\frac{y\,dy}{\sqrt{2Cy^2 - \mu}} = \pm\,dx$$

and integrating, $\frac{1}{2C}(2Cy^2 - \mu)^{\frac{1}{2}} = \pm(x + D)$, where D is a constant.

Squaring both sides and rearranging, we have

$$2Cy^2 - \mu = 4C^2(x + D)^2$$

and the solution is
$$y^2 = \frac{\mu}{2C} + 2C(x + D)^2$$

EXAMPLE 2

The Inverse Square Law. A particle moves along a straight line with an acceleration directed towards a point O in the straight line and of magnitude $\frac{\mu}{OP^2}$. Find the velocity of the particle at P if it starts from R with zero velocity.

Let $x = \overline{OP}$ and $\overline{OR} = r$ (Fig. 129). The equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$$

or

$$v\frac{dv}{dx} = -\frac{\mu}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the negative sign denoting that the acceleration is directed toward O.

Integrating,

$$\tfrac{1}{2}v^2 = +\frac{\mu}{x} + C \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

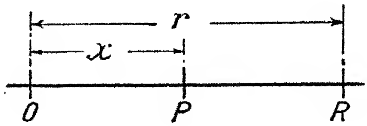


FIG. 129

When

$$x = r,\ v = 0$$

$\therefore$

$$0 = \frac{\mu}{r} + C$$

or

$$C = -\frac{\mu}{r}$$

Substituting in (2),

$$\tfrac{1}{2}v^2 = \mu\left(\frac{1}{x} - \frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which the velocity may be found for any value of x. The velocity v at P is therefore given by (3). If $r = \infty$, i.e. if the particle moves from infinity to P, $\frac{1}{r} = 0$ and (3) becomes

$$\tfrac{1}{2}v^2 = \frac{\mu}{x}$$

or

$$v = \sqrt{\frac{2\mu}{x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The attraction of the earth on a particle outside of it varies inversely as the square of its distance from the earth's centre, and so (3) and (4) above will apply to the case of a particle falling towards the earth. In this case, if a is the earth's radius, the acceleration of the particle when $x = a$ is g. Consequently, $\frac{\mu}{a^2} = g$, and from (3)

$$\tfrac{1}{2}v^2 = a^2g\left(\frac{1}{x} - \frac{1}{r}\right)$$

and as before, if r is infinite,

$$v^2 = \frac{2a^2g}{x}$$

which gives the velocity for any value of x of a particle falling from infinity with initial velocity zero. When the particle arrives at the earth's surface, we have $x = a$ and

$$v^2 = 2ag$$

or

$$v = \sqrt{2ag}$$

Writing $\frac{dx}{dt}$ for v in (3), and taking the square root of each side, we have

$$\frac{dx}{dt} = -\sqrt{2\mu}\left(\frac{1}{x}-\frac{1}{r}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (5)$$

We place the minus sign before the term on the right, because we are considering the case of a particle moving from R towards O. From (5)

$$-\int\sqrt{\frac{x}{r-x}}\,dx = \sqrt{\frac{2\mu}{r}}\int dt + C$$

$$\therefore \qquad \sqrt{\frac{2\mu}{r}}\,t = -C - \int\sqrt{\frac{x}{r-x}}\,dx$$

Now substitute $x = r\cos^2\theta$, $\sqrt{\frac{x}{r-x}} = \cot\theta$, and $dx = -2r\sin\theta\cos\theta d\theta$.

$$\therefore \qquad \sqrt{\frac{2\mu}{r}}\cdot t = -C + 2r\int\cos^2\theta d\theta$$

$$= -C + r\int(1+\cos 2\theta)d\theta$$

$$= -C + r(\theta + \tfrac{1}{2}\sin 2\theta)$$

$$\therefore \qquad \sqrt{\frac{2\mu}{r}}\,.\,t = -C + r\left(\cos^{-1}\sqrt{\frac{x}{r}} + \frac{1}{r}\sqrt{rx-x^2}\right)$$

Now $t = 0$ when $x = r$,

$$\therefore \qquad 0 = -C + r(\cos^{-1} 1)$$

$$\therefore \qquad C = 0$$

and

$$t = \sqrt{\frac{r}{2\mu}}\left\{r\cos^{-1}\sqrt{\frac{x}{r}} + \sqrt{rx-x^2}\right\} \quad . \quad . \quad (6)$$

(6) gives the time taken for the particle to move from R to any other point. We see from (3) that v is infinite when $x = 0$, so that the particle would reach the origin with infinite velocity. From (6), if $x = 0$ we have

$$t = \sqrt{\frac{r}{2\mu}}\cdot r\cos^{-1} 0 = \frac{r^{\frac{3}{2}}}{\sqrt{2\mu}} \times \frac{\pi}{2} = \frac{\pi r^{\frac{3}{2}}}{2\sqrt{2}\sqrt{\mu}}$$

the time taken to move from R to O.

EXAMPLE 3

Simple Harmonic Motion. Solve the differential equation

$$\frac{d^2x}{dt^2} + n^2x = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where t is time in seconds, x is the displacement of a particle along a line, and n is a constant. Show that the motion of the particle is periodic and of period $\frac{2\pi}{n}$ seconds. (See Art. 38.)

Multiplying both sides of (1) by $2\frac{dx}{dt}$, and transposing,

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2n^2x\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad (2)$$

Integrating with respect to t we have, since $2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)^2$,

$$\left(\frac{dx}{dt}\right)^2 = -n^2x^2 + C \quad . \quad . \quad (3)$$

Since the left-hand side of this is positive, C is also positive. Write $C = n^2a^2$.

Then $$\left(\frac{dx}{dt}\right)^2 = n^2(a^2 - x^2)$$

and $$\frac{1}{\sqrt{a^2 - x^2}}\,\frac{dx}{dt} = \pm n$$

Integrating this, $$\cos^{-1}\frac{x}{a} = \pm(nt + \alpha)$$

from which $$x = a\cos(nt + \alpha) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the double sign being unnecessary because $\cos(-A) = \cos A$. a and α are constants of integration, and their values are determined in any particular case from the particulars of the motion at any given time. Since α is an arbitrary constant, (4) may be written in the form

$$x = a\sin(nt + \alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where α' is equal to $\alpha + \frac{\pi}{2}$. Either (4) or (5) may be taken as the solution of (1), as also may

$$x = A\sin nt + B\cos nt \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is obtained by expanding $\sin(nt + \alpha')$ in (5) and writing A for $a\cos\alpha'$ and B for $a\sin\alpha'$.

The motion represented by (4), (5), or (6) is periodic, and its period is the time in which $nt + \alpha$ increases by 2π, i.e. $\frac{2\pi}{n}$ seconds. This motion is known as *Simple Harmonic Motion*. The angle α in (4) is known as the epoch; the angle $nt + \alpha$ is the *phase*. Equation (1) written in the form $\frac{d^2x}{dt^2} = -n^2x$ shows that the acceleration is proportional to the displacement, and that the former is always directed towards the origin $x = 0$.

EXAMPLE 4

Simple Pendulum. A particle C suspended from a fixed point O by a string of length l ft makes small oscillations under gravity about its equilibrium position. Find the periodic time.

In Fig. 130, OC shows the pendulum in a position in which the string makes an angle θ with the equilibrium position OB. Suppose the particle to be set oscillating to and fro between the extreme positions A and D, and let C be its position after time t sec. Let θ radians be the magnitude of the angle BOC and W lb the weight of the particle. The forces acting on the particle are (*a*) the tension in the string OC, which being at right angles to the direction of motion has no component along the tangent at C to the arc ACB, and (*b*) W lb, the weight of the particle which acts parallel to OB. The component of the latter force in the direction of motion is $W \sin\theta$ lb, and the equation of motion is

Force = mass × acceleration

or
$$-W\sin\theta = \frac{W}{g}\frac{d^2x}{dt^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

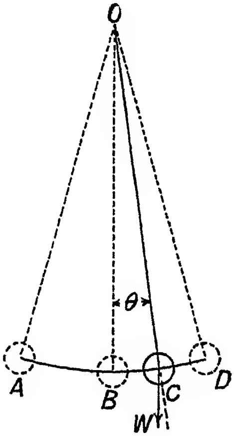


FIG. 130

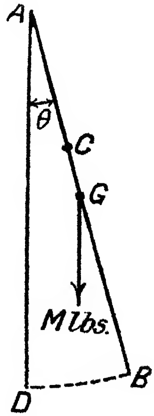


FIG. 131

where $x = \text{arc } BC = l\theta$. Substituting this value of x in (1), we have on transposing

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or substituting for θ in (2),

$$\frac{d^2x}{dt^2} + g\sin\frac{x}{l} = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As the oscillations are small, we may write $\frac{x}{l}$ for $\sin\frac{x}{l}$, thus obtaining

$$\frac{d^2x}{dt^2} + \frac{g}{l}x \quad 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where t is time in seconds, x is the displacement of a particle along a line, and n is a constant. Show that the motion of the particle is periodic and of period $\frac{2\pi}{n}$ seconds. (See Art. 38.)

Multiplying both sides of (1) by $2\frac{dx}{dt}$, and transposing,

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2n^2x\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating with respect to t we have, since $2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)^2$,

$$\left(\frac{dx}{dt}\right)^2 = -n^2x^2 + C \quad . \quad . \quad . \quad (3)$$

Since the left-hand side of this is positive, C is also positive. Write $C = n^2a^2$.

Then $$\left(\frac{dx}{dt}\right)^2 = n^2(a^2 - x^2)$$

and $$\frac{-1}{\sqrt{a^2 - x^2}}\frac{dx}{dt} = \pm n$$

Integrating this, $$\cos^{-1}\frac{x}{a} = \pm(nt + \alpha)$$

from which $$x = a\cos(nt + \alpha) \quad . \quad . \quad . \quad (4)$$

the double sign being unnecessary because $\cos(-A) = \cos A$. a and α are constants of integration, and their values are determined in any particular case from the particulars of the motion at any given time. Since α is an arbitrary constant, (4) may be written in the form

$$x = a\sin(nt + \alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where α' is equal to $\alpha + \frac{\pi}{2}$. Either (4) or (5) may be taken as the solution of (1), as also may

$$x = A\sin nt + B\cos nt \quad . \quad . \quad . \quad . \quad (6)$$

which is obtained by expanding $\sin(nt + \alpha')$ in (5) and writing A for $a\cos\alpha'$ and B for $a\sin\alpha'$.

The motion represented by (4), (5), or (6) is periodic, and its period is the time in which $nt + \alpha$ increases by 2π, i.e. $\frac{2\pi}{n}$ seconds. This motion is known as *Simple Harmonic Motion*. The angle α in (4) is known as the epoch; the angle $nt + \alpha$ is the *phase*. Equation (1) written in the form $\frac{d^2x}{dt^2} = -n^2x$ shows that the acceleration is proportional to the displacement, and that the former is always directed towards the origin $x = 0$.

EXAMPLE 4

Simple Pendulum. A particle C suspended from a fixed point O by a string of length l ft makes small oscillations under gravity about its equilibrium position. Find the periodic time.

In Fig. 130, OC shows the pendulum in a position in which the string makes an angle θ with the equilibrium position OB. Suppose the particle to be set oscillating to and fro between the extreme positions A and D, and let C be its position after time t sec. Let θ radians be the magnitude of the angle BOC and W lb the weight of the particle. The forces acting on the particle are (*a*) the tension in the string OC, which being at right angles to the direction of motion has no component along the tangent at C to the arc ACB, and (*b*) W lb, the weight of the particle which acts parallel to OB. The component of the latter force in the direction of motion is $W \sin \theta$ lb, and the equation of motion is

Force = mass × acceleration

or
$$-W \sin \theta = \frac{W}{g}\frac{d^2x}{dt^2} \qquad (1)$$

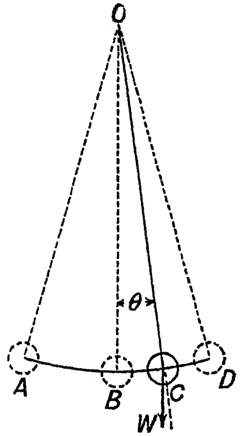


FIG. 130

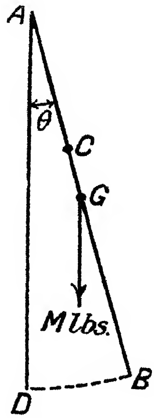


FIG. 131

where $x = \text{arc } BC = l\theta$. Substituting this value of x in (1), we have on transposing

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin \theta = 0 \qquad (2)$$

or substituting for θ in (2),

$$\frac{d^2x}{dt^2} + g \sin \frac{x}{l} = 0 \qquad (3)$$

As the oscillations are small, we may write $\frac{x}{l}$ for $\sin \frac{x}{l}$, thus obtaining

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0 \qquad (4)$$

Comparing this with equation (1) of the last example, we see that the motion is simple harmonic, and from (4) of the same example we see that the solution of this equation is

$$x = a \cos \left(\sqrt{\frac{g}{l}} t + \alpha \right) \quad . \quad . \quad . \quad . \quad (5)$$

where a and α are constants.

The periodic time is therefore $2\pi\sqrt{\frac{l}{g}}$

EXAMPLE 5

Compound Pendulum. A straight rod AB can rotate about a horizontal axis at A (Fig. 131). G is its centre of gravity and $AG = h$ ft. The radius of gyration of the rod about A is k ft. Find the periodic time for small oscillations about the equilibrium position (a) by means of the equation of motion and (b) by means of the energy equation.

If the mass of the rod is of M lb weight, and a mass of m lb weight is attached to the rod at a point C between A and B such that $AC = a$, find the new period of oscillation.

In the figure let AD be vertical and let angle $DAB = \theta$ radians.

(a) Suppose the rod to be oscillating about its mean position AD, and to be passing through the position AB in the sense of θ increasing at time t seconds. The forces acting on AB are (i) the reaction of the axis at A, and (ii) the weight M lb acting vertically downwards at G. The moment of the former about A is zero, and that of the latter is $- Mh \sin \theta$, the minus sign indicating that the sense is opposite to that of θ increasing. The equation of motion is therefore

Couple = moment of inertia × angular acceleration

i.e.
$$- Mh \sin \theta = \frac{Mk^2}{g} \frac{d^2\theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2} \sin \theta = 0 \quad . \quad . \quad . \quad . \quad (1)$$

For small oscillations we write θ for $\sin \theta$, and obtain

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2} \theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Comparing this with equation (4) of the last example, we see from (5) of that example that the solution is

$$\theta = \theta_1 \cos \left(\sqrt{\frac{gh}{k^2}} \, . \, t + \alpha \right) \quad . \quad . \quad . \quad . \quad (3)$$

θ_1 and α being constants. The time of an oscillation is therefore $t = 2\pi\sqrt{\frac{k^2}{gh}}$ which is the same as that of a simple pendulum of length $\frac{k^2}{h}$; $l = \frac{k^2}{h}$ is known as the *length of the equivalent simple pendulum.*

(b) Suppose that when the pendulum is in its extreme position, the angle between DA and AB is β radians.

The potential energy of AB in the position shown is $- Mh \cos \theta$ ft-lb.

The kinetic energy of AB in the position shown is $\frac{Mk^2}{2g}\left(\frac{d\theta}{dt}\right)^2$ ft-lb.

The total energy in this position is therefore

$$M\left\{\frac{k^2}{2g}\left(\frac{d\theta}{dt}\right)^2 - h\cos\theta\right\} \text{ ft-lb}$$

In the extreme position, the rod, being momentarily at rest, has no kinetic energy and its potential energy is $-Mh\cos\beta$ ft-lb. By the law of the conservation of energy, if we neglect friction, the total energy of AB is the same in all positions. Hence,

$$M\left\{\frac{k^2}{2g}\left(\frac{d\theta}{dt}\right)^2 - h\cos\theta\right\} = -Mh\cos\beta$$

which is known as *the equation of energy*. Dividing through by M, and differentiating with respect to t, we have

$$\frac{k^2}{g}\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + h\sin\theta\frac{d\theta}{dt} = 0$$

and dividing through by $\frac{k^2}{g}\frac{d\theta}{dt}$,

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2}\sin\theta = 0$$

which is the same as (1) above.

When the mass m is attached, the couple acting becomes

$$-Mh\sin\theta - ma\sin\theta = -(Mh + ma)\sin\theta$$

and the moment of inertia becomes

$$\frac{M}{g}k^2 + \frac{m}{g}a^2 = \frac{Mk^2 + ma^2}{g}$$

The equation of motion is therefore

$$-(Mh + ma)\sin\theta = \frac{Mk^2 + ma^2}{g}\frac{d^2\theta}{dt^2}$$

or $$\frac{d^2\theta}{dt^2} + \frac{g(Mh + ma)}{Mk^2 + ma^2}\sin\theta = 0$$

The length l of the equivalent simple pendulum is therefore

$$l = \frac{Mk^2 + ma^2}{Mh + ma}$$

and the periodic time is $2\pi\sqrt{\frac{l}{g}}$

Systems like those in Exs. 4 and 5 above to which the law of the conservation of energy applies are known as *conservative systems* and are free from frictional resistances or other energy dissipating influences. If there are elastic forces in the system and strains occur during the motion, the elastic energy stored up in the system must

be included in the total energy. If T, V and E_e are, respectively, the kinetic, potential, and elastic energy, the equation of energy is

$$T + V + E_e = C, \text{ where } C \text{ is a constant.}$$

In the example of Art. 33, $T = \dfrac{W}{2g}\left(\dfrac{dx}{dt}\right)^2$ ft-lb, $V = -Wx$ ft-lb, assuming the potential energy in the equilibrium position to be zero, and $E_e = \frac{1}{2}$ tension $\times$ stretch $= \frac{1}{2}(W + Ex)\left(\dfrac{W}{E} + x\right) = \dfrac{W^2}{2E} + Wx + \frac{1}{2}Ex^2$ ft-lb.

The energy equation is

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 - Wx + \frac{W^2}{2E} + Wx + \frac{1}{2}Ex^2 = C$$

i.e.

$$\left(\frac{dx}{dt}\right)^2 + \frac{Wg}{E} + \frac{gE}{W}x^2 = \frac{2gC}{W}$$

whence differentiating with respect to t and simplifying,

$$\frac{d^2x}{dt^2} + \frac{gE}{W}x = 0$$

is the equation of motion, the same as in the example. We could have ignored the potential energy in this example because the initial tension in the spring, i.e. that when $x = 0$, balances the pull of gravity on the weight and so the terms Wx and $-Wx$ cancel.

If we know that the motion is simple harmonic, we can find the frequency of vibration without forming the equation of motion, by equating the elastic energy in an end position, in which there is no kinetic energy, to the kinetic energy in the neutral position, in which there is no potential energy. The equation of energy in this case becomes

Maximum kinetic energy = Maximum potential energy.

In the above case, if the amplitude of vibration is a ft, and the angular frequency of vibration is ω radians per second, i.e. the time of a complete oscillation is $\dfrac{2\pi}{\omega}$ seconds, the maximum strain energy is $\frac{1}{2}Ea^2$ and the maximum kinetic energy is $\dfrac{W}{2g}\,\omega^2a^2$ and

$$\frac{W}{2g}\,\omega^2a^2 = \tfrac{1}{2}Ea^2$$

i.e.

$$\omega^2 = \frac{Eg}{W}$$

so that if t seconds is the period of oscillation,

$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{gE}}$$

As a more general illustration consider the case of a load W lb attached to some point in a light flexible structure and producing a deflection Δ ft at that point If the load is set oscillating in a vertical line with amplitude a ft, the load E lb to produce a deflection of 1 ft is $E = \frac{W}{\Delta}$ The energy equation is

Maximum kinetic energy = Maximum potential energy

$$\frac{W}{2g}\omega^2 a^2 = \frac{W}{2\Delta}a^2$$

and $$\omega^2 = \frac{g}{\Delta}$$

i e $$\omega = \sqrt{\frac{g}{\text{static deflection in ft}}} \qquad (g = 32\ 2)$$

or $$\omega = \sqrt{\frac{12g}{\text{static deflection in in}}}$$

EXAMPLE 6

Euler's Theory of Struts ACB (Fig 132) is a long uniform flexible rod or strut under the action of two equal and opposite thrusts P, P at its ends The straight line AB along which the thrusts act is the unstrained position of the rod, and the ends A and B of the rod are unconstrained in direction Find the relation between P and the deflection of C the mid-point of AB from its original position

Let $x = \overline{AM}$ and $y = \overline{XM}$ be the co-ordinates of any point X on the rod Let $l = \overline{AB}$, and let E and I be Young's Modulus and the moment of inertia of the cross-section of the rod about its neutral axis The bending moment at X is $-Py$ and by the theory of bending,

$$EI\frac{d^2y}{dx^2} = -Py \qquad (1)$$

or $$\frac{d^2y}{dx^2} = -\frac{P}{EI}y = -m^2y \text{ where } m = \sqrt{\frac{P}{EI}}$$

This equation corresponds to that in Ex 3 above Hence, by comparison, the solution is

$$y = A\cos mx + B\sin mx \qquad (2)$$

where A and B are arbitrary constants

Since $y = 0$ when $x = 0$, $0 = A$, and (2) reduces to

$$y = B\sin mx \qquad (3)$$

Again $y = 0$ when $x = l$ hence

$$B \sin ml = 0 \tag{4}$$

from which $\qquad B = 0$ or $\sin ml = 0$

As we are finding the value of P for which the strut will bend $B = 0$ is not in agreement with the conditions and consequently $\sin ml = 0$ is the required solution of (4) From $\sin ml = 0$ we have the infinite number of solutions $ml = \pi$, $ml = 2\pi$, $ml = 3\pi$, $ml = n\pi$ etc Since $m = \sqrt{\frac{P}{EI}}$ these give the solutions

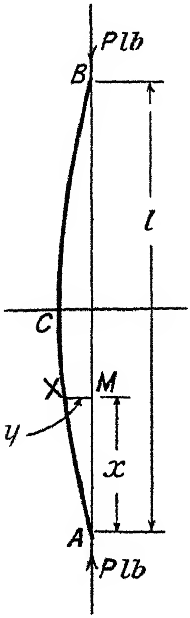


FIG 132

$$P = \frac{\pi^2 EI}{l^2},\ P = \frac{4\pi^2 EI}{l^2},\ P = \frac{9\pi^2 EI}{l^2}, \text{ etc} \tag{5}$$

The least value of these would cause the rod represented in the figure to deflect as shown (3) shows that the curve ACB is a sine curve The other values of P in (5) correspond to the cases in which the rod is so constrained that the curve into which its centre line is deflected forms respectively (1) a complete wavelength, (2) one and a half wavelengths, (3) two wavelengths, etc of a sine curve When P has any of the values in (5), (4) is satisfied for all values of B and we see from (3) that the maximum value of the deflection may have any value Thus, when $P = \frac{\pi^2 EI}{l^2}$, the rod will be in equilibrium in any position, and if whilst the rod is deflected, P is increased, the rod will collapse by buckling For values of P less than $\frac{\pi^2 EI}{l^2}$ the rod will remain in its unstrained position

It is clear that though we have apparently an infinite number of buckling loads for the strut, only one or two are possible in practice, as the value $\frac{P}{\text{Area of cross-section}}$ cannot exceed the safe working stress for the material

139 **Equations of the Type $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ in which a and b are Constants.** Consider the equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{XII 7}$$

and substitute $y = Ae^{\lambda x}$ From this

$$\frac{dy}{dx} = \lambda y \text{ and } \frac{d^2y}{dx^2} = \lambda^2 y$$

Substituting these values in (XII 7),

$$y(k^2 + ak + b) = 0$$

If, therefore, we give to k a value which satisfies the equation

$$k^2 + ak + b = 0 \qquad \text{(XII 8)}$$

$y = Ae^{kx}$ will satisfy (XII 7) Let k_1 and k_2 be the roots of the quadratic equation (XII 8), then $y = Ae^{k_1x}$ and $y = Be^{k_2x}$ will both satisfy the equation (XII 7) We have written B as the arbitrary constant in the second of these, as the constants may have any values whatever and need not be equal It is clear, further, that, since $y = Ae^{k_1x}$ and $y = Be^{k_2x}$ satisfy (XII 7) separately, their sum

$$y = Ae^{k_1x} + Be^{k_2x} \qquad \text{(XII 9)}$$

will also satisfy the equation As the latter contains two arbitrary constants it is, therefore, the full solution of (XII 7) (XII 8) is known as the *auxiliary equation* There are three cases to consider, (1) that in which k_1 and k_2 are real and unequal, (2) that in which k_1 and k_2 are real and equal, (3) that in which k_1 and k_2 are imaginary or complex quantities

CASE 1 AUXILIARY EQUATION WITH REAL AND UNEQUAL ROOTS If $a^2 > 4b$ in (XII 7), k_1 and k_2 are real quantities and (XII 9) is the final form of the solution, in which

$$k_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b})$$

and

$$k_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b})$$

If $b = 0$, k_1 is zero, and the solution becomes

$$y = A + Be^{-ax} \qquad \text{(XII 10)}$$

If $a = 0$ and $b \neq 0$, then b must be negative, and the roots of (XII 8) are $k_1 = \sqrt{-b}$ and $k_2 = -\sqrt{-b}$, which are both real

The solution is then

$$y = Ae^{\sqrt{-b}x} + Be^{-\sqrt{-b}x}$$

Writing c^2 for $-b$ and using the relations given in Art 11, we can express the above solution in the form

$$y = C\cosh cx + D\sinh cx \qquad \text{(XII 11)}$$

EXAMPLE 1

Solve the differential equations—

(i) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = 0$ (ii) $\dfrac{d^2y}{dx^2} = n^2y$, where n is a constant

(i) From (XII.8), the auxiliary equation is $k^2 - 5k + 3 = 0$, the roots of which are

$$k = \tfrac{1}{2}(5 \pm \sqrt{13})$$

i.e. $$k_1 = 4{\cdot}303 \text{ and } k_2 = 0{\cdot}697$$

The complete solution of the differential equation is therefore

$$y = Ae^{4\cdot303x} + Be^{0\cdot697x}$$

where A and B are arbitrary constants.

(ii) Here the auxiliary equation is $k^2 = n^2$, whence $k_1 = n$ and $k_2 = -n$.

The complete solution is therefore

$$y = Ae^{nx} + Be^{-nx}$$

CASE 2. AUXILIARY EQUATION WITH REAL AND EQUAL ROOTS. If $a^2 = 4b$, k_1 and k_2 are each equal to $-\tfrac{1}{2}a$, and (XII.9) reduces to

$$y = Ae^{k_1x} = Be^{k_1x}, \text{ or if } C = A + B,\ y = Ce^{k_1x}$$

which is not the full solution, as it contains only one arbitrary constant C.

In order to find the full solution, we adopt the following artifice. Assume that the roots are k_1 and $k_1 + h$, where h is a small quantity which will ultimately be made to approach the limit zero.

Then $$y = Ae^{k_1x} + Be^{(k_1+h)x}$$

$\therefore$ $$y = e^{k_1x}(A + Be^{hx})$$

On expanding e^{hx}, we obtain

$$y = e^{k_1x}\left[A + B\left(1 + hx + \frac{h^2x^2}{\underline{|2}} + \frac{h^3x^3}{\underline{|3}} + \ldots\right)\right]$$

i.e. $$y = e^{k_1x}\left[C + Bhx + Bh\left(\frac{hx^2}{\underline{|2}} + \frac{h^2x^3}{\underline{|3}} + \frac{h^3x^4}{\underline{|4}} + \ldots\right)\right]$$

where $C = A + B$.

The infinite series in the inner bracket converges for finite values of x, and is zero when $h = 0$. By letting the value of B approach infinity as the value of h approaches zero, we may make the value of Bh remain finite. Let D be this finite value. Then we have as the full solution of (XII.7)

$$y = (C + Dx)e^{-\frac{1}{2}ax} \quad . \quad . \quad . \quad \text{(XII.12)}$$

EXAMPLE 2

Solve the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

The auxiliary equation is $k^2 - 6k + 9 = 0$

i.e. $(k - 3)^2 = 0$

whence $k = 3$ (equal roots)

From (XII.12), the full solution is

$$y = (C + Dx)e^{3x}$$

CASE 3. AUXILIARY EQUATION WITH COMPLEX ROOTS. If $a^2 < 4b$, the roots of (XII.8) are complex and unequal. In this case $k_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$ and $k_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$. Writing c^2 for $4b - a^2$ and substituting in (XII.9), we have

$$y = Ae^{\frac{1}{2}(-a+ci)x} + Be^{\frac{1}{2}(-a-ci)x}, \text{ where } i = \sqrt{-1}$$

From this, $y = e^{-\frac{1}{2}ax}(Ae^{\frac{1}{2}icx} + Be^{-\frac{1}{2}icx})$, or making use of the relations (I.45) and (I.46),

$$y = e^{-\frac{1}{2}ax}\left(A \cos \frac{cx}{2} + Ai \sin \frac{cx}{2} + B \cos \frac{cx}{2} - Bi \sin \frac{cx}{2}\right)$$

Putting E for $A + B$ and F for $(A - B)i$, we have

$$y = e^{-\frac{1}{2}ax}\left(E \cos \frac{cx}{2} + F \sin \frac{cx}{2}\right) \quad . \quad \text{(XII.13)}$$

Alternative forms of this are

$$y = Re^{-\frac{1}{2}ax} \sin \left(\frac{cx}{2} + \alpha\right) \quad . \quad . \quad \text{(XII.14)}$$

and

$$y = Re^{-\frac{1}{2}ax} \cos \left(\frac{cx}{2} + \beta\right) \quad . \quad . \quad \text{(XII.15)}$$

These three relations represent damped oscillations when x represents time. If $a = 0$, the exponential factor becomes unity and disappears from the relations, which then represent undamped oscillations, or simple harmonic motion.

Another method of solution of this latter case is given in Example 3, Art. 138.

EXAMPLE 3

Auxiliary Equation with Complex Roots. Damped Oscillations. Solve the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + (n^2 + k^2)x = 0$, k being positive, and show that the successive maxima of x form a series in geometrical progression.

If the oscillations of a pendulum are determined by the above equation, the time of a complete oscillation being 1 second, and if the first and fifth displacements on the same side of the equilibrium position are in the ratio of 4 to 1, show that the time taken in swinging out from an equilibrium position to an extreme position is about 0·241 second. (U.L.)

In solving the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + (n^2 + k^2)x = 0$$

we have, as auxiliary equation

$$m^2 + 2km + n^2 + k^2 = 0$$

Solving for m,

$$m^2 + 2km + k^2 = -n^2$$

i.e. $$(m + k)^2 = -n^2i^2, \text{ where } i = \sqrt{-1}$$

i.e. $$m + k = \pm ni$$

i.e. $$m = -k \pm ni$$

The full solution is therefore, by (XII.14),

$$x = Re^{-kt}\sin(nt - \varepsilon) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R and ε are arbitrary constants.

Let us reckon from an instant when $x = 0$, i.e. $t = 0$ when $x = 0$. Then from (5), $0 = R\sin\varepsilon$, from which $\sin\varepsilon = 0$ and, therefore, $\varepsilon = 0$. Substituting in (5),

$$x = Re^{-kt}\sin nt \quad . \quad . \quad . \quad . \quad . \quad (2)$$

represents the motion.

The graph of this function is of the type shown in Fig. 74, y being the dependent variable there instead of x as in the present case.

To find the values of t for which x is a maximum, we put

$$\frac{dx}{dt} = 0$$

i.e. $$\frac{d}{dt}(Re^{-kt}\sin nt) = 0$$

or $$-k\sin nt + n\cos nt = 0$$

from which we find that $$\tan nt = \frac{n}{k}$$

If α is the least positive value of nt satisfying this, the roots of the equation in nt are

$$nt = \alpha, \pi + \alpha, 2\pi + \alpha, 3\pi + \alpha, \text{ etc.}$$

from which $$t = \frac{\alpha}{n}, \frac{\alpha + \pi}{n}, \frac{\alpha + 2\pi}{n}, \frac{\alpha + 3\pi}{n}, \text{ etc.} \quad . \quad . \quad . \quad (3)$$

Substituting these values in turn in (6), we obtain for x the values

$$x = Re^{-k\frac{\alpha}{n}}\sin\alpha,\ Re^{-k\left(\frac{\pi + \alpha}{n}\right)}\sin(\alpha + \pi),\ Re^{-k\left(\frac{2\pi + \alpha}{n}\right)}\sin(\alpha + 2\pi)$$

$$Re^{-k\frac{3\pi + \alpha}{n}}\sin(\alpha + 3\pi), \text{ etc.}$$

These are alternately maximum and minimum values, and the ratio of each maximum to the succeeding one is found by dividing the first value by the third, or the second by the fourth, which of these it is depending upon whether α is less than or greater than $\frac{\pi}{2}$. These ratios have both the same value, i.e. $e^{\frac{2\pi k}{n}}$ and, therefore, successive maxima of x form a series in geometrical progression.

We have seen that $x = \sin(nt + \varepsilon)$ represents simple harmonic motion, and that $x = Re^{-kt}$ is the decay function. By comparing (1) above with (5) of Ex. 3, Art. 138, we see that the motion is oscillatory and that the amplitude is continually decreasing. The time of an oscillation is the periodic time of $\sin(nt + \varepsilon)$, i.e. $\frac{2\pi}{n}$, and since this is 1 second, $n = 2\pi$. The ratio of two successive maximum displacements is therefore e^k, and that of the first displacement to the fifth on the same side of the equilibrium position is $(e^k)^4 = e^{4k}$. As this ratio is 4, we have $e^{4k} = 4$, from which

$$4 \times 0{\cdot}4343k = 0{\cdot}6021$$

or

$$k = 0{\cdot}3466$$

To find the time of the first outward swing in any oscillation we note that when the pendulum is in an equilibrium position $x = 0$, and therefore from (2), $nt = m\pi$, where m is 0 or a positive integer, and $t = \frac{m\pi}{n}$. At the end of the next outward swing the time is $\frac{m\pi + \alpha}{n}$, as seen from (3). Hence, the time T of the outward swing is given by $T = \frac{m\pi + \alpha}{n} - \frac{m\pi}{n} = \frac{\alpha}{n}$. Since $n = 2\pi$ and $k = 0{\cdot}3466$, we have

$$\tan\alpha = \frac{2\pi}{0{\cdot}3466} = 18{\cdot}13$$

and from tables,

$$\alpha = 1{\cdot}516 \text{ radians}$$

Hence,

$$T = \frac{1{\cdot}516}{2\pi} = 0{\cdot}241 \text{ second nearly.}$$

EXAMPLE 4

Solve the equation $\frac{d^2x}{dt^2} + n^2x = 0$ (see Ex. 3, Art. 138).

The auxiliary equation is $k^2 = -n^2$, whence $k = \pm ni$.

The full solution is then

$$x = E\cos nt + F\sin nt$$

which corresponds to (6) of the example quoted.

[In (XII.13) put $a = 0$, $c = 2n$, and $x = t$.]

The reader should notice that the equation $\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0$ represents oscillatory motion only if $a^2 < 4b$, and that the damping or dying away of the oscillations is caused by the presence of the term $a\frac{dy}{dt}$. As a decreases, the damping decreases, and if $a = 0$, we have simple harmonic motion, as in Ex. 3, Art. 138. If b is negative, as in Ex. 1 (ii) above, a^2 cannot be less than $4b$, and the motion is not oscillatory.

EXAMPLE 5

Solve $\dfrac{d^2y}{dx^2} = k^2y$ where k is a constant.

Substituting $y = Ae^{mx}$ and dividing through by Ae^{mx}, we have

$$m^2 = k^2$$

or
$$m = \pm k$$

The complete solution is therefore

$$y = Ae^{kx} + Be^{-kx}$$

140. Linear Equations of Higher Order than the Second. In order to solve a linear equation of the nth order with constant coefficients such as

$$\frac{d^ny}{dx^n} + a\frac{d^{n-1}y}{dx^{n-1}} + b\frac{d^{n-2}y}{dx^{n-2}} + \ldots + l\frac{dy}{dx} + my = 0 \quad \text{(XII.16)}$$

we substitute $y = Ae^{kx}$. Making this substitution and dividing through by Ae^{kx} we obtain the auxiliary equation (of the nth degree in k)

$$k^n + ak^{n-1} + bk^{n-2} + \ldots + lk + m = 0 \quad . \quad \text{(XII.17)}$$

Let the n roots of this be $k_1, k_2, k_3, \ldots, k_n$. Then the complete solution of (XII.16) is

$$y = Ae^{k_1x} + Be^{k_2x} + Ce^{k_3x} + \ldots + Me^{k_nx} \quad . \quad \text{(XII.18)}$$

where $A, B, C, \ldots, M$ are arbitrary constants. For the general method of solving (XII.16) when two or more roots of (XII.17) are equal, or when some of the roots are complex, readers are referred to treatises on differential equations. We shall solve two examples.

EXAMPLE 1

Solve
$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

Substituting $y = Ae^{kx}$ we obtain the auxiliary equation

$$k^3 - 2k^2 - k + 2 = 0$$

or
$$k^2(k - 2) - (k - 2) = 0$$

i.e.
$$(k - 2)(k^2 - 1) = 0$$

the roots of which are $k = -1$, $k = 1$, and $k = 2$. The complete solution of the differential equation is therefore

$$y = Ae^{-x} + Be^{x} + Ce^{2x}$$

EXAMPLE 2

Whirling of Shafts. In finding the whirling speeds of unloaded shafts it is necessary to solve the equation $\frac{d^4y}{dx^4} = m^4y$ where m is a known constant. y is the deflection from its statical position of a point on the shaft distant x from some point in the shaft. Solve this equation.

Substituting $y = Ae^{kx}$ in $\frac{d^4y}{dx^4} = m^4y$ we obtain the auxiliary equation $k^4 = m^4$, the roots of which are $k = \pm m$ and $k = \pm im$ where $i = \sqrt{-1}$. The complete solution of the differential equation is therefore

$$y = Ae^{mx} + Be^{-mx} + Ce^{mxi} + De^{-mxi} \quad . \quad . \quad . \quad (1)$$

As in Case 3, Art. 139, the sum of the last two terms on the right can be written in the form $E \cos mx + F \sin mx$ and (1) becomes

$$y = Ae^{mx} + Be^{-mx} + E \cos mx + F \sin mx \quad . \quad . \quad . \quad (2)$$

The arbitrary constants A, B, E, and F are determined in any particular case by the nature of the constraints on the shaft in a similar manner to that in which the values of the constants were determined in Ex. 6, Art. 138.

141. Use of Operators. If we write Dy for $\frac{dy}{dx}$, D^2y for $\frac{d^2y}{dx^2}$ and, generally, D^ny for $\frac{d^ny}{dx^n}$ in (XII.16) we obtain the relation

$$D^ny + aD^{n-1}y + bD^{n-2}y + \ldots + my = 0 \quad . \quad (XII.19)$$

or

$$(D^n + aD^{n-1} + bD^{n-2} + \ldots + m)y = 0 \quad . \quad (XII.20)$$

(XII.20) is merely a short way of writing (XII.19), a single y being written after the bracket to save writing y in each term. If $f(D)$ represents the quantity in the brackets in (XII.20) this relation may be written

$$f(D)y = 0 \quad . \quad . \quad . \quad (XII.21)$$

We saw in Art. 27 that

$$D(u + v) = Du + Dv$$

and we have seen also that in an expression such as that on the left of (XII.19) the order of operation is immaterial, i.e.

$$\left.\begin{aligned} (D + m)y &= (m + D)y \\ (D^2 + lD + m)y &= (D^2 + m + lD)y \\ &= (lD + D^2 + m)y = \text{etc.} \\ \text{Again,} \quad D(au) &= aDu \text{ if } a \text{ is a constant} \\ \text{and} \quad D^n \,.\, D^m u &= D^{n+m}u \end{aligned}\right\} \quad (XII.22)$$

We can therefore manipulate the symbol D in combination with constants and with positive integral powers of itself by the methods of ordinary algebra. Now let us write Iy for $\int y\,dx$, from which the constant of integration is omitted. Then, since $\frac{d}{dx}(\int y\,dx) = y$, we have $D \,.\, Iy = y$. Similarly,

$$\int \frac{d}{dx}(y)\,dx = y \text{ or } I \,.\, Dy = y$$

From these we see that

$$I = \frac{1}{D} = D^{-1} \text{ and } D = \frac{1}{I} = I^{-1}$$

Similarly, $$I^2 = D^{-2} \text{ and } I^n = D^{-n}$$

Some Important Relations

(1) To find a meaning for $D^n e^{\alpha x}$ and for $f(D)e^{\alpha x}$. It is easy to show that, if n is a positive integer, $D^n e^{\alpha x} = e^{\alpha x} \,.\, \alpha^n$ and that $D^{-n}e^{\alpha x} = I^n e^{\alpha x} = e^{\alpha x} \,.\, \frac{1}{\alpha^n} = \alpha^{-n}e^{\alpha x}$. Consequently, for all integral values of n,

$$D^n e^{\alpha x} = \alpha^n e^{\alpha x} \qquad \text{(XII.23)}$$

If $f(D) = a_1 D^n + a_2 D^{n-1} + a_3 D^{n-2} + \ldots + a_{n+1}$ it follows that

$$f(D)e^{\alpha x} = f(\alpha)e^{\alpha x} \qquad \text{(XII.24)}$$

(2) To find a meaning for $\frac{1}{D-a}e^{\alpha x}$ and for $\frac{1}{f(D)}e^{\alpha x}$. We have $(D-a) \,.\, \frac{1}{D-a}e^{\alpha x} = e^{\alpha x}$. Now write y for $\frac{1}{D-a}e^{\alpha x}$ and we have $(D-a)y = e^{\alpha x}$, an equation which we solved in Ex. 3, Art. 134. Putting $c = 0$ in that solution,

$$y = \frac{1}{\alpha - a}e^{\alpha x}$$

Thus, $$\frac{1}{D-a}e^{\alpha x} = \frac{1}{\alpha - a}e^{\alpha x}$$

Similarly, $$\frac{1}{(D-a_2)(D-a_1)}e^{\alpha x} = \frac{1}{D-a_2} \cdot \frac{e^{\alpha x}}{\alpha - a_1}$$

$$= \frac{1}{\alpha - a_1} \cdot \frac{1}{D - a_2}e^{\alpha x} = \frac{1}{(\alpha - a_1)(\alpha - a_2)}e^{\alpha x}$$

and if $f(D)$ is a function of D which can be written in the form $(D - a_1)(D - a_2)(D - a_3) \ldots (D - a_n)$

$$\frac{1}{f(D)} e^{\alpha x} = \frac{e^{\alpha x}}{f(\alpha)} \quad . \quad . \qquad \text{(XII.25)}$$

(3) To find meanings for $f(D^2)\sin(ax + b)$ and $f(D^2)\cos(ax + b)$

$D^2 \sin(ax + b), = - a^2 \sin(ax + b)$ and

$D^2 \cos(ax + b) = - a^2 \cos(ax + b)$. Hence,

$$\left.\begin{aligned} f(D^2)\sin(ax + b) &= \sin(ax + b) f(-a^2) \\ \text{and } f(D^2)\cos(ax + b) &= \cos(ax + b) f(-a^2) \quad . \end{aligned}\right\} \qquad \text{(XII.26)}$$

(4) To find a meaning for $\frac{1}{f(D^2)}\sin(ax + b)$ and $\frac{1}{f(D^2)}\cos(ax + b)$. First consider the meaning of $\frac{1}{D^2 - \alpha^2}\sin(ax + b)$.

Putting y for this expression, we have

$$(D^2 - \alpha^2)y = \sin(ax + b) \quad . \quad . \qquad \text{(XII.27)}$$

Assuming that $y = A\sin(ax + b)$, (XII.27) reduces to

$A(- a^2 - \alpha^2)\sin(ax + b) = \sin(ax + b)$ from which

$$A = \frac{1}{- a^2 - \alpha^2}$$

Thus, $$\frac{1}{D^2 - \alpha^2}\sin(ax + b) = \sin(ax + b) \cdot \frac{1}{(-a^2 - \alpha^2)}$$

Similarly,

$$\frac{1}{(D^2 - \beta^2)(D^2 - \alpha^2)}\sin(ax + b)$$

$$= \sin(ax + b) \cdot \frac{1}{(-a^2 - \alpha^2)(- a^2 - \beta^2)}$$

By continuing in this way we see that if we have $f(D^2)$, a function of D^2, which can be written in the form

$$(D^2 - \alpha^2)(D^2 - \beta^2)(D^2 - \gamma^2)(D^2 - \delta^2) \ldots$$

then $$\frac{1}{f(D^2)}\sin(ax + b) = \frac{\sin(ax + b)}{f(- a^2)} \quad . \qquad \text{(XII.28)}$$

and similarly,

$$\frac{1}{f(D^2)}\cos(ax + b) = \frac{\cos(ax + b)}{f(- a^2)} \qquad \text{(XII.29)}$$

142. Linear Equations. Complementary Function and Particular Integral. Consider now the equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X \qquad . \qquad . \qquad (XII.30)$$

in which P and Q are constants and X is a function of x. We have solved the equation in the case where $X = 0$, that is, the equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \qquad . \qquad . \qquad (XII.31)$$

Let $y = u$ be the complete solution of (XII.31). Then, if $y = v$ is any particular integral of (XII.30), the complete solution of the latter is

$$y = u + v \ . \qquad . \qquad . \qquad . \qquad (XII.32)$$

To prove this we substitute $u + v$ for y in (XII.30), obtaining

$$\left(\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu\right) + \left(\frac{d^2v}{dx^2} + a\frac{dv}{dx} + bv\right) = X$$

This is obviously true, for the quantity in the first bracket is zero, as $y = u$ satisfies (XII.31) and the quantity in the second bracket is equal to X because $y = v$ satisfies (XII.30). Hence, since u contains two arbitrary constants, $y = u + v$ is the complete solution of (XII.30). Thus the solution (XII.32) is the sum of two parts u and v, of which u is known as the *complementary function* and v as the *particular integral*. The method of solving (XII.30) is to find (1) the general solution of the equation obtained by putting $X = 0$, and (2) any particular solution of the equation. The sum of these two parts is the complete solution required. In finding the particular integral we can guess the same if possible, but we shall usually make use of operators, using the rules proved in Art. 141.

EXAMPLE 1

Solve the differential equations

$$\text{(i)}\ \frac{dy}{dx} + 4y = 2e^{3x}$$

$$\text{(ii)}\ \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = e^{6x}$$

Equation (i) can be solved by the method of Art. 134. Here it will be solved by operators.

Let $y = u + v$, where v is a particular solution of

$$\frac{dv}{dx} + 4v = 2e^{3x} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and u is the complete solution of

$$\frac{du}{dx} + 4u = 0 \qquad (2)$$

Determined by the method of Art. 131,

$$u = Ae^{-4x}$$

which is the complementary function

The particular integral v is found from

$$(D + 4)v = 2e^{3x}$$

$$\therefore \qquad v = \frac{1}{D+4} 2e^{3x}$$

$$= \frac{1}{3+4} 2e^{3x}$$

i.e. $$v = \frac{2}{7} e^{3x}$$

Hence, the complete solution of the equation is

$$y = Ae^{-4x} + \frac{2}{7}e^{3x}$$

In (ii) the complementary function u is the complete solution of

$$\frac{d^2u}{dx^2} - 8\frac{du}{dx} + 16u = 0$$

Determined by the method of Art. 139,

$$u = e^{4x}(Ax + B)$$

The particular integral v is found from

$$(D^2 - 8D + 16)v = e^{-6x}$$

$$\therefore \qquad v = \frac{1}{(D-4)^2} e^{-6x}$$

$$= \frac{1}{(-6-4)^2} e^{-6x}$$

i.e. $$v = 0{\cdot}01e^{-6x}$$

Hence, the complete solution of the equation (ii) is

$$y = e^{4x}(Ax + B) + 0{\cdot}01e^{-6x}$$

EXAMPLE 2

Forced Oscillations, Undamped. Solve the differential equation

$$\frac{d^2x}{dt^2} + \mu x = a \sin pt \qquad . \qquad . \quad (1)$$

Let $x = u + v$ where v is a particular solution of

$$\frac{d^2v}{dt^2} + \mu v = a \sin pt \qquad . \quad . \quad . \quad . \quad (2)$$

and u is the complete solution of

$$\frac{d^2u}{dt^2} + \mu u = 0 \qquad . \quad . \quad (3)$$

We found u in Ex. 3, Art. 138, where we saw that

$$u = A\cos(\sqrt{\mu}\,t + \alpha) \quad . \quad . \quad . \quad . \quad (4)$$

This is the complementary function.

To find a particular solution of (2), we have

$$(D^2 + \mu)v = a\sin pt$$

$$v = \frac{1}{D^2 + \mu}a\sin pt$$

$$= \frac{a\sin pt}{\mu - p^2}$$

Hence, the complete solution of $\dfrac{d^2x}{dt^2} + \mu x = a\sin pt$

is
$$x = A\cos(\sqrt{\mu}\,t + \alpha) + \frac{a\sin pt}{\mu - p^2} \quad . \quad . \quad . \quad (5)$$

The relation (5) represents the oscillations of a body which would execute simple harmonic motion if left to itself, but which is subjected to an impressed acceleration represented by $X = a\sin pt$.

EXAMPLE 3

Forced Oscillations, Damped. Solve the differential equation

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 50y = \sin\omega t$$

A spring is loaded with 32 lb weight, its point of support has a motion given by $\sin\omega t$ (ft), the resistance to the motion of the load is measured by ten times the speed in feet per second. If the spring extends $\frac{1}{50}$ ft per lb load and its mass may be neglected, give, when the oscillations have become steady and $\omega^2 = 50$, the amplitude of the oscillations. (U.L.)

We have
$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 50y = \sin\omega t \quad . \quad . \quad . \quad . \quad (1)$$

Let $y = u + v$ be the complete solution, u being the complementary function and v the particular integral of (1). To determine v we have from (1)

$$(D^2 + 10D + 50)v = \sin\omega t$$

i.e.
$$v = \frac{1}{D^2 + 10D + 50}\sin\omega t$$

$$= \frac{1}{-\omega^2 + 10D + 50}\sin\omega t, \text{ by (XII.28)}$$

$$= \frac{10D - (50 - \omega^2)}{100D^2 - (50 - \omega^2)^2}\sin\omega t$$

$$= -\frac{10D - (50 - \omega^2)}{2\,500 + \omega^4}\sin\omega t$$

$$= \frac{(50 - \omega^2 - 10D)}{2\,500 + \omega^4}\sin\omega t$$

$$\therefore \quad v = \frac{(50 - \omega^2)\sin\omega t - 10\omega\cos\omega t}{2\,500 + \omega^4} \quad . \quad . \quad . \quad . \quad (2)$$

The complementary function u is the complete solution of

$$\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 50u = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Putting $u = Ae^{kt}$ we obtain the auxiliary equation

$$k^2 + 10k + 50 = 0$$

the roots of which are $k = -5 + 5i$ and $-5 - 5i$.

The complete solution of (3) is therefore

$$u = Ae^{(-5+5i)t} + Be^{(-5-5i)t}$$

$$= Re^{-5t}\cos(5t + \varepsilon) \text{ as in Ex. 3, Art. 139.}$$

and the solution of (1) is

$$y = Re^{-5t}\cos(5t + \varepsilon) + \frac{(50 - \omega^2)\sin\omega t - 10\omega\cos\omega t}{2\,500 + \omega^4} \quad . \quad (4)$$

To obtain the equation of motion of the weight, we have, if y ft is the displacement of the lower end of the spring from its position of rest,

Stretch of spring $= y - \sin\omega t$ ft

Tension in spring $= 50(y - \sin\omega t)$ lb

Frictional resistance $= 10\dfrac{dy}{dt}$ lb

Total force resisting motion $= 50(y - \sin\omega t) + 10\dfrac{dy}{dt}$ lb

But Force = mass × acceleration

$$\therefore \quad -\left\{50(y - \sin\omega t) + 10\frac{dy}{dt}\right\} = \frac{32}{32} \times \frac{d^2y}{dt^2}$$

Hence,

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 50y = 50\sin\omega t$$

and the solution of this is

$$y = Re^{-5t}\cos(5t + \varepsilon) + \frac{50}{2\,500 + \omega^4}[(50 - \omega^2)\sin\omega t - 10\omega\cos\omega t]$$

or, since $\omega^2 = 50$,

$$y = Re^{-5t}\cos(5t + \varepsilon) - \frac{5}{\omega}\cos\omega t \quad . \quad . \quad . \quad (5)$$

The first term on the right of (5) represents the free oscillations. These decay and soon become negligible, leaving

$$y = -\frac{5}{\omega}\cos\omega t$$

to represent the subsequent motion when the oscillations have become steady. Putting in the value $\omega = \sqrt{50}$ we have $y = -\dfrac{\sqrt{2}}{2}\cos 5\sqrt{2}t$, and the amplitude is therefore 0·707 ft.

EXAMPLE 4

Solve the equation $L\dfrac{di}{dt} + Ri = E_0 \cos \omega t$, using operators.

A similar equation was solved by another method in Ex. 6, Art. 135. Dividing through by L we have

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\cos \omega t \quad . \quad . \quad . \quad . \quad (1)$$

The solution is $i = u + v$ where $i = u$ is the complete solution of

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

and $i = v$ is a particular solution of (1). To find the complementary function u we have

$$\frac{du}{dt} + \frac{R}{L}u = 0 \quad . \quad . \quad . \quad (2)$$

and therefore

$$\frac{1}{u}\frac{du}{dt} = -\frac{R}{L}$$

the solution of which is

$$\log_e u = -\frac{R}{L}t + \log_e A$$

or

$$u = Ae^{-\frac{R}{L}t} \quad . \quad . \quad . \quad (3)$$

To determine the particular integral v we have

$$\frac{dv}{dt} + \frac{R}{L}v = \frac{E_0}{L}\cos \omega t$$

or

$$v = \frac{E_0}{L}\frac{1}{D + \frac{R}{L}}\cos \omega t$$

In order that we may be able to use the relation

$$\frac{1}{f(D^2)}\cos \omega t = \cos \omega t \cdot \frac{1}{f(-\omega^2)}$$

we must convert $\dfrac{1}{D + \frac{R}{L}}$ into a fraction whose denominator is a function of D^2.

We do this by multiplying numerator and denominator by $D - \dfrac{R}{L}$. Thus

$$v = \frac{E_0}{L}\,\frac{D - \frac{R}{L}}{D^2 - \left(\frac{R}{L}\right)^2}\cos \omega t$$

$$= \frac{E_0}{L\left(-\omega^2 - \frac{R^2}{L^2}\right)}\left(D - \frac{R}{L}\right)\cos \omega t$$

or since $\quad D = \dfrac{d}{dt}, \; v = -\dfrac{LE_0}{\omega^2L^2 + R^2}\left(-\omega \sin \omega t - \dfrac{R}{L}\cos \omega t\right)$

i.e. $$v = \frac{E_0}{\omega^2L^2 + R^2}(\omega L \sin \omega t + R \cos \omega t)$$

or $$v = \frac{E_0}{\sqrt{\omega^2L^2 + R^2}} \cos(\omega t - \alpha)$$

where $$\alpha = \tan^{-1}\frac{\omega L}{R}$$

The complete solution is then

$$i = \frac{E_0}{\sqrt{\omega^2L^2 + R^2}} \cos(\omega t - \alpha) + Ae^{-\frac{R}{L}t}$$

EXAMPLES XII

Solve the following differential equations and evaluate the constants of integration from the given data.

(1) $\dfrac{d^2x}{dt^2} = g$, where g is a constant. $x = x_0$ and $\dfrac{dx}{dt} = \omega_0$ when $t = 0$.

(2) $\dfrac{d^2x}{dt^2} = 3 + 4t$. $\dfrac{dx}{dt} = 2$ and $x = 4$ when $t = 1$.

(3) $\dfrac{d^2y}{dt^2} = kt^2$. $\dfrac{dy}{dt} = 0$ and $y = 0$ when $t = t_0$. k is a constant.

(4) $\dfrac{d^2y}{dx^2} = k(ax - x^2)$. $\dfrac{dy}{dx} = 0$ and $y = y_0$ when $x = 0$. k and a are constants.

(5) $\dfrac{d^2y}{dx^2} = -9 \sin x - 4 \cos x$. $y = 2 - \dfrac{\pi}{2}$ when $x = \dfrac{\pi}{2}$ and $y = 1$ when $x = 0$.

(6) $\dfrac{d^2y}{dx^2} = 6y$. $\dfrac{dy}{dx} = 0$ when $x = 0$ and $y = 6$ when $x = 0$.

(7) $\dfrac{d^2y}{dx^2} = -\dfrac{4}{y^3}$. $\dfrac{dy}{dx} = \dfrac{1}{4}$ when $x = 0$ and $y = 8$ when $x = 0$.

(8) $\dfrac{d^2x}{dt^2} + 9x = 0$. $\dfrac{dx}{dt} = 18$ when $t = 2\pi$ and $x = -6$ when $t = \pi$.

(9) $\dfrac{d^2y}{dx^2} = \dfrac{36}{y^2}$. $\dfrac{dy}{dx} = 0$ when $x = 0$ and $y = 8$ when $x = 0$.

(10) $\dfrac{d^2y}{dt^2} = -\dfrac{16}{y^2}$. $\dfrac{dy}{dt} = 0$ and $y = 8$ when $t = 0$.

(11) $\dfrac{d^2y}{dt^2} + y = 6$. $\dfrac{dy}{dt} = 0$ when $t = \pi$ and $y = 12$ when $t = 0$.

[*Hint.* Substitute a new variable Y for $y - 6$.]

(12) $\dfrac{d^2y}{dx^2} + 4y = 20$. $\dfrac{dy}{dx} = 6$ if $x = \pi$ and $y = 8$ if $x = \dfrac{\pi}{2}$.

(13) Solve $\frac{d^2y}{dx^2} - 8y = 20$.

(14) Solve the example in Art. 137, for the cases in which the load diagram is (i) an isosceles triangle with the length of the beam as base, (ii) a trapezium with its non-parallel sides each of length l and its parallel sides horizontal, the upper side being half the lower side which extends from support to support.

(15) Solve the equations: (i) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$.

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 17y = 0$.

(16) Solve the equations: (i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

(ii) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 25y = 0$.

(17) Solve $\frac{d^2x}{dt^2} - 2a\frac{dx}{dt} + (a^2 + k^2)x = 0$.

(18) Solve $\frac{d^2x}{dt^2} + 3a\frac{dx}{dt} - 4a^2x = 0$.

(19) Solve $\frac{d^2x}{dt^2} + n^2x = 0$.

(20) Write down the equation of motion for small oscillations of a simple pendulum about its equilibrium position, and show that its periodic time t in seconds is $t = 2\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum in feet and g is the acceleration due to gravity in feet per second per second.

(21) Prove that $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$ where y and z are continuous functions of x.

If s is the distance traversed and v the speed of a body at any time t, show that $\frac{d^2s}{dt^2} = v\frac{dv}{ds}$.

The force on a body of mass M moving in a straight line is directed towards a fixed point O in that line, and is of magnitude kM/x^2, where x is the distance from O at time t. If E is the kinetic energy at $x = a$, find the kinetic energy at any distance x. Reckoning t from the instant when $x = a$ and the body is moving towards O, find x in terms of the time if $E = kM/a$. (U.L.)

(22) A small ring of mass m which can move on a circular wire, centre O, is acted on by a force directed from the point A on the circumference of the circle, the force when the ring is at P being equal to $m\mu(1 - AP/AO)$. Show that there are three positions of equilibrium, and find the period of a small oscillation when the position of equilibrium is stable. (U.L.)

(23) A mass of W lb is suspended at the end of a spiral spring which stretches through a distance l ft under the action of a tensile force of E lb weight. Find the period of small vertical oscillations of the weight (i) neglecting the mass of the spring, (ii) assuming the spring to have a mass of w lb.

(24) A uniform rod AB of mass m is free to turn about a smooth vertical axis through the end A. A spring, with axis horizontal and perpendicular to the rod, has one end fixed and the other end attached to B. Show that the period of small oscillations of the rod is the same as the period of a mass $\frac{1}{3}m$ at the end of the same spring. (U.L.)

(25) A uniform horizontal rod AB of mass W lb is free to turn in a vertical plane about A. A mass w lb is attached at B. The rod is held in a horizontal position by a vertical spring attached to the rod at C between A and B, such that $AC = a$ ft, the length AB being l ft. The spring is such that it would stretch 1 ft under the application of a tensile force of E lb weight. Find the period of small oscillations of the rod (neglect the weight of the spring).

(26) Assuming the equation $EI\dfrac{d^2y}{dx^2} = M$ for the curvature of a loaded beam, show that, if a uniform horizontal beam of length $2a$ is simply supported at the ends and carries a variable distributed load which at a distance x from the centre has the value $w(1 - x^2/a^2)$ per unit length, the central deflexion of the beam is $(61Wa^3)/(480EI)$, where W denotes the total load. (U.L.)

(27) Show that the inclination θ to the vertical at any time t of a simple pendulum of length l making small oscillations in a medium in which the resistance per unit mass is k times the linear velocity is given by

$$l\frac{d^2\theta}{dt^2} + kl\frac{d\theta}{dt} + g\theta = 0$$

The time of a complete oscillation of a pendulum making small oscillations *in vacuo* is 2 seconds; if the angular retardation due to the air is 0·04 × angular velocity of the pendulum, and the initial amplitude is 1°, show that in ten complete oscillations the amplitude will be reduced to approximately 40′. (U.L.)

(28) If x is the displacement of a particle oscillating under the action of a force proportional to the displacement, and a small frictional resistance proportional to the velocity, the equation of motion is $\dfrac{d^2x}{dt^2} + k\dfrac{dx}{dt} + n^2x = 0$ and $k^2 < 4n^2$. Show that the ratio of the amplitude of any oscillation to that of the preceding one is constant. Give the solution of the above equation.

(29) A simple pendulum is formed of a particle of mass M suspended from a fixed point by a light wire of length l. If a particle of mass m is fixed to the wire at a distance a from the fixed point, show that the time of a small oscillation is decreased and that it is possible to choose the distance a so that the time of oscillation is a minimum. (U.L.)

(30) A sphere can rotate freely about a horizontal axis distant a ft from the centre of the sphere. If the radius of the sphere is r ft and its mass M lb, find the period of small oscillations about the position of stable equilibrium.

(31) Apply the equations of angular momentum and energy to the following problem. A mass M hangs from the end of a string which is passed up through a hole in a smooth horizontal table so that a length a lies straight on the table. A mass m, attached to the end on the table, is projected with a speed V in the horizontal direction at right angles to the string. Show that, in the subsequent motion, M will move up and down within definite limits, and find the value of V so that just half the length of the string on the table may be pulled through the hole. (U.L.)

(32) A weight, lying on a rough horizontal table, has attached to it a spring the axis of which is horizontal. If the free end of the spring is set in motion in

the direction of its axis with a velocity v, which is kept constant, show that, after the weight has begun to move, its motion relative to the free end is simple-harmonic, with a maximum relative velocity v. (Assume that the friction has the same magnitude when the weight is at rest as when it is in motion.) (U.L.)

(33) Solve the equation $\dfrac{d^2x}{dt^2} + 36x = 3\cos 6t$.

A spiral spring of negligible mass, which stretches 1 in. under a tension of 2 lb, is fixed at one end and carries a mass of M lb at the other end. Find M, given that the period of the free vertical oscillations of the mass is $\dfrac{\pi}{3}$ sec. If, when the mass M is at rest, a periodic force $X = 2\cos 6t$ lb acts upon it, prove that the displacement x ft in time t sec is given by $x = \frac{1}{4}t\sin 6t$. (Neglect frictional forces.)

(34) A mass oscillates under the action of a spiral spring, which is fixed at one end, and against a constant frictional resistance which is always contrary to the velocity. Show that the period is independent of the friction, and that the amplitude decreases by the same amount in each half-oscillation. (U.L.)

(35) A sphere of radius a is projected with linear velocity V up a plane inclined to the horizontal at an angle α. The sphere initially having no rotational velocity, show that the time which elapses before the sphere rolls without slipping on the plane is

$$\frac{V}{g(\sin\alpha + \frac{7}{2}\mu\cos\alpha)}$$

and obtain the time when the sphere is at the highest point on the plane. μ is the coefficient of friction between the sphere and the plane, and is greater than $\frac{2}{7}\tan\alpha$. (U.L.)

(36) Solve the equation $\dfrac{d^3y}{dx^3} - 2a\dfrac{d^2y}{dx^2} - a^2\dfrac{dy}{dx} + 2a^3y = 0$, where a is a constant.

(37) Solve completely the equation $\dfrac{d^2s}{dt^2} + 6\dfrac{ds}{dt} - 16s = \cos 3t$.

(38) If y satisfies the differential equation $EI\dfrac{d^2y}{dx^2} + Py = 0$ (E, I, P constants) and $y = 0$ when $x = 0$ and $x = l$ show that y is zero for all values of x unless P has one of the values $\dfrac{n^2\pi^2}{l^2}EI$, where n denotes an integer. (U.L.)

(39) Show how to solve the differential equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0$, a and b being constants.

If $y = \phi(x)$ is a particular integral of the equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = f(x)$, explain how to solve this equation.

Solve completely the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = e^x$. (U.L.)

(40) Solve the differential equations: (i) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} - 7y = 0$

(ii) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

(iii) $\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} + 12y = 0$

(41) Solve the differential equations: (i) $\dfrac{d^2x}{dt^2} \quad 25x = e^{7t}$

(ii) $\dfrac{d^2x}{dt^2} + 25x \quad 20e^{6t}$

(iii) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 20y - 1 - e^{4}$

(iv) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y \quad \cos 5x$

(42) The equation for the discharge of a condenser is

$$L\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Cx = 0$$

If the charge is q when the circuit is closed, i.e. if $x = q$ and $\dfrac{dx}{dt} = 0$ when $t = 0$, find the value of x in terms of t.

What relation exists between the constants L, R, and C if no periodic terms occur in the value of x? (U.L.)

(43) Solve the equations

(i) $\dfrac{d^2y}{dx^2} - 4y = 10$

(ii) $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{K} \quad E$

where $R = 100$, $L = 0{\cdot}005$, $K = 10^{-6}$, $E = 1\,000$. (U.L.)

(44) Integrate the equation $\dfrac{d^2x}{dt^2} + x \quad a \sin t$.

(45) The motion of a weight at the lower end of a spring is given by $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 25y = \sin 2t$, the other end of the spring having a simple harmonic motion. Solve this equation and point out the part of it which gives a steady motion when t is large.

Also solve $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + a^2y = e^{ax}$ (U.L.)

(46) Verify that the solution of the equation $\dfrac{d^2y}{dt^2} + n^2y \quad a \sin pt$ is $y = \dfrac{a}{n^2 - p^2} \sin pt + \kappa \sin (nt + \alpha)$, where κ and α are arbitrary constants.

If $n = 2p$, find the values of κ and α in the case for which y and $\dfrac{dy}{dt}$ are both zero for $t = 0$. Draw the graph of y for values of t from 0 to $2\pi/p$. (U.L.)

(47) Find the value of u which satisfies the differential equation $\dfrac{d^2u}{d\theta^2} + u = 2k \cos \theta$, and the two following conditions—

(i) u has the same value when $\theta = \quad \cdot \frac{1}{2}\pi$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} u \, d\theta = 0$ (U.L.)

(48) Solve the equation $\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + (a^2 + b^2)x = k \sin pt$ subject to the conditions that $x = h$ and $\frac{dx}{dt} = 0$ when $t = 0$ (U.L.)

(49) If x is the displacement of a particle, t the time, and a, b, and c constants, the equation of motion of the particle can be written $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$. Solve this equation, (i) when $b^2 > 4ac$, (ii) when $b^2 = 4ac$, and (iii) when $b^2 < 4ac$. In each case sketch the graph, showing the relation between x and t.

(50) A condenser of capacity C is discharged through a circuit of resistance R and inductance L. Prove that the charge Q at any time t is given by $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$; hence, show that if R be sufficiently small, the discharge is oscillatory, and determine the period of the oscillation. Calculate the frequency if the capacity is 0·02 microfarad, the inductance 0·0003 henry, and the resistance negligible. (U.L.)

(51) If $a\frac{d\theta}{dt} + 2b\theta + u = 0$ and $\theta = \frac{dx}{dt}$ and $x = ku$, find the differential equation connecting x, t, and the constants a, b, and k. Express the relation between a, b, and k so that if x denotes a displacement and t the time, the motion may just be non-oscillatory. On this supposition, show that

$$x = e^{-\frac{b}{a}t}\left\{x_0 + \frac{b}{a}x_0t + \theta_0t\right\}$$

where x_0 and θ_0 are the values of x and θ when $t = 0$. (U.L.)

(52) A thin uniform wooden rod, 4 ft long, can turn freely about one end in a vertical plane. A cross-piece in the shape of a thin metal rod can be attached to the wooden rod at any point at right-angles to it and with its mid-point on it in the vertical plane in which the wooden rod moves. The weight of the cross-piece is eight times that of the wooden rod. When the system makes small oscillations about the fixed end, compare the times of these oscillations (i) when the cross-piece is at the other end, (ii) when it is at the middle of the wooden rod. (U.L.)

(53) A rod diameter d in., length l ft, weight W lb, is suspended horizontally by two parallel vertical wires of length L ft. The wires have their upper ends attached to a ceiling and the lower ends are attached to the rod at points whose horizontal distances from the centre of the rod are a ft. The axis of the rod executes small oscillations about the vertical through its centre. Find the time of one oscillation.

(54) Obtain the equations of motion of a body rotating about a fixed axis.

A wheel is suspended with its axis vertical by two equal vertical wires, each of length 3 ft, attached to points on the wheel which are diametrically opposite and each 4 in. from the axis. When the wheel is set oscillating about its axis through a small angle, the period of a complete oscillation is found to be 3·26 sec. Find the radius of gyration of the wheel, assuming $g = 32{\cdot}2$ ft per sec per sec. (U.L.)

(55) If $x + iy = ae^{ipt^2}$, a and p being constants and t denoting time, show that the point (x, y) describes a circle with uniformly accelerated angular velocity.

The co-ordinates of a particle moving in the xy plane satisfy the equations $\frac{dx}{dt} = -ny$, $\frac{dy}{dt} = nx$. Prove that the particle is describing a circle with constant speed. (U.L.)

CHAPTER XIII

STATISTICS AND CORRELATION

143. **Statistics.** Recent developments in the engineering industry have been more and more on the lines of repetition work. A single machine, or a set of machines, gives a continuous output of apparently identical articles. Owing to various causes such as variations in the quality of the material and in its physical characteristics, tool wear, shaft and bearing wear, fluctuating conditions of working, variations in control due either to neglect or to fatigue or to insufficiency of personnel, and to many other causes some of which cannot be specified, the articles are not identical. Causes which can be specified are *assignable* causes; those which cannot are *chance* causes. The former can be reduced to a minimum by careful methods of control, but the latter will remain. As knowledge of the variations increases, some of the chance causes may become assignable.

In order that the effective control may be applied, it is necessary to study variations in the measures of the characteristics which it is intended to control—such characteristics being dimensions, weights, density, hardness, ductility, tensile or compressive strengths, etc.—and the effect of the variations on the quality of the product, as well as to locate, as far as is possible, the causes of excessive variation. The whole collection of apparently similar articles produced is known as the *population* or *universe.* For our purpose the term *statistics* means the collection of numbers representing the observed measurements of one or more particular characteristics in the whole, or part, of the population. *Statistical method* is the method of obtaining useful information from the statistics. Statistics and statistical method are both included when the former term is used in a general sense as in the heading to this section.

144. **Frequency Distribution.** Suppose that the numbers representing values of a characteristic over a population are arranged in ascending order of magnitude, and that the scale of numbers is divided into a number of intervals, usually equal. The number of characteristic values falling in an interval is known as the *frequency* in that interval. We also speak of the frequency between any two specified limits and of the total frequency, which is the number of the population. The difference between the highest and the lowest

of the characteristic values is the *range*. The interval, or the characteristic value at the middle of an interval, is associated with the frequency, and the whole of these pairs of numbers is called a *frequency distribution*.

We shall illustrate the above by considering the characteristic values in Table I below, which give the tensile strength in quarter-ounces of single threads of cotton yarn, all of which are supposed to be of the same degree of fineness, i.e. 50's counts. The numbers are in order down the columns from left to right.

TABLE I

16	18	19	25	21	19	22	17	17	20
18	17	22	17	22	17	20	19	20	20
20	22	19	21	26	20	23	20	18	21
18	17	23	18	18	16	23	20	18	22
19	18	20	16	23	19	18	17	23	20
19	20	22	20	25	20	18	19	19	20
20	15	24	21	24	18	19	21	24	19
22	15	24	20	20	17	20	20	24	20
22	18	21	22	20	22	20	20	24	18
22	16	27	23	22	18	17	18	23	25

As the numbers are correct to the nearest unit, the least number 15 may represent any value between 14·5 and 15·5, and the greatest number 27 any value between 26·5 and 27·5. The range is therefore $27{\cdot}5 - 14{\cdot}5 = 13$. We shall divide the range into 13 equal intervals each of length $\frac{13}{13} = 1$. Table II gives the frequency distribution.

TABLE II

Interval . . .	14·5–15·5	15·5–16 5	16·5–17·5	17·5–18 5	18·5–19·5	19·5–20·5	20·5–21·5
Frequency . .	2	4	9	15	11	23	6

Interval . . .	21·5–22·5	22·5–23·5	23·5–24·5	24·5–25·5	25·5–26·5	26·5–27·5	
Frequency . .	12	7	6	3	1	1	

In Fig. 133 characteristic values are set off to scale along the horizontal axis and frequencies along the vertical axis, and rectangles are constructed whose bases represent the intervals or *sub-ranges* and whose heights represent the corresponding frequencies. The interval

being unity in this case, the area of each rectangle is equal to the corresponding frequency. The stepped figure is called a *histogram*.

Another form of representation of the frequency distribution is the *frequency polygon*. This is obtained by joining the tops of the frequency ordinates at the middle of the intervals, and is shown

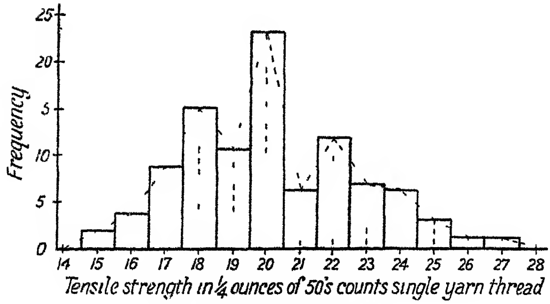


FIG. 133. HISTOGRAM

dotted in Fig. 133. For the frequency polygon all the class intervals should be equal. If the number in the population is increased and the class interval reduced, both the histogram and the frequency

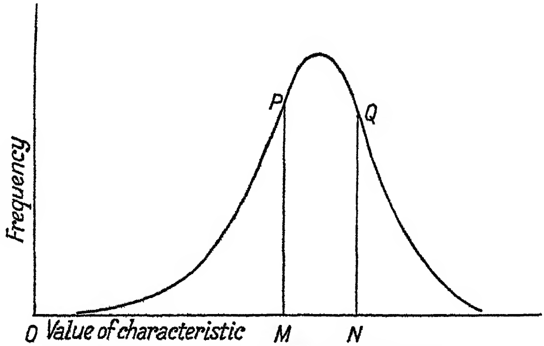


FIG. 134. TYPICAL DISTRIBUTION CURVE

polygon will approach more closely to a smooth curve, and in the limit when the population becomes infinite they will both merge into the smooth curve known as the *frequency distribution curve*, such as that in Fig. 134.

The area under the curve between any two ordinates PM and QN represents the number of articles having values between $\overline{ON} = x_2$

and $O\bar{M} = x_1$. The curve shown, which is bell-shaped, is roughly typical of those obtained by plotting frequency against characteristic value in the case of most engineering characteristics. It is roughly symmetrical about its maximum ordinate though in many practical cases this symmetry is not so pronounced. If we had enough observations in the above distribution table to plot a smooth curve, it appears likely from the histogram (Fig. 133) that the hump in the curve would be roughly over the value 20 which is to the left of the centre of the range. This lack of symmetry is called *skewness*, and the distribution is called a *skew* distribution. Not all distribution curves are bell-shaped; some have two or more maxima, some are U-shaped, and some are of the shape of the letter J or J reversed. The hundred values of yarn strength given above constitute a *sample* taken from the population, and represent only a small portion of the population. Tests on a sample or samples are intended to provide information about the whole population. It is necessary, therefore, to ensure that there is no bias in selecting the articles, i.e. they should be selected in such a way that every article has the same chance of selection. If there is no bias and if the size of the sample is sufficiently large, the histogram or frequency curve will be practically the same for the whole population as for the sample, except for the vertical scale. An unbiased sample is known as a *random sample*. The larger the sample, the more representative it is of the total population.

What can we take as the value of the characteristic, i.e. thread strength, in the example given? If we are to be accurate, we can only say that the thread strength lies between 14·5 and 27·5 quarter-ounces. We see that the greatest frequency occurs when the strength is 20 quarter-ounces, and that as a whole the frequency diminishes as the strength diminishes or increases from this value; hence the strength in quarter-ounces of any particular thread is much more likely to be near to 20 than to 15 to 27.

145. **Constants of a Frequency Distribution.** The *arithmetic mean* is the sum of all the characteristic values divided by the total frequency. Let $x_1, x_2, x_3, \ldots, x_n$ be the different values of the characteristics, and $f_1, f_2, f_3, \ldots, f_n$ their respective frequencies, and let $\bar{x}$ be the arithmetic mean and $N = \Sigma f$ the total frequency.

Then $$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots + f_n}$$

or $$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{\Sigma fx}{N} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.1)}$$

It follows from (XIII.1) that

$$\Sigma f(x - \bar{x}) = 0 \quad . \quad . \quad . \quad \text{(XIII.2)}$$

The *median* is the central value of the characteristic when the values are arranged in order of magnitude. If the number of values is even, the median is half the sum of the two central values. In a grouped frequency the value of the median is found by the method of proportional parts. The *mode* is the value of the characteristic at the middle of the interval in which the frequency is greatest. This is not easy to determine as the choice of interval is arbitrary and it is not much used in engineering statistics. The mean is usually taken as the best estimate of the true value of the characteristic. In the determination of the mean all the values are used, whereas the mode and the median would both remain unchanged if either the lower or the higher frequency groups had their values increased or decreased.

A measure of the scatter or *dispersion* of values about the mean is also needed. Of several possible measures the following are the most important.

The *range* is not a good measure as it takes account of the two end values only and does not indicate whether or not the values are closely packed about the mean.

The *mean deviation* from the mean, or the *average error*, is the mean numerical difference between the value of the characteristic and its mean; thus

$$\text{Mean deviation} = \frac{\Sigma f(x \sim \bar{x})}{\Sigma f} \quad . \quad . \quad . \quad \text{(XIII.3)}$$

The sign $\sim$ in (XIII.3) indicates that all differences are to be taken positive.

The mean deviation from any other value, say $x = \zeta$, is

$$\frac{\Sigma f(x - \zeta)}{\Sigma f}$$

Then, mean deviation (from $x = \zeta$)

$$\begin{aligned}
&= \Sigma f(x - \zeta) \div \Sigma f \\
&= \Sigma f(x - \bar{x} + \bar{x} - \zeta) \div \Sigma f \\
&= \{\Sigma f(x - \bar{x}) + \Sigma f(\bar{x} - \zeta)\} \div \Sigma f \\
&= \Sigma f(\bar{x} - \zeta) \div \Sigma f \quad \text{[using (XIII.2)]} \\
&= (\bar{x} - \zeta)\Sigma f \div \Sigma f \\
&= \bar{x} - \zeta
\end{aligned}$$

$\therefore$ Mean deviation (from $x = \zeta$)

$$\frac{\Sigma f(x-\zeta)}{\Sigma f} = \bar{x} - \zeta \quad . \qquad \text{(XIII.4)}$$

The *standard deviation* is the root mean square of the deviation from the mean. If σ = standard deviation, then

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{N}} \quad . \qquad \text{(XIII.5)}$$

We use σ to denote the standard deviation in the whole population and s that in a sample. If the sample is a very large one, the value of s may be taken as that of σ.

Note that σ is the radius of gyration of the area under the frequency curve about a vertical axis through its centroid.

The root mean square value of the deviation from any given value of x, say $x = \zeta$, is s', where

$$s' = \sqrt{\frac{\Sigma f(x-\zeta)^2}{\Sigma f}} \quad . \quad . \quad . \qquad \text{(XIII.6)}$$

$$\begin{aligned}
\therefore\ s'^2\Sigma f &= \Sigma f(x-\zeta)^2 \\
&= \Sigma f\{(x-\bar{x}) + (\bar{x}-\zeta)\}^2 \\
&= \Sigma f(x-\bar{x})^2 + 2\Sigma f(x-\bar{x})(\bar{x}-\zeta) + \Sigma f(\bar{x}-\zeta)^2 \\
&= \sigma^2\Sigma f + 2(\bar{x}-\zeta)\Sigma f(x-\bar{x}) + (\bar{x}-\zeta)^2\Sigma f
\end{aligned}$$

Now $\Sigma f(x-\bar{x}) = 0$; hence, writing d for $\bar{x}-\zeta$ and dividing through by Σf, we have

$$s'^2 = \sigma^2 + d^2 \quad . \quad . \quad . \qquad \text{(XIII.7)}$$

Note that, if both sides of (XIII.7) are multiplied by A (the area under the frequency curve), the relation obtained is equivalent to the theorem of parallel axes for moments of inertia of area. (XIII.7) shows that s'^2 has minimum value σ^2, which occurs when $d = 0$, i.e. when $\zeta = \bar{x}$, so that the sum of the squares of the deviations is a minimum when these deviations are reckoned from the mean.

The square of the standard deviation is known as the *variance*. The standard deviation is generally taken as the measure of dispersion. $\frac{1}{N}\Sigma fx$ and $\frac{1}{N}\Sigma fx^2$ are called the *first moment* and the *second moment* respectively about $x = 0$, and $\frac{1}{N}\Sigma f(x-a)$ and $\frac{1}{N}\Sigma f(x-a)^2$ are those about $x = a$.

When estimating the standard deviation of a population from a

portion, or sample, of it, we introduce an error by assuming the sample mean to be the population mean. This gives too small a value of the variance by c^2, where c is the distance between the two means. For, if ζ above is assumed to be the population mean, then $\bar{x} - \zeta = c$, i.e. $d = c$, and (XIII.7) becomes in this case $s_1^2 = \sigma^2 + c^2$. As the population mean is not known, c is not known. It is shown in Art. 152 that the above error is corrected by writing $\Sigma f - 1$ for Σf, or $N - 1$ for N, in (XIII.5). This correction is necessary for small values of N, but becomes unimportant when N is large.

If the values of x are not grouped, we replace Σfx by Σx, $\Sigma f(x - \bar{x})$ by $\Sigma(x - \bar{x})$, $\Sigma f(x \sim \bar{x})$ by $\Sigma(x \sim \bar{x})$, and $\Sigma f(x - \bar{x})^2$ by $\Sigma(x - \bar{x})^2$ in the above relations.

EXAMPLE 1

Calculate the mean value, the standard deviation, and the mean deviation of the frequency distribution in Table II above, which gives the grouped frequency distribution formed from the values in Table I.

We set up Table III as follows—

TABLE III

1	2	3	4	5	6	7
x = strength in quarter-ounces	f frequency	fx	$d = x - \bar{x}$	fd	d^2	fd^2
15	2	30	− 5·09	− 10·18	25·908	51·82
16	4	64	− 4·09	− 16·36	16·728	66·91
17	9	153	− 3·09	− 27·81	9·548	85·93
18	15	270	− 2·09	− 31·35	4·368	65·52
19	11	209	− 1·09	− 11·99	1·188	13·07
20	23	460	− 0·09	− 2·07	0·008	0·18
21	6	126	0·91	5·46	0·828	4·97
22	12	264	1·91	22·92	3·648	43·78
23	7	161	2·91	20·37	8·468	59·28
24	6	144	3·91	23·46	15·288	91·73
25	3	75	4·91	14·73	24·108	72·32
26	1	26	5·91	5·91	34·928	34·93
27	1	27	6·91	6·91	47·748	47·75
Totals .	$\Sigma f = 100$	$\Sigma fx = 2\,009$		$\Sigma fd = 0$		$\Sigma fd^2 = 638{\cdot}19$

The headings indicate how the columns are filled in.

Total frequency $= \Sigma f = 100$

Mean value $\bar{x} = \dfrac{\Sigma fx}{\Sigma f} = \dfrac{2\,009}{100} = 20{\cdot}09$ quarter-ounces

This value is substituted for $\bar{x}$ at the head of column 4.

Standard deviation $= \sqrt{\dfrac{\Sigma fd^2}{\Sigma f - 1}} = \sqrt{\dfrac{638{\cdot}19}{99}} = 2{\cdot}54$ quarter-ounces.

The total of column 5 should be zero. If the signs are all taken positive in this column, the total is 199·52.

$\therefore$ Mean deviation $= \dfrac{199{\cdot}52}{\Sigma f} = \dfrac{199{\cdot}52}{100} = 2{\cdot}00$ quarter-ounces.

EXAMPLE 2

The lengths in inches of 100 brass plugs are given in column 1 of Table IV, and the frequencies in column 2. Find the mean value and the standard deviation of the length.

Here we simplify the arithmetic by using class intervals, as given in column 3, instead of characteristic values. We also measure deviation from some value $x = \zeta$, say, which is estimated to be near the mean value. Thus we take 0·751 as a first estimate of the mean value.

TABLE IV

$x =$ length in inches	$f =$ frequency	$x' =$ class interval	$d = x' - 6$ = deviation in class intervals	fd	d^2	fd^2
0·746	1	1	5	5	25	25
0·747	2	2	4	8	16	32
0·748	2	3	3	6	9	18
0·749	6	4	− 2	12	4	24
0·750	13	5	− 1	13	1	13
0·751	28	6	0	0	0	0
0·752	23	7	1	23	1	23
0·753	16	8	2	32	4	64
0·754	4	9	3	12	9	36
0·755	3	10	4	12	16	48
0·756	1	11	5	5	25	25
0·757	1	12	6	6	36	36
Totals .	100		21 − 15 6	90 − 44 46		344

By (XIII.1), $\bar{x}' - 6 = \dfrac{\Sigma fd}{\Sigma f}$ class intervals

$$= \frac{46}{100} \text{ class intervals}$$

$$x - 0.751 = 0.46 \times 0.001 = 0.00046 \text{ in.}$$

i.e., $$\bar{x} = 0.751 + 0.0005 = 0.7515 \text{ in.}$$

Also by (XIII 7), $$\sigma^2 = s'^2 - d^2$$

$$= \frac{344}{99} - \left(\frac{46}{100}\right)^2$$

$$= 3 \cdot 47 - 0.21$$

$$= 3 \cdot 26$$

$$\sigma = \sqrt{3.26} = 1.80 \text{ class divisions}$$

$$\sigma = 1.8 \times 0.001 \text{ in.}$$

$$0.0018 \text{ in.}$$

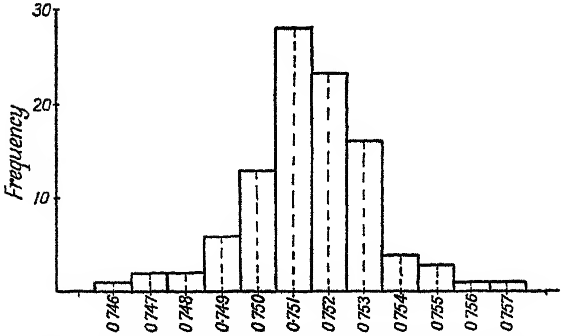


FIG 135. HISTOGRAM FOR LENGTHS OF BRASS PLUGS

The mean value is 0.7515 in., and the standard deviation is 0.0018 in. The histogram for the above distribution is shown in Fig. 135.

In both the above examples the total frequency is too small to give an approximation to a smooth frequency curve. Both are somewhat skew.

146. Sheppard's Correction for the Second Moment. We assumed above that the frequency in each class-interval was collected at its centre. The usual type of frequency distribution is like that represented in Fig. 134, in which the frequency is greater near the middle and tails off to zero, or to very small values, at the ends. A more accurate value of the second moment, or of the standard deviation, will be found if we assume the frequency distribution in each class-interval to be represented by a trapezium, as in Fig. 136, rather than

by the mid-ordinate of a rectangle as assumed above. Let the class-interval be the constant h and let the ordinates from left to right be $y_1, y_2, y_3, \ldots, y_{n+1}$, n being the number of class-intervals. Also let $ABCD$ be the rth strip of area A_r, so that $\overline{AD} = y_r$ and $\overline{BC} = y_{r+1}$. The total frequency is $N = \Sigma A_r$. The distance of O from the mid-point of AB is $d_r = h(r - \frac{1}{2})$. The trapezium $ABCD$ may be regarded as made up of a rectangle of the same area A_r on AB as base and two

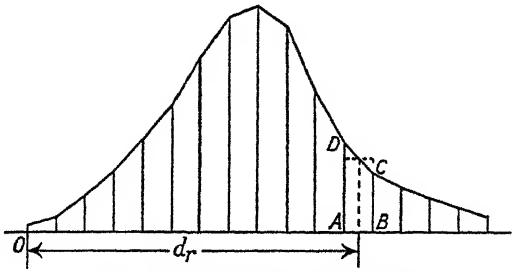


FIG. 136 SHEPPARD'S CORRECTION

congruent right-angled triangles one of which is added to and the other taken away from the rectangle (see Fig. 136). The second moment of the rectangle about O is $A_r \times \frac{1}{12}h^2 + A_r \times d_r^2$. The horizontal distances of the centroids of the triangles from O are $d_r \pm \frac{1}{3}h$, and the net second moment of these triangles about O is

$$\tfrac{1}{2} \times \tfrac{1}{2}(y_r - y_{r+1}) \times \frac{h}{2}\{(d_r - \tfrac{1}{3}h)^2 - (d_r + \tfrac{1}{3}h)^2\} + I_G - I_G$$

(where I_G is the second moment of each triangle about a vertical axis through its centroid),

i.e. $$\frac{h}{8}(y_r - y_{r+1}) \times 2d_r \times -\tfrac{2}{3}h$$

i.e. $$\tfrac{1}{6}h^3(r - \tfrac{1}{2})(y_{r+1} - y_r), \quad \text{since } d_r = h(r - \tfrac{1}{2})$$

Hence, the second moment of the area of $ABCD$ about O is

$$A_r d_r^2 + \tfrac{1}{12}A_r h^2 + \tfrac{1}{6}h^3(r - \tfrac{1}{2})(y_{r+1} - y_r)$$

If μ_2 is the second moment of the whole distribution about O, then

$$A\,\mu_2 = \Sigma A_r d_r^2 + \tfrac{1}{12}h^2 \Sigma A_r + \tfrac{1}{6}h^3 \Sigma (r - \tfrac{1}{2})(y_{r+1} - y_r)$$

i.e. $$A\,\mu_2 = \Sigma A_r d_r^2 + \tfrac{1}{12}Nh^2 + \tfrac{1}{6}h^3 \Sigma (r - \tfrac{1}{2})(y_{r+1} - y_r) \quad . \quad \text{(XIII.8)}$$

Now $\Sigma(r - \frac{1}{2})(y_{r+1} - y_r)$

$$= \tfrac{1}{2}(y_2 - y_1) + \tfrac{3}{2}(y_3 - y_2) + \tfrac{5}{2}(y_4 - y_3) + \ldots + \frac{2n-3}{2}(y_n - y_{n-1}) + \frac{2n-1}{2}(y_{n+1} - y_n)$$

$$= -(\tfrac{1}{2}y_1 + y_2 + y_3 + y_4 + \ldots + y_n) + \frac{2n-1}{2}y_{n+1}$$

$$= -(\tfrac{1}{2}y_1 + y_2 + y_3 + y_4 + \ldots + y_n + \tfrac{1}{2}y_{n+1}) + ny_{n+1}$$

But, by the trapezoidal rule, the quantity in the brackets is $\dfrac{\Sigma A_r}{h} = \dfrac{N}{h}$, and the term ny_{n+1} is negligible since y_{n+1} is small, so that (XIII.8) becomes

$$N\mu_2 = \Sigma A_r d_r^2 + \tfrac{1}{12}Nh^2 - \tfrac{1}{6}Nh^2$$

i.e. $$N\mu_2 = \Sigma A_r d_r^2 - \tfrac{1}{12}Nh^2 \quad . \quad . \quad \text{(XIII.9)}$$

Thus, in assuming above that the second moment is $\dfrac{1}{N}\Sigma A_r d_r^2$, we have introduced an error which can be corrected by deducting $\frac{1}{12}h^2$ from the result obtained.

In order to find the second moment about the mean value $\bar{x}$ we deduce $N\bar{x}^2$ from that about O, and the error is the same for the second moment about the mean value. Since this latter is σ^2, the error in σ^2 is $-\frac{1}{12}h^2$.

In Example 1 of the last section we found that $\sigma^2 = \dfrac{638 \cdot 19}{99} = 6 \cdot 446$, so that the corrected value is $\sigma^2 = 6 \cdot 446 - \frac{1}{12} = 6 \cdot 363$, and $\sigma = 2 \cdot 52$ quarter-ounces. Similarly, since by working in class-intervals in Example 2 of the same section we found $s'^2 = \dfrac{344}{99} = 3 \cdot 475$, the corrected value is $s'^2 = 3 \cdot 475 - \frac{1}{12} = 3 \cdot 392$.

Hence, $\sigma^2 = 3 \cdot 392 - 0 \cdot 46^2 = 3 \cdot 180$, and $\sigma = 1 \cdot 783$ class divisions. Thus, the corrected value of σ is $\sigma = 1 \cdot 783 \times 0 \cdot 001 = 0 \cdot 00178$ in.

147. **Probability or Chance.** Suppose there is a large number N of occasions on which a certain event is equally likely to happen, and that the event happens on a number n of the N occasions. In such a case we should say that the chance, or probability, of the event happening on any one of the N occasions is n/N or alternatively 1 in N/n. If, for instance, we toss a coin in such a way that it is equally likely to turn up a head or a tail, the probability, or chance, of a head (or alternatively a tail) turning up is $\frac{1}{2}$ or 1 in 2. This does not mean that, if we continue to toss the coin, heads and tails will occur alternatively, but that if we toss it a large number of times, say 1 000 or

10 000 times, the number of times a head (or a tail) turns up will be approximately 500 or 5 000. There may be a considerable difference between the number of heads and the number of tails, but in a very large number of tosses the ratio of the number of heads (or of tails) to the number of tosses will approach very closely to $\frac{1}{2}$. If the production of a machine is carefully controlled and it is found that of a large number of articles produced p per cent are defective, the chance, or probability, that a single article taken at random as it comes from the machine will be defective is $p/100$ or 1 in $100/p$.

In the above we have used the undefined terms "equally likely to happen," "carefully controlled," "at random." The meanings of these terms should become clearer to the reader as he studies the subject matter of this chapter. For a full discussion of the laws of probability readers should consult textbooks of algebra, or books which deal particularly with the subject itself. In the following we shall only make assumptions about probability when these appear to be reasonable. For example, if we have a frequency distribution like that of Table IV, but with a very much larger frequency in each class, we shall assume that the chance, or probability, of any article chosen at random having a length between 0·7485 in. and 0·752 in. is the sum of the frequencies in the groups between those lengths divided by the total frequency. For the distribution of Fig. 134 the probability of the characteristic falling between the values $\overline{OM}$ and $\overline{ON}$ is the ratio of the area of the strip $PQNM$ to that under the whole curve.

148. **The Binomial and the Poisson Distributions.** Consider the products of a machine which are tested either by limit gauges such as "go" and "not go" gauges or by visual inspection. Those articles which do not pass the test are known as *defectives*. From past records it is possible to find the proportion p of defectives in a large number of the articles. If $q = 1 - p$, then q is the proportion of satisfactory articles produced. The engineer wishes to find from these values of p and q how many rejects he can expect to find in a random sample of a given size.

Suppose we take N articles at random from the products. Of these qN will be satisfactory on the average and pN will be defective. Suppose now we take a second set N at random and pair them one by one with the first set. This will produce N pairs. Of the pairs formed with the qN articles of the first set, $q \times qN = q^2N$ will have no rejects and $p \times qN = qpN$ will have one reject, whilst of those formed with the pN articles of the first set $q \times pN = qpN$ will have

one reject and $p \times pN = p^2N$ will have two rejects. Thus the number of pairs with 0, 1, 2 rejects respectively are q^2N, $2qpN$, p^2N which are the terms of the expanded form of $N(q + p)^2$.

Now suppose that the pairs are transformed into N groups of three by combining a third random set of N one by one with each of the N pairs. In the fraction q of the groups of three the numbers of defectives will be the same as in the original pairs, whilst in the fraction p the number of defectives will be increased by one.

Thus we have for N groups of three

	Number of Defectives			
	0	1	2	3
In qN groups .	q^3N	$2q^2pN$	qp^2N	
In pN groups .		q^2pN	$2qp^2N$	p^3N
In all N groups .	q^3N	$3q^2pN$	$3qp^2N$	p^3N

These are the terms in the expansion of $N(q + p)^3$.

Similarly, if we take a fourth random set N and combine the articles one by one with the N groups of three, the resulting groups of four will be such that in qN of the groups the numbers of defectives will be the same as in the groups of three, whilst in the remaining pN of the groups the numbers of defectives will be one more than in the corresponding groups of three. Thus we have for N groups of four—

	Number of Defectives				
	0	1	2	3	4
In qN groups .	q^4N	$3q^3pN$	$3q^2p^2N$	qp^3N	
In pN groups .		q^3pN	$3q^2p^2N$	$3qp^3N$	p^4N
In all N groups .	q^4N	$4q^3pN$	$6q^2p^2N$	$4qp^3N$	p^4N

These are the terms in the expansion of $N(q + p)^4$.

By proceeding thus and forming groups of 5, 6, 7, etc., . . . $\overline{n-1}$, n articles in succession we see by analogy with algebraic multiplication that for N samples or groups each including n articles the numbers in the groups containing 0, 1, 2, 3, 4, etc., . . . $\overline{n-1}$,

n defectives respectively are the terms taken in order of the expansion of $N(q+p)^n$, i.e.

$$Nq^n, Nnq^{n-1}p, \frac{Nn(n-1)}{2!}q^{n-2}p^2, \frac{Nn(n-1)(n-2)}{3!}q^{n-3}p^3, \ldots,$$
$$Nnqp^{n-1}, Np^n \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.10)}$$

If the factor N is omitted from each term, we have the series

$$q^n, nq^{n-1}p, \frac{n(n-1)}{2!}q^{n-2}p^2, \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3, \ldots, nqp^{n-1}, p^n$$
$$\text{(XIII.11)}$$

the $n+1$ terms of which represent respectively the chance or probability that any random group of n should contain 0, 1, 2, 3, . . ., $n-1$, n defectives.

EXAMPLE

Ten per cent of the articles from a certain machine are defective What is the chance that there will be six defectives in a sample of 25 ?

$p = 0{\cdot}1$, $q = 1 - p = 0{\cdot}9$, and $n = 25$. Thus the required chance is the seventh term in the expansion of $(0{\cdot}9 + 0{\cdot}1)^{25}$.

This term is $\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}(0{\cdot}9)^{19} \times (0{\cdot}1)^6$ where $n = 25$

$= 177\,100 \times (0{\cdot}9)^{19} \times (0{\cdot}1)^6$

Chance $= 0{\cdot}024$

In only 2·4 per cent of a large number of samples of 25 will there be 6 defectives in a sample.

The distributions (XIII.10) and (XIII.11) are known as *binomial distributions*. If $p = q$, they are symmetrical distributions as dealt with in Art. 149, but if $p \neq q$, they are skew. If n is large enough to make $p \sim q$ negligible compared with $\sqrt{npq}$, the distributions become nearly symmetrical and approximate closely to that shown in Fig. 137. In practical cases p is often less than 0·1 and for samples of from 3 to 100 the distributions are skew. If we assume that p is small compared with 1, $q^n = (1-p)^n = \left(1 - 1\Big/\frac{1}{p}\right)^{\frac{1}{p}np}$, which is approximately equal to e^{-np}, and the series (XIII.11) becomes

$$e^{-np}\left[1, \frac{np}{q}, \frac{n(n-1)}{2!}\left(\frac{p}{q}\right)^2, \frac{n(n-1)(n-2)}{3!}\left(\frac{p}{q}\right)^3, \ldots \text{to } n+1 \text{ terms}\right]$$

If n is sufficiently large we may write n for $n-1$, $n-2$, $n-3$, etc., in the early terms of the series and the later terms become unimportant because of the small value of p/q. Also since q is nearly 1 we

may write p for p/q. With these changes the distribution becomes that of the terms of

$$e^{-np}\left[1 + np + \frac{n^2p^2}{2!} + \frac{n^3p^3}{3!} + \frac{n^4p^4}{4!} + \ldots \text{ to } n + 1 \text{ terms}\right]$$

or putting m for np

$$e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \frac{m^4}{4!} + \ldots \text{ to n} + 1 \text{ terms}\right] \quad \text{(XIII.12)}$$

This is known as the *Poisson distribution.* As it stands it is a probability distribution, the $(n + 1)$th term of which represents the probability of n defectives. If the expression is multiplied throughout by N, the number of samples, the terms represent the distribution of the samples, i.e. the numbers of samples having 0, 1, 2, 3, . . . etc., defectives respectively.

We shall now find the mean value and the standard deviation of each distribution. Assuming an interval of 1 between successive frequencies, we associate the numbers of the distribution with the values $x = 0, 1, 2, 3, \ldots$ etc., respectively. Let μ_1 and μ_2 be the first and second moments about $x = 0$. Then for the binomial distribution (XIII.11)

$$\mu_1 = \left[q^n \times 0 + nq^{n-1}p \times 1 + \frac{n(n-1)}{2!}q^{n-2}p^2 \times 2 + \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 \times 3 + \ldots \text{ to } n + 1 \text{ terms}\right]$$

$$= np\left[q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2!}q^{n-3}p^2 + \ldots \text{ to } n \text{ terms}\right]$$

$$= np(q + p)^{n-1}, \text{ or, since } q + p = 1,$$

$$\mu_1 = np \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.13)}$$

$$\mu_2 = \left[q^n \times 0 + nq^{n-1}p \times 1^2 + \frac{n(n-1)}{2!}q^{n-2}p^2 \times 2^2 + \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 \times 3^2 + \ldots \text{ to } n + 1 \text{ terms}\right]$$

$$= np\left[q^{n-1} + (n-1)q^{n-2}p \times 2 + \frac{(n-1)(n-2)}{2!} \times q^{n-3}p^2 \times 3 + \ldots \text{ to } n \text{ terms}\right]$$

$= np \times$ first moment about $x = -1$ of the terms of $(q + p)^{n-1}$

i.e. $\mu_2 = np[(n - 1)p + 1]$ . . . . . . (XIII.14)

The mean value is $\mu_1 = np$ . . . . . . (XIII.15)

The mean square s'^2 about $x = 0$ is

$$s'^2 = \mu_2 = np[\overline{n - 1}\, p + 1] \quad . \quad . \quad \text{(XIII.16)}$$

Substituting this in (XIII.7) and writing np for d

$$\sigma^2 = n^2p^2 - np^2 + np - n^2p^2$$
$$= np(1 - p)$$

and $$\sigma = \sqrt{npq} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.17)}$$

For the Poisson distribution (XIII.12)

$$\mu_1 = e^{-m}\left[1 \times 0 + m \times 1 + \frac{m^2}{2!} \times 2 + \frac{m^3}{3!} \times 3 + \ldots + \text{to } n + 1 \text{ terms}\right]$$

$$= me^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots + \text{to } n \text{ terms}\right]$$

$\therefore \quad \mu_1 = me^{-m} \times e^m$, approximately when n is large

$$\mu_1 = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.18)}$$

The mean value is $\mu_1 = m = np$ as for the binomial distribution.

$$\mu_2 = e^{-m}\left[1 \times 0 + m \times 1^2 + \frac{m^2}{2!} \times 2^2 + \frac{m^3}{3!} \times 3^2 + \ldots \text{to } n + 1 \text{ terms}\right]$$

$$= me^{-m}\left[1 \times 1 + m \times 2 + \frac{m^2}{2!} \times 3 + \frac{m^3}{3!} \times 4 + \ldots \text{to } n \text{ terms}\right]$$

$= m \times$ first moment about $x = -1$ of the first n terms of (XIII.12)

i.e. $\mu_2 = m(m + 1)$ approximately . . . . . (XIII.19)

The mean square s'^2 about $x = 0$ is $\mu_2' = m(m+1)$ and

$$\sigma^2 = s'^2 - d^2$$
$$= m(m+1) - m^2$$
$$= m$$

and $$\sigma = \sqrt{m} = \sqrt{np} \quad . \qquad . \text{ (XIII.20)}$$

From (XIII.18) and (XIII.20) we see that for the Poisson Distribution the standard deviation is the square root of the mean. The value of σ from (XIII.20) is the same as that from (XIII.17) if q is put equal to unity.

EXAMPLE

On the average 3 per cent of the articles produced by a machine are defectives. Find the probability that there will be c defectives in a sample of 40 taken at random from the production, where c has the values 0, 1, 2, 3, 4, 5, 6 in succession.

Assuming the distribution over a large number of samples to be binomial, the probabilities of 0, 1, 2, 3, etc., defectives are found by substituting $n = 40$, $p = 0{\cdot}03$, $q = 0{\cdot}97$ in the successive terms of (XIII.11). These probabilities are given in the second row of Table VI. Similarly, assuming the distribution to be Poisson's, we substitute in (XIII.12) putting $m = np = 1{\cdot}2$; the values of the successive terms are given in the third row of the Table.

TABLE VI

Number of defectives in samples of 40 from population with 3 per cent defectives .	0	1	2	3	4	5	6
P_B = Probability in Binomial Distribution . .	0·2957	0·3668	0·2206	0·0866	0·0248	0·0055	0·0010
P_P = Probability in Poisson Distribution . .	0·3012	0·3614	0·2169	0·0867	0·0260	0·0062	0·0012

More accurate calculation is needed to find the probabilities of samples with more than 6 defectives. Table VII refers only to the Poisson distribution. The first row gives the cumulative sums of the numbers in the third row of the above table.

TABLE VII

POISSON DISTRIBUTION

Cumulative Sums of probabilities in above table = S	0·3012	0·6626	0·8795	0·9662	0·9922	0·9984	0·9996
Values of $1-S$	0·6988	0·3374	0·1205	0·0338	0·0078	0·0016	0·0004

Consider the value $S = 0{\cdot}8795$. This is the sum of the probabilities of 0, 1, and 2 defectives so that $1-S = 1-0{\cdot}8795$ is the probability of 3 or more defectives. Values of $1-S$ represent in order the probabilities of 1 or more, 2 or more, 3 or more, etc., defectives as indicated in Table VIII.

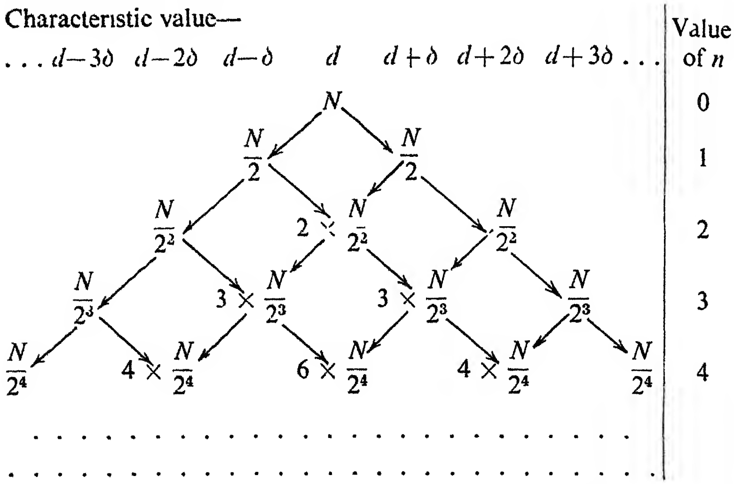
TABLE VIII

Number of defectives in samples of 40 with 3 per cent defective	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more
Probability in Poisson Distribution	0·6988	0·3374	0·1205	0·0338	0·0078	0·0016	0·0004

These probabilities apply to all cases in which $np = 1{\cdot}2$, e.g. $n = 30$, $p = 0{\cdot}04$; $n = 20$, $p = 0{\cdot}06$; etc. For small values of n or large values of p, the Poisson distribution does not apply and probabilities should be calculated from the Binomial distribution.

149. **Cumulative Effect of Small Errors: Normal Frequency Curve.** We shall assume that the deviations from the true value d, say, amongst the members of a population N are due to a large number of small errors affecting each characteristic. For simplicity we shall assume that all the errors are equal in absolute magnitude and of amount δ, and that errors are just as likely to be negative as positive. Due to the first error half the population $\frac{1}{2}N$ will have the characteristic value $d-\delta$, and the other $\frac{1}{2}N$ will have the characteristic value $d+\delta$. Due to the second error the former group will form two groups, the one $\frac{1}{2^2}N$ with the value $d-2\delta$ and the other $\frac{1}{2^2}N$ with the value $d-\delta+\delta$, i.e. d; whilst the latter group will also form two groups, the one $\frac{1}{2^2}N$ with the value d and the other

$\frac{1}{2^2}N$ with the value $d + 2\delta$. Due to a third error, each of these groups will be split into two equal groups, one with its value decreased by δ and the other with its value increased by δ. This process is continued for successive errors, as shown up to the fourth error in the following scheme—



The arrows indicate that each half of any group splits up into two equal parts, one with value increased by δ and the other with value decreased by δ. The successive rows are seen to be the terms in the expansion of $N(a + b)^n$, where $a = b = \frac{1}{2}$ and n has the successive values 0, 1, 2, 3, 4, ... For n errors the corresponding row will give the successive terms in the expansion of $N(\frac{1}{2} + \frac{1}{2})^n$, i.e. $\frac{N}{2^n}(c + d)^n$, where c and d are each given the value unity. Thus, the terms of the distribution are those of the binomial series

$$\frac{N}{2^n}\left[1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \ldots\right.$$

$$\left. + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} + \ldots + \frac{n(n-1)}{2!} + n + 1\right]$$

The $(r + 1)^{\text{th}}$ term is

$$\frac{N}{2^n} \cdot \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \text{ or } \frac{N}{2^n}\,{}^nC_r$$

We shall now find the limiting form of this distribution as n becomes very large, and we shall assume that n is even and equal to $2k$. Let y_0 be the value of the central frequency and y_x the value of the xth frequency to the right of the central one.

Then $\quad y_0 =$ central term in expansion of $\frac{N}{2^{2k}}(1+1)^{2k}$

$$= (k+1)\text{ term in expansion of } \frac{N}{2^{2k}}(1+1)^{2k}$$

$$= \frac{N}{2^{2k}} \times \frac{2k(2k-1)(2k-2)\ldots(2k-k+1)}{\underline{|k}}$$

i.e.
$$y_0 = \frac{\underline{|2k}}{\underline{|k}\,\underline{|k}} \times \frac{N}{2^{2k}} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.21)}$$

Also y_x is the $(k+x+1)^{\text{th}}$ term in expansion of $\frac{N}{2^{2k}}(1+1)^{2k}$

$$\therefore \quad y_x = \frac{N}{2^{2k}} \times \frac{2k(2k-1)(2k-2)\ldots(2k-k-x+1)}{\underline{|k+x}}$$

i.e.
$$y_x = \frac{\underline{|2k}}{\underline{|k+x}\,\underline{|k-x}} \times \frac{N}{2^{2k}} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.22)}$$

From these two relations,

$$y_x = y_0 \times \frac{\underline{|k}\,\underline{|k}}{\underline{|k+x}\,\underline{|k-x}}$$

Now
$$\frac{\underline{|k}}{\underline{|k+x}} = \frac{1}{(k+x)(k+x-1)(k+x-2)\ldots(k+1)},$$

and
$$\frac{\underline{|k}}{\underline{|k-x}} = k(k-1)(k-2)\ldots(k-x+1)$$

$$\therefore \quad y_x = y_0 \times \frac{k(k-1)(k-2)\ldots(k-x+1)}{(k+x)(k+x-1)\ldots(k+1)}$$

Dividing numerator and denominator by k^x and rearranging, we have

$$y_x = y_0 \times \frac{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\left(1-\frac{3}{k}\right)\ldots\left(1-\frac{x-1}{k}\right)}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\ldots\left(1+\frac{x-1}{k}\right)\left(1+\frac{x}{k}\right)}$$

$$. \quad . \quad . \quad . \quad . \quad \text{(XIII.23)}$$

For small values of z, $\log_e(1+z) \simeq z$; hence, taking logarithms of both sides in (XIII.23) and neglecting the second and higher powers of $\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \ldots, \frac{x}{k}$, we have

$$\log_e \frac{y_x}{y_0} = -\left(\frac{1}{k}+\frac{2}{k}+\frac{3}{k}+\ldots+\frac{x-1}{k}\right)$$
$$-\left(\frac{1}{k}+\frac{2}{k}+\frac{3}{k}+\ldots+\frac{x}{k}\right)$$
$$-\frac{2}{k}\left(1+2+3+\ldots+x-1\right)-\frac{x}{k}$$
$$=-\frac{x(x-1)}{k}-\frac{x}{k}$$

i.e.
$$\log_e \frac{y_x}{y_0} = -\frac{x^2}{k}$$

$\therefore$
$$y_x = y_0 e^{-\frac{x^2}{k}}$$

or, dropping the suffix x,
$$y = y_0 e^{-\frac{x^2}{k}} \qquad \text{(XIII.24)}$$

y is positive for all values of x, and the values of x extend from $-\infty$ to $+\infty$.

The standard deviation σ is given by (XIII.17) with $p = q = \frac{1}{2}$. Thus $2\sigma^2 = k$ and substituting this value of k in (XIII.24)

$$y = y_0 e^{-\frac{x^2}{2\sigma^2}} \qquad \text{(XIII.25)}$$

This is called the *normal function* and its graph, the *normal curve*, is symmetrical about OY, has a maximum ordinate y_0 when $x = 0$, and lies entirely above OX (Fig. 137). It is easy to show that it has points of inflexion at $x = \pm\sigma$ and is asymptotic to $X'OX$ at $x = \pm\infty$. The whole area under the graph is the total frequency N, so that

$$N = \int_{-\infty}^{\infty} y_0 e^{-\frac{x^2}{2\sigma^2}} dx \qquad \text{(XIII.26)}$$

This integral is evaluated in Volume II, its value being $\sigma y_0\sqrt{2\pi}$, hence $N = \sigma y_0\sqrt{2\pi}$.

Substituting in (XIII.25),
$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad \text{(XIII.27)}$$

Putting $N = 1$,
$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad \text{(XIII.28)}$$

This is the probability distribution. If the whole area under the graph is unity and $a < b$, $\int_a^b y\,dx$ represents the probability P_{ab} that a single characteristic value, chosen at random, lies between a and b. Thus

$$P_{ab} = \frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2\sigma^2}}\,dx \quad . \quad . \quad (\text{XIII.29})$$

Approximate values of P_{ab} may be found by expanding the integrand in powers of x^2 and integrating term by term.

$$P_{ab} = \frac{1}{\sigma\sqrt{2\pi}}\int_a^b \left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{4\sigma^4 2!} - \frac{x^6}{8\sigma^6 3!} + \ldots\right) dx \quad . \quad (\text{XIII.30})$$

i.e.

$$P_{ab} = \frac{1}{\sigma\sqrt{2\pi}}\left[x - \frac{x^3}{2\,.\,3\,.\,\sigma^2} + \frac{x^5}{4\,.\,5\,.\,\sigma^4\,.\,2!} - \frac{x^7}{8\,.\,7\,.\,\sigma^6\,.\,3!} \quad \ldots\right]_a^b$$

$$. \quad . \quad (\text{XIII.31})$$

For example the probability P that the value lies between $x = 0$ and $x = \sigma/2$ is

$$P = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{\sigma}{2} - \frac{\sigma}{8\,.\,2\,.\,3} + \frac{\sigma}{32\,.\,4\,.\,5\,.\,2!} - \ldots\right) = 0{\cdot}1915$$

Table IX gives values of A, the areas to the right of the ordinates in Fig. 137, $N = 1$. Since OA bisects the area, which is unity, when $x = 0$, $A = 0{\cdot}5000$ and when $x = \sigma/2$, $A = 0{\cdot}5000 - 0{\cdot}1915 = 0{\cdot}3085$ as in the table. Values of $1 - A$ will give the areas to the left of the ordinates. More comprehensive tables are given in textbooks on statistics. The numbers to the left of the ordinates in Fig. 137 are σ times the lengths of the ordinates for the case $N = 1$ calculated from (XIII.28). The mean deviation of the distribution is

$$\begin{aligned}
\text{Mean deviation} &= \Sigma xy\Delta x \div \Sigma y\Delta x \\
&= 2\int_0^\infty xy\,dx, \text{ since } \Sigma y\Delta x = 1 \\
&= \frac{2}{\sigma\sqrt{2\pi}}\int_0^\infty x e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{2}{\sigma\sqrt{2\pi}}\left[-\sigma^2 e^{-\frac{x^2}{2\sigma^2}}\right]_0^\infty
\end{aligned}$$

$$\text{i.e. mean deviation} = \sigma\sqrt{\frac{2}{\pi}} = 0{\cdot}7979\sigma$$

or, very nearly, mean deviation $= \frac{4}{5}$ standard deviation.

TABLE IX

x/σ	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
A	0·5000	0·4602	0·4207	0·3821	0·3446	0·3085	0·2743	0·2420	0·2119
x/σ	0·9	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7
A	0·1841	0·1587	0·1357	0·1151	0·0968	0·0808	0·0668	0·0548	0·0446
x/σ	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5	2·6
A	0·0359	0·0287	0·0228	0·0179	0·0139	0·0107	0·0082	0·0062	0·0047
x/σ	2·7	2·8	2·9	3·0	3·1	3·2	3·3	1·96	3·09
A	0·0035	0·0026	0·0019	0·00135	0·00097	0·00069	0·00048	0·0250	0·00100

From Table IX the area between any two ordinates may be found by subtraction as follows, the total area under the curve being taken as unity—

The area between—

$x = -\ \sigma$ and $x = +\ \sigma$ is $2(0{\cdot}5 - 0{\cdot}1587) = 2 \times 0{\cdot}3413 = 0{\cdot}6826$

$x = -\ 2\sigma$ and $x = +\ 2\sigma$ is $2(0{\cdot}5 - 0{\cdot}0228) = 2 \times 0{\cdot}4772 = 0{\cdot}9544$

$x = -\ 3\sigma$ and $x = +\ 3\sigma$ is $2(0{\cdot}5 - 0{\cdot}00135) = 2 \times 0{\cdot}49865 = 0{\cdot}9973$

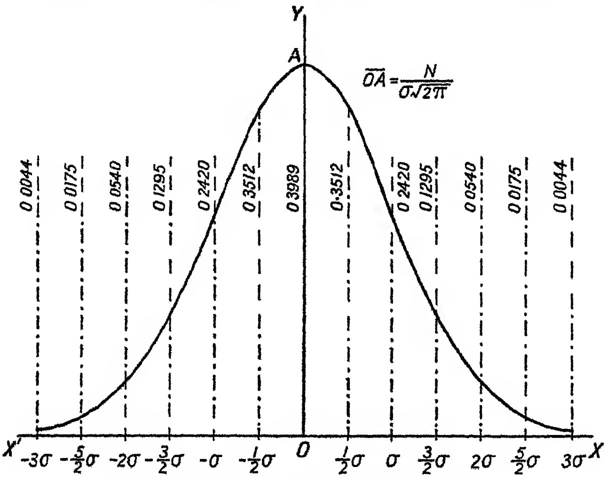


FIG. 137. NORMAL DISTRIBUTION CURVE

These numbers represent the proportions of the characteristic values which fall inside the given limits.

68·26 per cent of the values fall between the limits $x = \pm\sigma$
95·44 per cent of the values fall between the limits $x = \pm 2\sigma$
and 99·73 per cent of the values fall between the limits $x = \pm 3\sigma$

Two limits of importance are $x = -1{\cdot}96\sigma$ to $x = +1{\cdot}96\sigma$ and $x = -3{\cdot}09\sigma$ to $x = +3{\cdot}09\sigma$. The areas between these limits can be found from the values in Table IX.

Area between $x = -1{\cdot}96\sigma$ and $x = +1{\cdot}96\sigma$
$$= 2(0{\cdot}50000 - 0{\cdot}02500) = 0{\cdot}95000$$
Area between $x = -3{\cdot}09\sigma$ and $x = +3{\cdot}09\sigma$
$$= 2(0{\cdot}50000 - 0{\cdot}00100) = 0{\cdot}99800$$

These show that in a normal distribution only 5 per cent, i.e. 1/20th of the values, fall outside the limits $x = \pm 1{\cdot}96\sigma$, and that of these half fall beyond each limit; also that only 0·002 per cent or 2/1000ths of the values fall outside the limits $x = \pm 3{\cdot}09\sigma$ and that of these half fall beyond each limit. The chance or probability that one characteristic value chosen at random from a normal distribution shall fall between given limits is the ratio of the frequency contained between those limits to the total frequency, that is, the ratio of the area between those limits to the total area. As the total area is assumed to be unity the chance is given by the area between the limits. With limits $\pm 1{\cdot}96\sigma$ the chance that a value taken from the population will fall between the limits is 95/100; the chance that it will fall below the lower limit or above the upper limit is 1/40 in each case. With limits $\pm 3{\cdot}09\sigma$ the chance that the value will fall between the limits is 998/1 000; the chance that it will fall below the lower limit or above the upper limit is 1/1 000 in each case.

EXAMPLE 1

In a normal frequency distribution containing 1 000 characteristic values the mean is 45 and the standard deviation is 5. What would be the frequency of values between the values 50 and 60? What is likely to be the whole range of values?

$$\bar{x} = \text{mean} = 45,\ \sigma = 5,\ 50 = \bar{x} + \sigma,\ 60 = \bar{x} + 3\sigma$$

From the Table, if P is the probability of a value falling between $\bar{x} + \sigma$ and $\bar{x} + 3\sigma$,
$$P = 0{\cdot}1587 - 0{\cdot}0013 = 0{\cdot}1574$$
and the frequency of values between 50 and 60 is $1\,000 \times 0{\cdot}1574$, say 157. If we take the limits $\bar{x} \pm 3{\cdot}09\sigma$ the range will contain all but 2 of the 1 000 values, one at each end. The range is therefore very nearly $45 \pm 5 \times 3{\cdot}09$, i.e. 29·5 to 60·5.

EXAMPLE 2

Assume that the diameters of 1 000 brass plugs taken consecutively from a machine form a normal distribution with mean 0·7515 in. and standard deviation

$\sigma = 0{\cdot}0020$ in. How many of the plugs are likely to be rejected if the diameter is to be $0{\cdot}752 \pm 0{\cdot}004$ in.?

Greatest diameter allowable $= 0{\cdot}756$ in.

Least diameter allowable $= 0{\cdot}748$ in.

Greatest diameter—actual mean diameter $= 0{\cdot}0045$ in.

$$= \frac{45}{20}\sigma = 2{\cdot}250\sigma \text{ in.}$$

By proportional parts of the values in Table IX the area to the right of the ordinate $x = 2{\cdot}25\sigma$ in Fig. 137 is 0·0122 and the number of rejects at the upper limit is $1\,000 \times 0{\cdot}0122 = 12$, say.

Actual mean diameter—least diameter $= 0{\cdot}7515 - 0{\cdot}748$

$$= 0{\cdot}0035$$

$$= \frac{35}{20}\sigma = 1{\cdot}75\sigma \text{ in.}$$

From the Table, as before, the area to the right of the ordinate at $x = 1{\cdot}75\sigma$ is 0·0401 and the number of rejects at the lower limit is $1\,000 \times 0{\cdot}0401 = 40$, say. The total number of rejects is about 52 or roughly 5 per cent.

The normal distribution occurs very often in practice though there are other types of distribution. For example, the diameters of a large number of balls for ball bearings, produced under control by a single machine, would form a close approximation to a normal distribution. The weights of these balls, being proportional to the diameter cubed, would form a skew distribution with the longer tail on the right. The reader is warned against assuming without sufficient evidence that any distribution occurring in engineering statistics is normal and against insisting too strongly on the strict numerical accuracy of his results.

150. **Quartiles, Deciles.** If in a frequency distribution we determine two characteristic values Q_1 and Q_3 so that $\frac{1}{4}$ of the values are less than the former and $\frac{1}{4}$ of them are greater than the latter, then Q_1 and Q_3 are known respectively as the *lower quartile* and the *upper quartile*. With the median they divide the distribution into four parts each containing a quarter of the values. $\frac{1}{2}(Q_3 - Q_1)$ is known as the *quartile deviation* or the *semi-interquartile range* and is a measure of dispersion sometimes used in place of standard deviation or mean deviation. The characteristic values which divide the population arranged in order of magnitude into ten numerically equal groups are known as *deciles*. In a symmetrical distribution the two quartiles Q_1, Q_3 are equidistant from the median M_t and the quartile deviation is equal to $Q_3 - M_t$ and to $M_t - Q_1$. In this case the median and the mean coincide and half the deviations from the mean will be greater than, and half less than, the quartile deviation. On this account the quartile deviation used to be known as the *probable error*,

a term now little used. For a normal distribution the quartiles can be found by interpolation in Table IX and are such that

$$\text{Quartile deviation} = 0{\cdot}6745 \times \text{standard deviation}$$

151. **Sampling Distribution of the Mean.** When characteristic values can be measured without injuring the articles, as in the case of a measured dimension, 100 per cent inspection is possible though usually inexpedient because of its cost. Where tests injure the articles, as in tests of strength or durability, only comparatively few of the articles can be tested. In any case the usual procedure is to take from production a series of samples each containing the same number of articles, usually from 2 to 12, for engineering dimensions, but up to 100 or more in some cases. The mean value of the characteristics is found for each sample and the range of values in each sample. The distribution of the mean value over a number of samples is called *the sampling distribution of the mean.* In the same way we can form a sampling distribution of the range or of the standard deviation of the samples. The sampling distribution of the mean over a large number of samples is a close approximation to the normal distribution, much closer than is the parent distribution, and we usually assume that with production under careful control the sampling distribution of the mean is normal. Suppose we take a large number m of samples of n each from a universe of values. Let the values in any one sample be $x_1, x_2, x_3, \ldots x_{n-1}, x_n$ all measurements being made from the universe mean $\bar{X}$, i.e. if X_r is an actual measurement $x_r = X_r - \bar{X}$. The sample mean of the sample is $\dfrac{1}{n}\sum\limits_{r=1}^{r=n} x_r$ and its square is $\dfrac{1}{n^2}\sum\limits_{r=1}^{r=n} x_r^2 + \dfrac{2P_1}{n^2}$, where P_1 is the sum of the products of all possible pairs of values in the sample. If we find the square of the means in all the m samples and add them we have

$$\text{Sum of squares of sample means} = \frac{1}{n^2}\sum_{r=1}^{r=mn} x_r^2 + \frac{2}{n^2}(P_1 + P_2 + P_3 + \ldots + P_m) \quad \text{(XIII.32)}$$

where P_r is the sum of the products of pairs of values of x in the rth sample. The quantity in the brackets is the sum of a large number of products of pairs of values of x selected at random from the population; hence there is no correlation between the sets of values forming the pairs. Since about half the products are positive and half negative their sum is negligible compared with Σx_r^2. The sum of the squares of the sample means of which there are m is $m\sigma_{\bar{x}}^2$ where $\sigma_{\bar{x}}$ is the

standard deviation of the mean and $\sum_{i=1}^{i=mn} x_i^2$ is $mn\sigma^2$ where, since m is large, σ is a close approximation to the standard deviation of the universe. From (XIII.32) then we have

$$m\,\sigma_{\bar{x}}^2 = \frac{1}{n^2}\,mn\sigma^2$$

or
$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$$

from which
$$\sigma_{\bar{x}} = \sigma/\sqrt{n} \quad . \quad . \quad . \quad \text{(XIII.33)}$$

i.e. the standard deviation of the sampling distribution of the mean, also known as the *standard error of the mean*, is $\frac{1}{\sqrt{n}}$ times the standard deviation of the whole population. The principal use of statistical theory to the engineer lies in its providing a method for the careful supervision of a machine and its products so as to prevent avoidable production of articles not up to standard. This method is known as *quality control.*

EXAMPLE

The mean strength of yarn found from Table I was shown in Ex. 1, Art. 145, to be 20·09 quarter-ounces and its standard deviation 2·54 quarter-ounces. How accurate is this mean?

The values $x = 20{\cdot}09$ and $\sigma = 2{\cdot}54$ are those of a single sample of 100 and not those of the population. With a sample of this size the values will be fairly close to the population values. We have on substitution in (XIII.33)

$$\sigma_{\bar{x}} = \frac{2{\cdot}54}{\sqrt{100}}$$
$$= 0{\cdot}254 \text{ quarter-ounces}$$

and
$$3{\cdot}09\sigma_{\bar{x}} = 0{\cdot}785$$

As only 1 in 1 000 values fall beyond each of the limits $\bar{x} \pm 3{\cdot}09\sigma_{\bar{x}}$ it is practically certain that the correct value of $\bar{x}$ lies between $20{\cdot}09 + 0{\cdot}79$ and $20{\cdot}09 - 0{\cdot}79$, i.e. between 19·3 and 20·9 quarter-ounces.

152. **Standard Deviation Estimated from a Sample.** The estimate of σ, the standard deviation of a universe, found from a sample is in error because (1) it varies from sample to sample even when these have the same mean, and (2) the sample mean and the universe means are not usually the same. By (XIII.7) the value of σ^2 found from the sample is always less than it should be by the square of the difference between the two means. The error due to (1) is unbiased and is unavoidable. It can be reduced only by taking larger samples. That due to (2) is biased as it always reduces the value of σ and can be allowed for.

Suppose we estimate the value of σ from a sample of size n. Let $\bar{x}$ and $\bar{X}$ be the sample and universe means respectively. Then we can improve on the estimate from the sample by taking deviation from the universe mean, which is unknown. This seems impossible but there is a way out of the difficulty. Let σ_e be the estimate of σ from the sample about the universe mean; then

$$\begin{aligned} n\sigma_e^2 &= \Sigma(x - \bar{X})^2 \\ &= \Sigma[(x - \bar{x}) + (\bar{x} - \bar{X})]^2 \\ &= \Sigma(x - \bar{x})^2 + 2(\bar{x} - \bar{X})\,\Sigma(x - \bar{x}) + n(\bar{x} - \bar{X})^2 \end{aligned}$$

The first term on the right is $n\sigma_s^2$, where σ_s is the standard deviation in the sample itself, and the second term is zero. We have then

$$\sigma_e^2 = \sigma_s^2 + (\bar{x} - \bar{X})^2$$

Over a large number of terms $(\bar{x} - \bar{X})^2$ would form a sampling distribution of which the mean value would be σ^2/n. Substituting this mean value for $(\bar{x} - \bar{X})^2$ we have

$$\sigma_e^2 = \sigma_s^2 + \sigma^2/n$$

$\sigma_s^2 = \dfrac{1}{n}\,(x - \bar{x})^2$ and σ_e is our estimate of σ, hence we may write

$$\sigma^2 = \frac{(x - \bar{x})^2}{n} + \frac{\sigma^2}{n} \text{ which reduces to}$$

$$\sigma^2 = \frac{(x - \bar{x})^2}{n - 1} \qquad . \quad . \quad . \quad . \quad . \quad . \text{(XIII.34)}$$

Thus instead of (XIII.5) we have

$$\sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{\Sigma f - 1}} = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{N - 1}} \qquad \text{(XIII.35)}$$

153. **Correlation.** When finding laws of graphs in Chapter VIII the type of law was usually suggested by theory and values of one variable were assumed to be known exactly whilst those of the other were subject to error, though actually values of both variables are subject to error. We also assumed that for any given value of one variable there is a unique value of the other. Many cases occur in social and industrial statistics in which this last is not true. If, for example, the two variables are the height and weight of the boys in a school, or the output and the expense of upkeep per week of a works, we shall find that a given value of one variable will usually be associated with two or more values of the other. This is not so because of errors in the estimates of the variables but occurs because of other variations not directly connected with the two variables. In such cases the variables are called *variates* and it is necessary to express in

some form the relationship between the variates. Fig. 138 shows the kind of diagram, known as a *scatter diagram*, which might be obtained on plotting values of pairs of variates. The points seem to have the trend of a curve, or it might be a straight line, and if the equation of this curve is found it will represent the cluster of points only in a very limited manner. In some cases there will be no apparent trend.

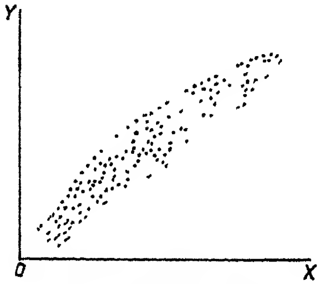


FIG. 138. SCATTER DIAGRAM

When there is a definite trend we say that the two variates are correlated; the more definite the trend, the closer the *correlation*. We shall consider only the case where the trend is along a straight line which we call *linear correlation*. Where the trend is curvilinear it may be possible by a change of variate or variates to change it to a linear trend. With a linear trend we could determine the equations of two straight lines between which all the points just lie but many pairs of such lines could be found and the use of two equations to represent the trend would unduly complicate any further analysis. We try therefore to find a single relation which represents the trend and look upon deviations from this relation as "errors."

Suppose we have n pairs of corresponding values of two variates X and Y and that on plotting these we find the points distributed more or less closely about a straight line. We wish to find the best expression $Y = mX + c$ between the variates. We use the method of least squares varied so as to suit cases in which n is large. As we have no reason to treat one variable as constant in preference to the other, we shall find the equation in two ways (1) treating X as exact and (2) treating Y as exact.

(1) Let $Y = mX + c$ be the equation to the best line assuming values of X to be exact, and those of Y to be subject to error. Let E be the sum of the squares of the errors, i.e. of the differences between the given values of Y and those calculated from the equation. Then as in Example 1, Art. 105,

$$E = \Sigma(Y - \overline{mX + c})^2 \quad . \quad . \quad (\text{XIII.36})$$

For E to be a minimum $\dfrac{\partial E}{\partial c} = 0$ and $\dfrac{\partial E}{\partial m} = 0$ from which, after division by -2, we have respectively

$$\Sigma(Y - mX - c) = 0 \quad . \quad . \quad . \quad (\text{XIII.37})$$

and $$\Sigma[X(Y - mX - c)] = 0 \quad . \quad . \quad . \quad \text{(XIII.38)}$$

From (XIII.37) $$\Sigma Y = m\Sigma X + nc$$

But, if $\bar{X}$ and $\bar{Y}$ are the mean values of X and Y respectively, $\Sigma Y = n\bar{Y}$ and $\Sigma X = n\bar{X}$,

$$\therefore \qquad \bar{Y} = m\bar{X} + c \quad . \quad . \quad . \quad \text{(XIII.39)}$$

which shows that the line passes through the point $(\bar{X}, \bar{Y})$ called the *mean centre.*

From (XIII.38), $$\Sigma XY - m\Sigma X^2 - c\Sigma X = 0$$

$$\therefore \qquad \Sigma XY - m\Sigma X^2 - (\bar{Y} - m\bar{X})\Sigma X = 0$$

i.e. $$m = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2} \quad . \quad . \quad . \quad \text{(XIII.40)}$$

The equation to the best straight line is therefore

$$Y - \bar{Y} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2}(X - \bar{X}) \quad . \quad . \quad \text{(XIII.41)}$$

(2) In the same way, assuming values of Y to be exact and those of X to be in error, the equation of the best line is

$$X - \bar{X} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma Y^2 - n\bar{Y}^2}(Y - \bar{Y}) \quad . \quad . \quad \text{(XIII.42)}$$

Now change the origin to the point $(\bar{X}, \bar{Y})$, keeping the direction of the axes unchanged. Let x and y be the new co-ordinates, i.e., $x = X - \bar{X}$, $y = Y - \bar{Y}$; then

$$\begin{aligned} \Sigma XY &= \Sigma(x + \bar{X})(y + \bar{Y}) \\ &= \Sigma xy + \bar{Y}\Sigma x + \bar{X}\Sigma y + n\bar{X}\bar{Y} \\ &= \Sigma xy + n\bar{X}\bar{Y}, \text{ since } \Sigma x = \Sigma y = 0 \end{aligned}$$

Hence $\Sigma XY - n\bar{X}\bar{Y} = \Sigma xy$

Also by (XIII.7)

$$\Sigma X^2 - n\bar{X}^2 = n\sigma_x^2$$

and $\Sigma Y^2 - n\bar{Y}^2 = n\sigma_y^2$, where σ_x and σ_y are, respectively, the standard deviations of X and Y. Substituting these values in (XIII.41) we have for the best line

$$y = \frac{\Sigma xy}{n\sigma_x^2} \cdot x \quad . \quad . \quad . \quad \text{(XIII.43)}$$

or $$\frac{y}{\sigma_y} = \frac{\Sigma xy}{n\sigma_x\sigma_y} \cdot \frac{x}{\sigma_x} \quad . \quad . \quad . \quad \text{(XIII.44)}$$

Similarly (XIII.42) becomes

$$x = \frac{\Sigma xy}{n\sigma_y^2} \cdot y \quad . \quad . \quad . \quad . \quad (\text{XIII.45})$$

or

$$\frac{x}{\sigma_x} = \frac{\Sigma xy}{n\sigma_x\sigma_y} \cdot \frac{y}{\sigma_y} \quad . \quad . \quad . \quad (\text{XIII.46})$$

Writing

$$r = \frac{\Sigma xy}{n\sigma_x\sigma_y} = \frac{\Sigma xy}{\sqrt{\Sigma x^2\,\Sigma y^2}} \quad . \quad (\text{XIII.47})$$

in (XIII.44) and (XIII.46),

$$\frac{y}{\sigma_y} = r\frac{x}{\sigma_x} \text{ or } Y - \bar{Y} = \frac{r\sigma_y}{\sigma_x}(X - \bar{X}) \quad . \quad (\text{XIII.48})$$

and

$$\frac{x}{\sigma_x} = r\frac{y}{\sigma_y} \text{ or } X - \bar{X} = \frac{r\sigma_x}{\sigma_y}(Y - \bar{Y}) \quad . \quad (\text{XIII.49})$$

These are the *equations of regression* of y on x (or Y on X) and of x on y (or X on Y) respectively and their graphs are the *lines of regression* of y on x and of x on y respectively. If σ_x is taken as the unit for the x values and σ_y as that for the y values, then with these *standard units* the equations of regression become

$$y = rx \text{ or } Y - \bar{Y} = r(X - \bar{X}) \quad . \quad . \quad (\text{XIII.50})$$

and

$$x = ry \text{ or } X - \bar{X} = r(Y - \bar{Y}) \quad . \quad . \quad (\text{XIII.51})$$

and the lines of regression plotted on the same scale along both axes are equally inclined to the line $y = x$ when r is positive and to the line $y = -x$ when r is negative. When $r = 1$ they coincide with $y = x$ and when $r = -1$ they coincide with $y = -x$. In both cases a change in one variate produces an equal change in the other, and we say that there is *perfect correlation* between the variates, positive in the former case and negative in the latter. If $r = 0$ the lines of regression become the axes of co-ordinates $y = 0$ and $x = 0$ respectively and, as these are perpendicular lines, a change in one variate is not associated with a change in the other and there is no correlation between the variates.

From (XIII.36), changing the co-ordinates to x and y and putting $m = \frac{r\sigma_y}{\sigma_x}, c = 0,$

$$E = \Sigma\left(y - rx\frac{\sigma_y}{\sigma_x}\right)^2$$

$$= \Sigma y^2 - \frac{2r\sigma_y}{\sigma_x}\Sigma xy + \frac{r^2\sigma_y^2}{\sigma_x^2}\Sigma x^2$$

$$= \Sigma y^2 - 2nr^2\sigma_y^2 + nr^2\sigma_y^2$$

or

$$E = (1 - r^2)\Sigma y^2$$

This is the least value of the sum of the squares of the errors on assumption (1) above and is, therefore, essentially positive. Thus r can only have values between -1, when there is perfect negative correlation, and $+1$, when there is perfect positive correlation; there is no correlation when $r = 0$. Thus r provides us with a measure of the degree of correlation between the variates and is known as the *coefficient of correlation.*

It can be shown on the assumption that r is distributed normally that its standard deviation is $\dfrac{(1-r^2)}{\sqrt{n}}$. This value may be used for large values of n when r is not too small. There is, therefore, practical certainty that r lies within the range $r \pm \dfrac{3(1-r^2)}{\sqrt{n}}$ and a 1 in 20 chance that r will fall outside the range $r \pm \dfrac{2(1-r^2)}{\sqrt{n}}$. If, for example, $r = 0{\cdot}2$ and $n = 81$, then $r \pm \dfrac{2(1-r^2)}{\sqrt{n}} = 0{\cdot}2 \pm 0{\cdot}214$ and there is a 1 in 20 chance that r lies outside the limits $-0{\cdot}014$ to $0{\cdot}414$. As this range includes negative values the value $r = 0{\cdot}2$ is not significant and there is not satisfactory evidence of correlation. We usually conclude that if r is greater than twice its standard deviation the correlation is significant though this test is not always reliable. For small values of n special methods are required for deciding which values of r are significant. Table X gives values of r which are just significant by the above tests for values of n from 10 to 100. For higher values of r the correlation is more significant.

TABLE X

n	10	15	20	25	30	35	40	45	50
r	0·63	0·51	0·44	0·40	0·36	0·34	0·31	0·29	0·28

n	55	60	65	70	75	80	85	90	100
r	0·27	0·25	0·24	0·23	0·22	0·22	0·21	0·20	0·20

Where the value of n is very large the above method of finding the correlation coefficient and the regression equations becomes very tedious and an alternative method is used. This consists of dividing the range of each variate into an equal number of sub-ranges, not

necessarily the same for the two variates. By drawing a horizontal line to represent one variate and a vertical line to represent the other, completing the rectangle, and drawing horizontal and vertical lines through the sub-range division points, we have a cell for each combination of sub-ranges, one from each variate. By making a dot in each square for each pair of variates falling in the two sub-ranges and counting the dots, the frequency is found for each cell. It thus becomes a simple matter to calculate the quantities involved in the regression equations and in the expression for r. For examples of this method readers are referred to textbooks on the subject.

The above method of finding the correlation coefficient is known as the *product-moment* method. Table XI gives, in the first and second columns, 25 pairs of corresponding values of two variates X and Y. We show how to calculate the value of r and how to find the equations to the lines of regression.

TABLE XI

X	Y	X^2	Y^2	XY
0·2	8·8	0·04	77·44	1·76
0·7	10·2	0·49	104·04	7·14
1·3	9·5	1·69	90·25	12·35
1·8	10·7	3·24	114·49	19·26
2·2	11·6	4·84	134 56	25·52
2·9	10·5	8·41	110·25	30 45
3·3	11·3	10 89	127·69	37·29
3 8	12·9	14·44	166·41	49 02
4·3	11·3	18 49	127·69	48 59
4·6	12 5	21·16	156 25	57 50
5·4	14 4	29·16	207·36	77·76
5·9	12·5	34·81	156·25	73·75
6·2	13 3	38·44	176·89	81·26
6·6	14·8	43·56	219·04	97·68
7·1	14·3	50·41	204·49	101·53
7·7	14·0	59·29	196·00	107·80
8·2	15·4	67·24	237·16	126·28
8·9	14·8	79·21	219 04	131·72
9·3	16·4	86·49	268·96	152·52
9·8	14·8	96·04	219·04	145·04
10·3	16·7	106·09	278·89	172·01
10 5	15·8	110·25	249·64	165·90
11 0	15·5	121·00	240·25	170·50
11·4	17·3	129·96	299·29	197·22
11·9	16·9	141·61	285·61	201·11
155·3	336·2	1277·25	4666·98	2290·96 TOTAL

$$\bar{X} = \frac{\Sigma X}{n} = 6·212$$

$$\bar{Y} = \frac{\Sigma Y}{n} = 13·448$$

$$\Sigma xy = \Sigma XY - n\bar{X}\bar{Y}$$
$$= 2\,290\,96 - 25 \times 6·212 \times 13·448$$
$$= 202\,49$$

$$\sigma_x^2 = \frac{\Sigma x^2}{25} = \frac{\Sigma X^2}{25} - \bar{X}^2$$
$$= \frac{1\,277·25}{25} - 6\,212^2$$
$$= 12·501$$
$$\sigma_x = 3·537$$

$$\sigma_y^2 = \frac{\Sigma y^2}{25} = \frac{\Sigma Y^2}{25} - \bar{Y}^2$$
$$= \frac{4\,666\,98}{25} - 13\,448^2$$
$$= 5·819$$
$$\sigma_y = 2·412$$

From (XIII.47) $r = \dfrac{\Sigma xy}{n\sigma_x\sigma_y}$

$$r = \frac{202·49}{25 \times 3·537 \times 2·412}$$
$$= 0·949$$

From the calculations at the side of the table we see that $r = 0{\cdot}95$ indicating a very high degree of correlation between X and Y.

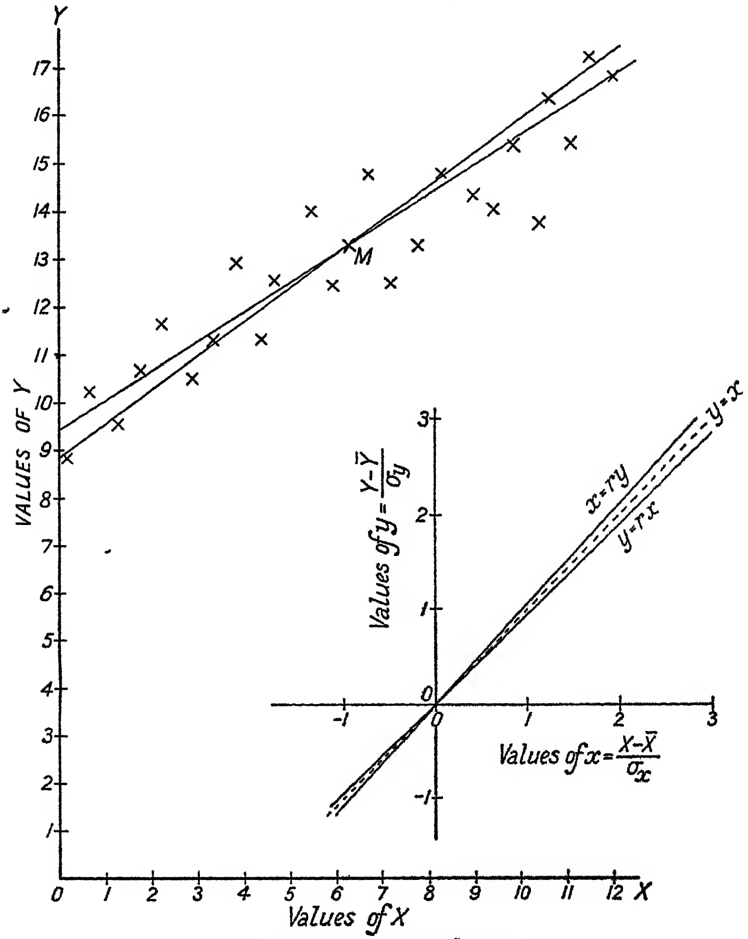


FIG. 139. REGRESSION LINES

From (XIII.48) on substitution

$$Y - 13{\cdot}448 = 0{\cdot}949 \times \frac{2{\cdot}412}{3{\cdot}537}(X - 6{\cdot}212)$$

i.e.
$$Y = 0{\cdot}647X + 9{\cdot}43$$

This is the equation of the line of regression of Y on X.
From (XIII.49),

$$X - 6{\cdot}212 = 0{\cdot}949 \times \frac{3{\cdot}537}{2{\cdot}412}(Y - 13{\cdot}448)$$

i.e.
$$X = 1{\cdot}392Y - 12{\cdot}51$$

is the equation to the line of regression of X on Y. The graphs of these lines are drawn in Fig. 139. Since both lines pass through $M(6{\cdot}21, 13{\cdot}45)$ only one other point need be plotted on each. The inset figure shows the regression lines with standard units and equal scales.

154. **Correlation by Rank.** Sometimes it is more convenient to deal with the rank or order of the values of the two sets of variables than with their actual values. In Table XI the order of the values of X in order of magnitudes is 1, 2, 3, etc. . . . up to 25, whilst the order of the values of Y is as given in Table XII. Note that where two or more values are equal they are given an average rank as follows. The two values 11·3 of Y would, if slightly different, have occupied the sixth and seventh places; they are each given the rank 6·5. Similarly the three values of 14·8 would occupy the sixteenth, seventeenth, and eighteenth places; they are each given the rank 17. These ranks could be taken instead of the values of X and Y and the coefficient of correlation found from them by the above method. This method is not so accurate as the general one but can be simplified so as to be carried out more rapidly. Suppose we have two sets of numbers each containing the digits 1, 2, 3, etc. . . . up to N in this order in one set but in a different order in the other. Looking upon these as our values of X and Y respectively we have

$$\bar{Y} = \bar{X} = \frac{\Sigma N}{N} = \frac{N(N+1)}{2N} = \frac{N+1}{2} \quad . \quad \text{(XIII.52)}$$

also
$$\sigma_x^2 = \sigma_y^2 = \frac{\Sigma X^2}{N} - \bar{X}^2$$

But
$$\Sigma X^2 = 1^2 + 2^2 + 3^2 + \ldots + N^2$$
$$= \frac{N(N+1)(2N+1)}{6}$$

$\therefore$
$$\sigma_x^2 = \sigma_y^2 = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$$

i.e.
$$\sigma_x^2 = \sigma_y^2 = \frac{N^2-1}{12} \quad . \quad . \quad . \quad \text{(XIII.53)}$$

Also
$$\Sigma(X - Y)^2 = \Sigma X^2 - 2\Sigma XY + \Sigma Y^2$$
$$= \frac{N(N+1)(2N+1)}{3} - 2\Sigma XY$$

Now write d for $X - Y$, the difference between the orders of X and Y.

Then $$\Sigma d^2 = \frac{N(N+1)(2N+1)}{3} - 2\Sigma XY$$

Thus we have

$$r = \frac{\Sigma XY - N\bar{X}\bar{Y}}{N\sigma_x\sigma_y}$$

$$= \frac{\frac{N(N+1)(2N+1)}{6} - \frac{\Sigma d^2}{2} - \frac{N(N+1)^2}{4}}{\frac{1}{12}N(N^2-1)}$$

i.e. $$r = 1 - \frac{6\Sigma d^2}{N(N^2-1)} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.54)}$$

TABLE XII

X .	1	2	3	4	5	6	7	8	9	10	11	12	13
Y	1	3	2	5	8	4	6·5	11	6·5	9·5	15	9·5	12
d .	0	1	1	1	3	2	0·5	3	2·5	0·5	4	2·5	1
d^2 .	0	1	1	1	9	4	0·25	9	6·25	0·25	16	6·25	1

X	14	15	16	17	18	19	20	21	22	23	24	25
Y	17	14	13	19	17	22	17	23	21	20	25	24
d .	3	1	3	2	1	3	3	2	1	2	1	1
d^2 .	9	1	9	4	1	9	9	4	1	4	1	1

$$\Sigma d^2 = 108$$

Hence $$r = 1 - \frac{648 \cdot 0}{25 \times 624}$$
$$= 0 \cdot 96$$

which approximates closely to the value found by the more tedious but more exact method.

Correlation by rank is not as reliable as that by measurements; a difference of one in rank may indicate an almost negligible difference or a very significant difference in measurement. The formula given above for the standard deviation of r does not apply to the rank correlation coefficient.

EXAMPLES XIII

(1) The following numbers are records of the weights in ounces of 100 articles, produced by the same machine, all of which are intended to weigh 12 ounces.

11·5	12·1	11·5	11·6	12·2	11·4	11·0	11·5	11·7	12·2
11·7	11·0	11·8	12·0	12·1	11·1	11·0	11·2	10·8	11·5
11·5	12·5	11·5	11·7	12·1	12·0	11·8	12·1	11·8	12·5
11·8	11·8	11·7	11·3	11·6	12·5	12·5	12·5	12·5	12·2
11·6	12·0	11·2	12·3	11·5	11·7	11·6	11·6	11·7	12·1
11·7	11·6	11·0	11·6	11·8	10·8	11·5	11·5	11·1	11·8
11·8	11·1	11·8	12·0	11·8	12·2	11·5	11·3	11·6	12·3
12·2	11·6	11·8	11·7	11·8	12·3	12·1	12·2	12·0	12·6
11·7	12·1	12·3	12·3	12·2	12·1	12·5	12·6	11·8	12·0
11·5	12·4	12·8	12·2	11·7	11·5	11·7	11·5	12·0	11·5

Form these numbers into a frequency distribution of eleven equal class-intervals each of 0·2 ounces covering the range 10·75 to 12·95. From this find the mean weight and the standard deviation (i) by the method of Table III, (ii) by the class interval method. Draw a histogram showing the distribution. Give the value of σ corrected by Sheppard's rule.

(2) Prove that the root mean square deviation of a frequency distribution from a given characteristic value x_0 is $\sqrt{\sigma^2 + (\bar{x} - x_0)^2}$ where σ is the standard deviation and $\bar{x}$ the mean value.

(3) A frequency distribution graph is a rectangle, one side of which is of length l and lies along the axis of values. Show that the standard deviation σ, is given by $12\sigma_r^2 = l^2$. If the rectangle is divided up into a number of vertical strips each of width h, and each rectangle is treated as a grouped frequency at its mid-ordinate, show that σ_g, the standard deviation of the grouped frequency, is given by $\sigma_g^2 = \sigma_r^2 - h^2/12$.

(4) Prove that the second relation of Example 3 is approximately true for the case of a frequency distribution like that shown in Fig. 134.

(5) On the same chart draw the graphs of $y = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}$ for the values $\sigma = \frac{1}{2}$, $\sigma = 1$, $\sigma = 2$, the range for x to be $x = -6$ to $x = +6$.

(6) Find the points of inflexion of the graph in Fig. 137 and show that the tangents at these points cut the x-axis where $x = -2\sigma$ and $x = +2\sigma$.

(7) Find the probability that in a normal distribution a value of the characteristic chosen at random will fall between $x = -3\sigma/2$ and $x = 2\sigma$.

(8) Write down the first four terms of the expansion of $e^{-\frac{x^2}{2\sigma^2}}$. Find an approximate value of $\int_0^{\frac{\sigma}{2}} e^{-\frac{x^2}{2\sigma^2}}\,dx$, and from this find the area under the curve in Fig. 137 between $x = 0$ and $x = \dfrac{\sigma}{2}$. By the same method find the area between $x = 0$ and $x = \sigma$.

(9) Find a normal distribution which has the same total frequency, mean value, and standard deviation as the distribution in Table III, Art. 145. Draw on the same chart and with the same axes and scales the normal curve and the histogram.

(10) A frequency distribution consists of the terms of $384(a + b)^6$ where $a = b = \frac{1}{2}$, the corresponding values being 0, 1, 2, 3, 4, 5, and 6 respectively.

Plot the distribution curve and superpose the normal distribution curve with the same total frequency, mean, and standard deviation.

(11) $y_1, y_2, y_3, \ldots y_n$ are all estimates of a quantity whose true value is y. Show that the method of least square of errors gives the average of y_1, y_2, etc., as the best value of y.

(12) Show that $\int_0^{\infty} e^{-t^2}\, dx = \sqrt{\pi}/2$. Prove that in a normal distribution the mean deviation is very nearly four-fifths of the standard deviation.

(13) Find the standard deviation of the sampling distribution of the mean formed by taking the numbers in Table I, page 417. Take samples of four and assume the samples to be taken in order down the columns from left to right. Take sub-ranges from 15·5 to 16·5, 16·5 to 17·5, etc., and draw up a table showing central values of the mean strength and corresponding frequencies. Where a value falls half-way between central values add one-half frequency to each group. Draw the histogram.

(14) Find the ratio of the mean deviation to the standard deviation for the following symmetrical distributions (i) rectangular, (ii) triangular, (iii) parabolic, vertex upwards.

(15) A rectangular frequency distribution is formed by equal numbers of the digits 1, 3, 5, 7, 9. Find its standard deviation.

(16) The following digits were obtained by pricking at random between the pages of a book of 500 pages and taking note each time of the last digit in the right-hand page number. When grouped the page digits form a rectangular distribution of 50 in each group. The digits found by pricking are as follows, commas dividing them into groups of four for use as samples—

1539, 5715, 1179, 5999, 7377, 3177, 7911, 7583, 3975, 5533, 9731, 9131,
5979, 5973, 3311, 5357, 9573, 7193, 7917, 9379, 9733, 3115, 5755, 1791,
3599, 9151, 9155, 1755, 3771, 9135, 9319, 9371, 5113, 9959, 1931, 1919,
5733, 9519, 3133, 9137, 5777, 1151, 3379, 1399, 3333, 3135, 1119, 3577.

(*a*) Find the frequency for each digit and draw a frequency polygon.

(*b*) Find the sample means and ranges and from these the mean and mean range of the samples. Compare the mean with that of the parent distribution.

(*c*) Calculate σ, the standard deviation, and compare it with that found in Example 14.

(*d*) From the sample means form a frequency distribution and find its standard deviation. Compare this with the value found from (XIII.33).

(17) Leaving out the last two digits in the above table, divide the digits into samples of 5 and work out parts (*b*), (*c*), and (*d*) as in the last Example.

(18) Assuming the binomial distribution in 1 000 samples of 5 from an infinite population containing 6 per cent defectives, how many samples will contain (*a*) no defectives, (*b*) three defectives. Find also the answers to (*a*) and (*b*) assuming a Poisson distribution.

(19) Two per cent of the articles produced by a machine in control are defective. Find the probability that a random sample of 20 will contain no defectives on the assumption that the distribution is (*a*) binomial, (*b*) the Poisson.

(20) Write down in succession the terms of $N(p + q)^4$ where $p + q = 1$. Taking these as class frequencies the characteristic values of which in order are 0, 1, 2, 3, 4, find (*a*) the mean, (*b*) the standard deviation, and show that your results agree with (XIII.15) and (XIII.17).

(21) In a certain village which has remained practically unchanged during the last 20 years there have been five ordinary house fires during that period needing the attention of the local fire brigade. Estimate the chance that if conditions remain unchanged there will be at least one fire during the next four years. Use the Poisson series taking $m = 5 \times 4/20 = 1$ the average number of fires in four years.

(22) Using the following table of values, find the correlation coefficient between X and Y. Is it significant?

X	5	8	10	13	16	18	20	23	26	28
Y	0·82	1·02	1·32	0·75	1·12	1 04	1·24	0·97	1·37	1·13
X	30	32	35	38	40	44	48	52	54	56
Y	1·38	1·35	1·68	1·54	1·39	1·75	1·92	1·76	1·69	1·96

(23) Find the lines of regression in the last example. Plot the points and draw the graphs of the regression lines.

(24) If the correlation coefficient r is twice its standard deviation $(1 - r^2)/\sqrt{N}$ prove that $4r = \sqrt{N + 16} - \sqrt{N}$. Hence find values of r for $N = 112$ and for $N = 128$ for which r is significant. Test this formula against the value given in the table for $n = 10$. Why do not the two values agree?

(25) From 12 pairs of values of X and Y the correlation coefficient is found to be $r = 0{\cdot}16$. Is this significant? Derive the formula for rank correlation from (XIII.47).

(26) Successive samples of four from the production of steel cylinders have diameters of 0·750 in. plus the following in ten thousandths parts of an inch, 75, 78, 76, 76, 79, 80, 83, 58, 57, 70. The corresponding ranges in the samples are respectively in ten-thousandths of an inch, 2, 7, 4, 3, 10, 1, 6, 7, 7, 4. Find the correlation coefficient between the mean and the range. Is this significant?

(27) Find the correlation coefficient between X and Y from the values given. Find also the two regression equations and draw their graphs.

X	1	2	3	4	5	6	7	8	9	10
Y	10	12	16	28	25	36	41	49	40	50

(28) As in (27) but reverse the order of the values of Y.

(29) Take the integers 1 to 10 in order and find the rank correlation with each of the following sets (*a*) 6, 7, 8, 9, 10, 1, 2, 3, 4, 5; (*b*) 1, 6, 2, 7, 3, 8, 4, 9, 5, 10; (*c*) 1, 10, 2, 9, 3, 8, 4, 7, 5, 6; (*d*) 10, 9, 8, 7, 6, 5, 4, 3, 2, 1.

(30) Putting order, or rank, instead of the values of Y in Examples (27) and (28), i.e. the lowest value of Y will be replaced by 1, the next lowest by 2, and so on, find the rank correlations.

(31) If the average of 140 measurements of the lengths of steel studs produced by a machine is 0·1506 in. and the standard deviation of the measurements is 0·0038 in., between what limits does the mean almost certainly lie? Between what limits will it lie with a probability of 0·95?

CHAPTER XIV

MISCELLANEOUS

155. **Scalar and Vector Quantities.** A physical quantity which has direction as well as magnitude is called a *vector* quantity, and one with magnitude but without direction is a *scalar* quantity. A vector has a scalar part, its magnitude, and is represented graphically by a straight line drawn in the direction of the vector, of a length proportional to its magnitude, and marked by an arrow to indicate in which way the vector is directed along its line of direction. A vector has therefore (1) magnitude, (2) direction, and (3) *sense* of direction, and methods of finding sums, differences, products, and quotients of vectors differ from those for dealing with scalar quantities. In Fig. 140 we suppose a particle to be displaced from A to B and then from B to C. The resulting change of position of the particle is from A to C. These displacements are vectors, and we have added two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ obtaining $\overrightarrow{AC}$. Thus, addition of vectors is defined by the vector equation

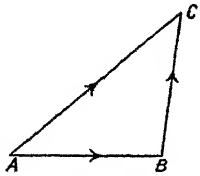


FIG. 140.
ADDITION OF VECTORS

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad . \quad . \quad . \quad \text{(XIV.1)}$$

Since $\overrightarrow{BC}$ when added to $\overrightarrow{AB}$ produces $\overrightarrow{AC}$ we say, by analogy with our treatment of scalar quantities, that $\overrightarrow{BC}$ is the difference $\overrightarrow{AC} - \overrightarrow{AB}$ and subtraction is defined by the equation

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} \qquad . \quad . \quad . \quad \text{(XIV.2)}$$

or

$$\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC}$$

We have supposed the displacements to take place successively, but we can make them occur simultaneously by moving the particle at a uniform rate in the direction of AB and at the same time moving it parallel to BC at another uniform rate, these rates being adjusted so as to cause the particle to move along AC to C. If the displacement from A to C takes place in unit time, the vectors become

velocities. If the velocities are not constant but are produced in unit time, the vectors become accelerations, and, since the force acting on a particle is proportional to the acceleration produced, the vectors may represent forces. The meaning of the difference of two vectors depends upon the kind of vector. If the vectors are displacements and we consider two particles at A to be given displacements $\overrightarrow{AC}$ and $\overrightarrow{AB}$ respectively, $\overrightarrow{BC}$ represents the displacement of the first particle relative to the second. If the vectors represent velocities, or accelerations, $\overrightarrow{BC}$ represents the velocity, or acceleration, of the first particle relative to the second and $\overrightarrow{CB}$ that of the second relative to the first. If the vectors represent forces acting at a point, $\overrightarrow{BC}$ is the force which along with the force $\overrightarrow{AB}$ will have the same effect as $\overrightarrow{AC}$.

The reader is supposed to be familiar with the applications of the triangle, parallelogram, and polygon of forces to systems of concurrent forces. Some vectors are fixed in position as well as in magnitude, direction, and sense, as, for instance, a force acting on a rigid body. Two forces alike in every respect except that they have different parallel lines of action, have different rotational effects on the body. Such vectors are *localized vectors*; those which are not localized are *free vectors*. Systems of free vectors, or of concurrent localized vectors, are combined in the same way as are systems of concurrent forces, and the triangle, parallelogram, and polygon of forces are particular cases of the triangle, parallelogram, and polygon of vectors.

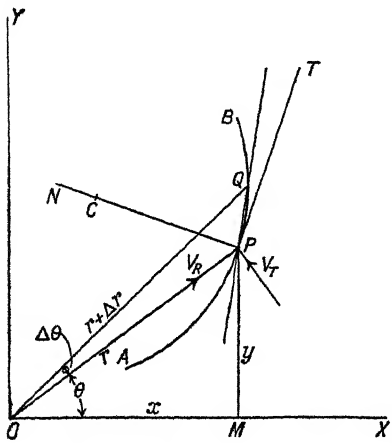


FIG. 141. MOTION ON A PLANE CURVE

156. Motion along a Plane Curve. In Fig. 141 APB is a plane curve and P a point on it whose polar co-ordinates are (r, θ) with respect to the origin O and the initial line OX, and whose rectangular co-ordinates are (x, y) with respect to the axes OX and OY. PM is perpendicular to OX and $\overline{OM} = x = r \cos \theta$. Suppose a particle to trace the curve

A at time $t = 0$, P at time t seconds and Q at time
ds. Lengths are measured in feet so that velocities
ons are given in ft per sec and ft per sec per sec

A_P be the velocity and acceleration respectively of the
at P, and let V_R and V_T be the radial and transverse
espectively of V_P, i.e. along OP and perpendicular to
Let A_R and A_T, not shown, be the radial and transverse
f A_P. Then

= rate of change of the position of the particle

$$= \text{limit}_{\Delta t \to o} \frac{\overrightarrow{OQ} - \overrightarrow{OP}}{\Delta t}$$

$$= \text{limit}_{\Delta t \to o} \frac{\overrightarrow{PQ}}{\Delta t}$$

and sense of V_P are those of the limiting position of

tangent PT in the sense P to T. The magnitude of V_P

$$= \underset{\Delta t \to o}{\text{limit}} \frac{PQ}{\Delta t}$$

re,

$\bar{V} = r \cos \theta$, and differentiating with respect to t

$$os\ \theta - r\,\dot{\theta} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.3)}$$

aw the initial line OX in any direction through O, we
oincide with OP. Putting $\theta = 0$ makes V_R identical
from (XIV.3), $V_R = \dot{r}$, which is self-evident, being the
$_R$. Similarly by putting $\theta = \pi/2$ we make $\dot{x}$ and $- V_T$

$$- V_T = - r\dot{\theta}$$
$$V_T = \dot{\theta} r = \omega r,$$

e angular speed of rotation of OP. This result is also

oth sides of (XIV.3) with respect to t; then

$$= \ddot{r} \cos \theta - 2\dot{r}\,\dot{\theta} \sin \theta - r \cos \theta \,.\, \dot{\theta}^2 - r\,\ddot{\theta} \sin \theta \quad \text{(XIV.4)}$$

tting $\theta = 0$,

$$\left.\begin{aligned} &= \ddot{r} - r\dot{\theta}^2 \\ &= \ddot{r} - \omega^2 r \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.5)}$$

and putting $\theta = \pi/2$,

$$-A_T = -2\dot{r}\dot{\theta} - r\ddot{\theta}$$

$$\left.\begin{aligned} \text{i.e.} \quad A_T &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\ \text{or} \quad A_T &= 2\omega\dot{r} + \dot{\omega}r \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.6)}$$

If the curve APB is a circle, centre at O, $\dot{r} = \ddot{r} = 0$ and (XIV.5) and (XIV.6) give $A_R = -\omega^2 r = -\dfrac{v_P^2}{r}$ and $A_T = \dot{\omega}r = \dot{v}_P$. If the particle moves along the circumference with uniform speed, $\dot{v}_P = 0$, and the acceleration of P is directed inwards along the radius PO, its magnitude is $\omega^2 r$ or v^2/r where v is the constant speed.

From (XIV.5) we see that the acceleration A_R is the sum of two accelerations, $\ddot{r}$ due to the increasing radius and $-\omega^2 r$ as with constant radius. Similarly A_T is the sum of two accelerations $\dot{\omega}r$ due to increasing speed at constant radius and $2\omega\dot{r}$. This latter component known as the *coriolis* acceleration is the transverse acceleration due to the increasing radius. Many engineering contrivances contain mechanisms in which two links rotate together but have relative sliding motion and the coriolis acceleration gives the relative transverse acceleration produced by the sliding.

157. **Normal and Tangential Accelerations.** Let PN, Fig. 141, be the normal to APB at P, C on PN being the centre of curvature at P. Suppose O to be instantaneously coincident with C. Since the curve and the circle of curvature have the same curvature at P, the particle will have the same acceleration at P when moving on the curve as it would have when moving on the circle of curvature with the same speed. But in the latter case we have seen that if the tangential speed is v_P, the normal velocity at P is zero. Also the tangential acceleration at $P = \dot{v}_P$ along TP and the normal acceleration $= \dfrac{v_P^2}{\rho}$ along PN, where $\rho = PC$ is the radius of curvature at P.

158. **Graphical Method of Finding the Acceleration in Uniform Circular Motion. The Hodograph.** Consider the motion of a particle moving, in the clockwise sense, along the circumference of a circle of radius r ft, with uniform speed v ft per sec.

Let P be the position of the particle at time t sec (Fig. 142). In the small figure on the left, let op be drawn parallel to the tangent TP at P and let the magnitude and sense of $\overrightarrow{op}$ be the same as those of the vector $\overrightarrow{PT}$ which represents V_P and whose magnitude is v. If this is done for all positions of P we obtain a series of lines through o all

of the same length. Thus as P traces the large circle, p will trace the small circle whose radius represents v to scale. The rate of change of V_P as the particle passes through P is the rate of change of $\overrightarrow{op}$ due to its rotation about o. OP and op rotate at the same angular speed $\omega = v/r$ radians per second because op is always perpendicular to OP. The magnitude of the rate of change of $\overrightarrow{op}$ is the speed of p due

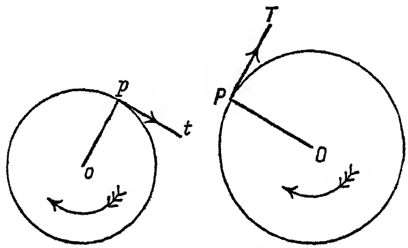


FIG. 142. HODOGRAPH FOR UNIFORM CIRCULAR MOTION

to its rotation about o which is $\omega\overline{op} = \omega v = v^2/r$ and this is the magnitude of the acceleration of P. If pt is the tangent at p, the direction of the acceleration of P is along pt from p to t, i.e. along PO inwards. The acceleration of the particle is directed from P towards the centre O of the circle and its magnitude is v^2/r ft per sec per sec. The small figure is called the *hodograph* for the motion. If the angular velocity is not constant, the hodograph is not a circle; if, for instance, it increases uniformly with time, the hodograph will be an equidistant spiral successive turns of which will divide any radius through o into an equal number of parts. The hodograph of a projectile moving without friction is a vertical straight line because its acceleration is always vertical.

NOTE. We give below a collection of examples on the application of mathematics to practical problems. Many such examples have already been included in their appropriate places and the following examples are intended to supplement the earlier ones as well as to provide some revision work. On pages 313 and 314, as well as elsewhere, we have stated some of the laws and principles on which methods of solution are based. For other principles the reader must rely on his knowledge of engineering subjects or must refer to textbooks on those subjects. Once the laws and principles have been expressed in mathematical form the solution of the problem depends upon the successful application of methods of analysis treated in this book.

EXAMPLES XIV

(1) A plate, whose boundary has the form of the curve given by the polar equation $r = 2 + \cos\theta$, lies on a fixed rough horizontal surface, and can rotate about a vertical axis through the pole of the curve. If W is the weight of the plate and if the pressure is supposed evenly distributed, find the smallest couple that will turn the plate for a given coefficient of friction between the plate and the surface. (U.L.)

(2) A particle moves so that its rectangular co-ordinates at time t are given by $x = a\cos pt$, $y = a\cos(pt + \alpha)$, where a and α are constants. Show that the path of the particle is an ellipse, and find the maximum and minimum distances of the particle from the origin.

Find also the maximum and minimum speeds of the particle in its path, and show that they occur at the minimum and maximum distances of the particle from the origin. (U.L.)

Eliminate t. $y = x\cos\alpha - \sin\alpha\sqrt{a^2 - x^2}$, which simplifies to

$$x^2 - 2xy\cos\alpha + y^2 = a^2\sin^2\alpha \quad . \quad . \quad . \quad . \quad (1)$$

Take new axes OX', OY', as in Fig. 101. If x', y' are the new co-ordinates, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$. Substituting in (1) and simplifying

$$x'^2(1 - \cos\alpha\sin 2\theta) + y'^2(1 + \cos\alpha\sin 2\theta) - 2x'y'\cos\alpha\cos 2\theta = a^2\sin^2\alpha \quad (2)$$

Choose θ so that the term in $x'y'$ vanishes, i.e. $\theta = 45°$, and (2) becomes

$$x'^2(1 - \cos\alpha) + y'^2(1 + \cos\alpha) = a^2\sin^2\alpha \quad . \quad . \quad . \quad (3)$$

which represents an ellipse whose semi-axes are $\dfrac{a\sin\alpha}{\sqrt{1-\cos\alpha}} = \sqrt{2}\, a\cos\dfrac{\alpha}{2}$ and $\dfrac{a\sin\alpha}{\sqrt{1+\cos\alpha}} = \sqrt{2}\, a\sin\dfrac{\alpha}{2}$. These are the maximum and minimum values of the distance from the origin respectively, if $\cos\alpha$ is positive and, in the reverse order, if $\cos\alpha$ is negative.

Since $x = \dfrac{1}{\sqrt{2}}(x' - y')$ and $y = \dfrac{1}{\sqrt{2}}(x' + y')$, we have $x' = \dfrac{1}{\sqrt{2}}(x + y)$ and $y' = \dfrac{1}{\sqrt{2}}(x - y)$, hence $\sqrt{2}x' = a\{\cos pt + \cos(pt + \alpha)\}$ or

$$x' = \sqrt{2}\, a\cos\frac{\alpha}{2}\cos\left(pt + \frac{\alpha}{2}\right)$$

Similarly,
$$y' = \sqrt{2}\, a\sin\frac{\alpha}{2}\sin\left(pt + \frac{\alpha}{2}\right)$$

If V is the velocity,
$$V^2 = \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2$$

$$= 2a^2p^2\left\{\cos^2\frac{\alpha}{2}\sin^2\left(pt + \frac{\alpha}{2}\right) + \sin^2\frac{\alpha}{2}\cos^2\left(pt + \frac{\alpha}{2}\right)\right\}$$

and
$$\frac{dV^2}{dt} = 2a^2p^3\sin(2pt + \alpha)\cos\alpha$$

V^2, and therefore V, has maximum and minimum values when $\sin(2pt + \alpha) = 0$, or when $pt + \dfrac{\alpha}{2} = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \pi$, etc. For these values $\dfrac{d^2V^2}{dt^2}$ has the values $4a^2p^4\cos\alpha$

and $-4a^2p^4\cos\alpha$ alternately. The values of V are therefore maxima and minima alternately in the same order as are the distances from the origin. The maximum and minimum speeds occur, therefore, at the maximum and minimum distances respectively.

(3) If the radius of gyration of a body about a certain axis through its centre of gravity is k_g, and that about a second parallel axis is k, prove that $k^2 = k_g^2 + d^2$, where d is the distance between the axes.

A solid uniform cube of which $ABCD$ is one square face has a part cut off by a plane perpendicular to $ABCD$ and bisecting the edges BC, CD. Find the radius of gyration of the remaining portion about the axis through A perpendicular to $ABCD$. (U.L.)

(4) A cylinder with any form of cross-section has plane ends at right angles to the generators, and is of length l and mass M. Prove that its moment of inertia about an axis in one terminal section is $M\left(\frac{l^2}{3} + k^2\right)$ where k is the radius of gyration of the terminal section about the same axis. (U.L.)

(5) An engine working at uniform horse-power draws a train against a uniform resistance R. If V is the maximum limit of the speed of the train, show that the kinetic energy of the train, after it has travelled a distance x from rest in time t, is $R(Vt - x)$. (U.L.)

Let v = speed in ft per sec at time t sec and P = horse-power exerted. The tractive force is $\frac{550P}{v}$ lb. When this becomes equal to the resistance R, the speed becomes constant, hence

$$\frac{550P}{V} = R \text{ or limiting speed } V = \frac{550P}{R}$$

The work done per second is $550P$ ft-lb. Rx ft-lb is used up in overcoming the constant resistance. The kinetic energy is increased by $550Pt - Rx$ and, since the initial velocity is zero, we have

$$\begin{aligned}\text{Increase of kinetic energy} &= 550Pt - Rx\\ &= VRt - Rx\\ &= R(Vt - x)\text{ ft-lb}\end{aligned}$$

(6) Suppose a racing car is resisted by a force proportional to the square of the speed and by another force which is constant, and suppose that the excess of the propulsive force over the constant part of the frictional resistance is fm, where f is constant and m is the mass of the car. Show that the car has then a terminal velocity V, and express, in terms of V, f, and t, the velocity v and the space s after the car has been moving t seconds from rest. (U.L.)

(7) A particle of mass m is acted on by gravity and by a resistance cmv, where v is the velocity of the particle at any instant and c is a constant. The particle passes through a point O with velocity V and in a direction inclined at angle ϕ above the horizontal, and T sec later the particle reaches its maximum vertical height above O. Show that

$$T = \frac{1}{c}\log_e\frac{g + cV\sin\phi}{g}$$

Take rectangular axes OX and OY through O, OY being vertical. Since the resistance is cmv, its horizontal and vertical components will be $cm\dot{x}$ and $cm\dot{y}$

respectively. (If the resistance were cmv^n, $n \neq 1$, it would not be possible to resolve the resistance in this way.) For vertical motion we have

$$\frac{m}{g}\ddot{y} = -m - mc\dot{y}$$

$$\therefore \qquad \ddot{y} = -(g + c\dot{y})$$

or
$$\frac{\ddot{y}}{\dot{y} + \dfrac{g}{c}} = -c$$

and integrating
$$\log_e\left(\dot{y} + \frac{g}{c}\right) = -ct + A \quad . \quad . \quad . \quad . \quad (1)$$

Since $\dot{y} = V \sin \phi$, when $t = 0$

$$\log_e\left(V \sin \phi + \frac{g}{c}\right) = A \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2)
$$\log_e \frac{c\dot{y} + g}{cV \sin \phi + g} = -ct$$

When the particle is at the highest point of its path, $\dot{y} = 0$ and $t = T$,

$$\therefore \qquad \log_e \frac{cV \sin \phi + g}{g} = cT$$

or
$$T = \frac{1}{c} \log_e \frac{g + cV \sin \phi}{g}$$

(8) One end of a spiral spring, whose axis is horizontal, is attached to a fixed point, and the other end to a mass which lies on a rough horizontal plane with which the coefficient of friction is μ. When the spring is unstressed, the plane suddenly begins to move with a constant velocity u in the direction of the axis of the spring. If the maximum friction between the mass and the plane will extend the spring by an amount l, show that, if $u^2 < \mu gl$, the mass gradually acquires a velocity u, which it retains for a time, and then begins to slip a second time. If t is measured from this moment of second slip, prove that, in the subsequent motion, the velocity of the mass is $u \cos kt$, where $k^2 = \mu g/l$. (U.L.)

(9) A rocket whose mass is initially M is ascending with initial velocity V. During the ascent the rocket ejects mass with a constant velocity v relative to the rocket. If p is the total mass which has been ejected at time t, show that the velocity is

$$V - gt + v \log_e \frac{M}{M - p}$$

If the rocket ejects mass at a constant rate m, assuming that the terms containing m^2 are negligible, show that the height to which it rises is

$$\frac{V^2}{2g}\left(1 + \frac{mv}{Mg}\right) \qquad \text{(U.L.)}$$

(10) OX, OY are rectangular axes in the plane of a lamina. A and B are the moments of inertia of the lamina about OX, OY respectively, and P is the product of inertia of the lamina for the axes OX, OY. Show that the principal axes of

the lamina at O make angles θ_1 and θ_2 with OX, where θ_1 and θ_2 are the roots of the equation

$$\tan 2\theta = \frac{2P}{B-A}$$

Find the directions of the principal axes at a corner of a rectangle, sides a and b, and deduce the directions of the principal axes at a corner of a square.

(11) AB is a thin uniform rod of length l and mass m. At a point C on the rod distant x from the end A, a particle of mass $\frac{4}{3}m$ is attached, and the rod is made to oscillate under gravity about a horizontal axis through A. Determine the length x so that the period of small oscillations may be a minimum, and also determine the minimum period.

$$I_A = \text{moment of inertia about } A$$

$$= \frac{ml^2}{3g} + \frac{4mx^2}{3g} = \frac{m}{3g}(l^2 + 4x^2)$$

$$h = \text{distance of centre of gravity from } A$$

then $$\frac{7}{3}mh = \frac{4}{3}mx + \frac{1}{2}ml \text{ or } h = \frac{4}{7}x + \frac{3}{14}l$$

$$L = \text{length of equivalent simple pendulum}$$

$$= \frac{K^2}{h}, \text{ where } K \text{ is the radius of gyration about } A$$

i.e. $$K^2 = \frac{m}{3g}(l^2 + 4x^2) \div \frac{7m}{3g} = \frac{1}{7}(l^2 + 4x^2)$$

Thus $$L = \frac{2(l^2 + 4x^2)}{8x + 3l}$$

The period is $2\pi\sqrt{\frac{L}{g}}$, and is a minimum when L is a minimum.

Now $$\frac{dL}{dx} = \frac{16\{x(8x + 3l) - (l^2 + 4x^2)\}}{(8x + 3l)^2}$$

$$= \frac{16(x + l)(4x - l)}{(8x + 3l)^2}$$

and $$\frac{d^2L}{dx^2} = \frac{16\{(8x + 3l)^3 - 16(8x + 3l)(4x - l)(x + l)\}}{(8x + 3l)^4}$$

As the value of x lies between 0 and l, the only solution of $\frac{dL}{dx} = 0$ is $x = \frac{l}{4}$ and for this value $\frac{d^2L}{dx^2}$ is positive. The length, and therefore the period, is a minimum when $x = \frac{l}{4}$. Substituting in the expression for L, we find $L = \frac{l}{2}$ and the minimum period is $2\pi\sqrt{\frac{l}{2g}}$

(12) Prove that the moment of inertia about a diagonal of a uniform rectangular lamina, with mass m and sides of lengths a and b, is $ma^2b^2/6(a^2 + b^2)$.

A right uniform solid pyramid of height h, whose base is a rectangle with sides of lengths a and b, is making small oscillations under gravity about a diagonal of the base as a horizontal axis. Show that the time of swing is the same as that of a simple pendulum of length $\frac{2}{5}\left\{h + \frac{a^2b^2}{h(a^2 + b^2)}\right\}$ (U.L.)

(13) A hollow cylinder of uniform density, internal diameter 1 ft, external diameter 2 ft, rolls without slipping down a plane inclined at 25° to the horizontal. Determine the linear velocity of the cylinder down the plane when it has described 10 ft of the plane, starting from rest.

(14) Prove that the periodic time of a compound pendulum is $2\pi k/\sqrt{gh}$, where k is the radius of gyration about the axis of suspension and h is the distance of the centre of gravity from that axis.

A non-uniform bar is suspended from one end and has a small movable mass m attached to it. When this mass is at a distance b from the top, the time of a small oscillation of the bar is T_1; when the mass is at a distance c from the top, the time of oscillation is T_2. Prove that the moment of inertia of the bar about its axis of suspension is

$$\frac{m\{(b-c)gT_1^2T_2^2 - 4\pi^2(b^2T_2^2 - c^2T_1^2)\}}{4\pi^2(T_2^2 - T_1^2)} \qquad \text{(U.L.)}$$

(15) A flywheel weighing 3 tons is suspended so as to oscillate about an axis perpendicular to its plane and 3 ft distant from the centre of the wheel. Find the radius of gyration of the wheel about its axis if the time of a small oscillation is 2·5 sec, and calculate the work required when the wheel is spinning about its axis to increase its velocity from 50 to 100 r.p.m. (U.L.)

(16) A uniform rod of length $2a$ and weight W turns about a vertical axis through its mid-point and perpendicular to it, with angular velocity ω. Find an expression for the tension in the rod at a distance x from the axis, and show that the maximum value of the tension is $\frac{W\omega^2a}{4g}$ (U.L.)

(17) An anchor ring is formed by the revolution of a circle of radius a about an axis distant $2a$ from its centre. Prove that the moment of inertia about the axis is $19Ma^2/4$, where M is its mass.

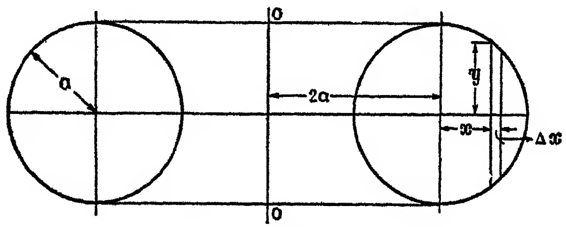


FIG. 143

If the ring rolls without slip down an inclined plane at an angle α to the horizontal, find its acceleration. (U.L.)

Consider the shell traced by the thin strip of the circle (Fig. 143) between chords distant x and $x + \Delta x$ from the centre. The height of the strip is $2y$ where $x^2 + y^2 = a^2$. The volume of the shell is $4\pi y(2a + x)\Delta x$, and its moment of

inertia about the axis OO is $4m\pi y(2a+x)^3\Delta x$, where m is the mass of unit volume in engineers' units.

Hence, $\quad V =$ total volume

$$= \pi a^2 \times 4\pi a \text{ (Art. 115)}$$
$$= 4\pi^2 a^3$$

From this $\quad M = 4\pi^2 a^3 m$ engineers' units . . . . . (1)

If I is the moment of inertia of the mass of the ring,

$$I = 4\pi m \int_{-a}^{a} (2a+x)^3\sqrt{a^2-x^2}dx$$

Put $x = a\sin\theta$, $\Delta x = a\cos\theta \,.\, \Delta\theta$, $\sqrt{a^2-x^2} = a\cos\theta$ and adjust the limits,

$$I = 4\pi m a^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2+\sin\theta)^3\cos^2\theta d\theta$$

and expanding the expression in the bracket,

$$I = 4\pi m a^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8\cos 2\theta + 12\sin\theta\cos^2\theta + 6\sin^2\theta\cos^2\theta + \sin^3\theta\cos^2\theta)d\theta$$

$$= 4\pi m a^5\left(4\pi + \frac{3}{4}\pi\right), \text{ or, substituting for } m,$$

$$I = \frac{19}{4}Ma^2$$

When the ring has velocity V, its angular velocity is $\omega = V/3a$. Its kinetic energy is $\frac{1}{2}I\omega^2 + \frac{1}{2}MV^2$. Assume that this has been acquired in falling from rest through a distance $s = h/\sin\alpha$ along the plane. Equating the loss of potential energy to the gain of kinetic energy,

$$\frac{1}{2}\times\frac{19Ma^2}{4}\frac{V^2}{9a^2} + \frac{MV^2}{2} = Mgh$$

from which $\quad \frac{55}{72}V^2 = gs\sin\alpha$

or $\quad V^2 = \frac{72}{55}\sin\alpha \times sg$

Comparing this with the formula $V^2 = 2fs$ for uniformly accelerated motion, we have

$$f = \text{acceleration}$$
$$= \frac{36}{55}g\sin\alpha$$

(18) A rigid body is rotating about a fixed axis with angular velocity ω. Prove that its kinetic energy is $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the fixed axis.

A square trap-door of side 3 ft and weight 40 lb is hinged at one side so as to be horizontal when closed and to open upwards. In the hinge is a spring which

tends to close the door, exerting a couple proportional to the angle through which the door has turned from its position when shut. The door is opened through 180° and can just be held in this position by its own weight, together with a weight of 20 lb placed on the edge farthest from the hinge. If this additional weight is suddenly removed, find the angular velocity of the door when it shuts. (U.L.)

(19) A uniform rod OA of mass $3m$ can turn freely about a fixed horizontal axis through the end O, and carries at A a small light hook. The rod falls from rest in the horizontal position, and when it is vertical the hook picks up a particle of mass m. Find the angle through which the rod will swing from the vertical before it next comes to rest. (U.L.)

(20) A truck has a body of mass M and four wheels, each of mass m, radius r, and radius of gyration k. It is driven by a torque T applied to the back axle. Find the acceleration, and the frictional forces between each wheel and the ground. (U.L.)

(21) A plank of weight W rests on two equal uniform cylindrical rollers, each of weight w, which rest on an inclined plane, making an angle α with the horizontal. The length of the plank is in the direction of the lines of greatest slope of the plane, and the axes of the cylinders are perpendicular to it. Show that, if there is no slip at any contact, the acceleration of the plank is

$$4g(W + w)\sin\alpha/(4W + 3w) \qquad \text{(U.L.)}$$

(22) A uniform flexible chain of length l and weight W hangs between two fixed points at the same level, and a weight W' is attached to its mid-point. If k is the sag in the middle, prove that the pull on either point of support is

$$\frac{kW}{2l} + \frac{lW'}{4k} + \frac{lW}{8k} \qquad \text{(U.L.)}$$

(23) Prove that a uniform, flexible, and inextensible chain hanging freely under gravity takes the form of the curve $y = c\cosh\dfrac{x}{c}$

If the chain is endless and hangs over a pulley of radius a, the tangent at the point where it leaves the pulley being inclined at the angle θ to the horizontal, and the length of the string being $2l$, prove that

$$\tan\theta = \sinh\left(\frac{a}{c}\sin\theta\right)$$

$$l = a(\pi - \theta) + c\tan\theta \qquad \text{(U.L.)}$$

(24) Obtain in any manner the series for $\sec\theta$ as far as the term in θ^4.

If a beam of length l is subject to a uniformly distributed load of intensity w and to an end thrust P, the central deflection is given by

$$\frac{wEI}{P^2}(\sec\tfrac{1}{2}ml - 1) - \frac{wl^2}{8P}$$

where $m^2 = \dfrac{P}{EI}$. Show that as $P \to 0$, this tends to the limit $\dfrac{5wl^4}{384IE}$ (U.L.)

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \text{ and by division, since } \sec\theta = 1/\cos\theta$$

$$\sec\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24}, \text{ neglecting terms beyond that in } \theta^4$$

The deflection is given by

$$D = \frac{wEI}{P^2}(\sec \tfrac{1}{2}ml - 1) - \frac{wl^2}{8P}, \; P = EIm^2$$

$$D = \frac{w}{m^4EI}\left(\frac{(\frac{1}{2}ml)^2}{2} + \frac{5(\frac{1}{2}ml)^4}{24} + \text{terms of higher degree in } m - \frac{m^2l^2}{8}\right)$$

$$= \frac{w}{EI}\left(\frac{5l^4}{384} + \text{terms in } m^2, m^4\ m^6, \text{etc.} \ldots\right)$$

As m approaches the limit zero, the sum of the convergent series of terms in powers of m^2 approaches the value zero, and in the limit

$$D = \frac{5wl^4}{384EI}$$

(25) A thin uniform bar of mass M is bent into the form of a semicircle of radius a and is pivoted at one end so that it can turn in a horizontal plane. Show that, when it is revolving with uniform angular velocity ω, the tension at the middle point is $Ma\omega^2 \dfrac{\pi + 2}{2\pi}$, and find an expression for the bending moment at the same point. (U.L.)

(26) A plane lamina moves about in its own plane. The rectangular co-ordinates of a point in the lamina referred to fixed axes are (x, y), and referred to axes moving with the lamina (X, Y); the moving axis of X makes an angle θ with the fixed axis of x, and the co-ordinates of the moving origin referred to the fixed axes are (ζ, η). Prove that

$$x = \zeta + X\cos\theta - Y\sin\theta$$

$$y = \eta + X\sin\theta + Y\cos\theta$$

and deduce by differentiating that at any moment the velocity of a point P moving with the lamina is proportional to IP, the distance of P from the point I, at which

$$x = \zeta - \frac{d\eta}{d\theta}$$

$$y = \eta + \frac{d\zeta}{d\theta}$$

and is also perpendicular to IP. (U.L.)

We shall first find the point in the lamina which is instantaneously at rest. This is the instantaneous centre (Art. 123). The component velocities of the moving point are $\dot{x}$ and $\dot{y}$ where, by differentiation with respect to time,

$$\dot{x} = \frac{d\zeta}{d\theta}\dot{\theta} - X\sin\theta\,\dot{\theta} - Y\cos\theta\,\dot{\theta} \quad . \quad . \quad . \quad (1)$$

$$= \dot{\theta}\left(\frac{d\zeta}{d\theta} - X\sin\theta - Y\cos\theta\right)$$

and similarly

$$\dot{y} = \dot{\theta}\left(\frac{d\eta}{d\theta} + X\cos\theta - Y\sin\theta\right) \quad . \quad . \quad . \quad (2)$$

If the point (xy) is at rest, $\dot{x}$ and $\dot{y}$ are each zero. Thus,

$$X \sin\theta + Y\cos\theta = \frac{d\zeta}{d\theta} \text{ and } Y\sin\theta - X\cos\theta = \frac{d\eta}{d\theta}$$

and the co-ordinates of the point are

$$\left.\begin{aligned} x &= \zeta - \frac{d\eta}{d\theta} \\ y &= \eta + \frac{d\zeta}{d\theta} \end{aligned}\right\} \text{ The co-ordinates of the instantaneous centre } I.$$

The co-ordinates x', y', of P with respect to new fixed axes through I are therefore

$$x' = X\cos\theta - Y\sin\theta + \frac{d\eta}{d\theta}$$

$$y' = X\sin\theta + Y\cos\theta - \frac{d\zeta}{d\theta}$$

Hence from (1) and (2) the component velocities of the point P are

$$\dot{x} = -\dot{\theta}\, y' \text{ and } \dot{y} = \dot{\theta} x'$$

The gradient of the path of P is $\dot{y}/\dot{x} = -\frac{x'}{y'}$, and the gradient of IP is $\frac{y'}{x'}$. Since the product of these is -1, the velocity of P is perpendicular to IP. Also, if v = velocity of P,

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= \dot{\theta}^2(x'^2 + y'^2) \\ &= \dot{\theta}^2\, \overline{IP}^2 \end{aligned}$$

or

$$v = \dot{\theta}\, \overline{IP}$$

that is, the velocity of P is proportional to $\overline{IP}$.

(27) A circular disc is rotating with uniform angular velocity ω about a horizontal axis through its centre, and the horizontal axis moves vertically downwards from rest with uniform acceleration f. If the path of the centre of the disc is taken as the x-axis, show that the space centrode of the disc is the parabola $y^2 = 2fx/\omega^2$.

[*Hint*. Find the co-ordinates after t sec of the point on a horizontal diameter which is instantaneously at rest and eliminate t.]

(28) Find the polar equation of the body centrode in the previous example.

(29) A uniform rod AB of length a can turn freely about a fixed hinge at A. A string is attached to the end B, passes over a smooth fixed peg at a height h ($> a$) vertically above A, and carries a load equal to half the weight of the rod. Show that there is a position of unstable equilibrium with the rod inclined to the vertical.

Discuss the stability of the positions in which the rod is vertical. (U.L.)

(30) A straight uniform rigid rod AB, of length a and mass m, is moving in a plane, its angular velocity being ω and the velocity of its middle point C being u, in a direction which makes an angle of 45° with CB at a certain instant when the point E half-way between C and B is brought suddenly to rest by an impulse applied at E. Find the magnitude and direction of the impulse and the loss of energy. (U.L.)

(31) Starting from rest, a body moves in a straight line with uniform acceleration f_1 until its velocity is V, then with uniform velocity for a time, and finally with uniform retardation f_2 until it comes to rest again. If T is the total time taken and d the total distance travelled, express T in terms of d, V, f_1, f_2, and determine the value of V for which T is a minimum.

(32) A rod of uniform cross-section floats in a liquid with its axis vertical and with 10 in. of its length immersed. Show that the period of small vertical oscillations of the rod is 1·01 sec.

(33) A reservoir discharging through sluices at a depth h below the water surface has a surface area A for various values of h as given below.

h	10	11	12	13	14 ft
A	950	1 070	1 200	1 350	1 530 sq ft

If t denotes time in minutes, the rate of fall of the surface is given by

$$\frac{dh}{dt} = -\frac{48\sqrt{h}}{A}$$

Using some graphical or numerical method, estimate the time taken for the water level to fall from 14 to 10 ft above the sluices. (U.L.)

(34) A vessel has the form obtained by rotating the curve $y = \frac{1}{2}x$ (the unit being 1 ft) about the axis of y, and is placed with its axis vertical and vertex downwards; 15 ft³ of water are placed in the bowl, and an outlet pipe of circular section and diameter 1 in. is then opened at the lowest point of the bowl. Assuming that the velocity of flow into the pipe is $0{\cdot}6\sqrt{2gz}$, when the depth of water in the bowl is z, find the time required for all the water to run out. (U.L.)

(35) Prove that the mass-centre of a uniform solid hemisphere of radius r is at a distance $\frac{3}{8}r$ from the centre.

A tank consists of a cylinder of diameter 4 ft with its axis horizontal, and has hemispherical ends. It is just full of oil of density 50 lb/ft³. Find the magnitude and direction of the resultant force on one of the ends.

(36) A horizontal boiler has a flat bottom and its ends consist of a square of 2 ft side, surmounted by a semicircle. Determine the centre of pressure of either end when the boiler is completely full. (U.L.)

(37) Show that the distance of the centre of gravity of a segment, of height h, of a uniform solid sphere of radius R, from the plane base of the segment is

$$\frac{h(4R - h)}{4(3R - h)}$$

A sphere of weight W and radius R floats in water with its centre at a depth $\frac{1}{2}R$ below the surface. Show that the work required to lift the sphere just clear of the water is $\frac{7}{8}WR$, assuming that there is no change in the level of the water. (U.L.)

(38) A uniform solid hemisphere of radius a is immersed in water with its centre at a depth h and its plane base (which is uppermost) inclined at an angle θ to the horizontal. Find the magnitude of the resultant thrust on the curved surface, and prove that, if ϕ is the inclination of the direction of the thrust to the horizontal, then

$$\tan\phi = \frac{2a + 3h\cos\theta}{3h\sin\theta} \qquad \text{(U.L.)}$$

(The resultant thrust is equal and opposite to the resultant of the water pressure on the flat face and the weight of the displaced water.)

(39) Prove that the volume of a paraboloid of revolution cut off by a plane perpendicular to its axis is half the volume of the surrounding cylinder.

A closed cylindrical vessel containing water is rotating with uniform angular velocity ω about its axis, which is vertical. The free surface of the water is known to be a paraboloid formed by the revolution of the parabola $y = \omega^2x^2/(2g)$ about its axis. Show that the difference in the heights of the lowest and highest points of the paraboloid varies as ω or ω^2, according as the water is or is not in contact with the top of the cylinder, provided that the cylinder contains sufficient water for the base to remain covered. (U.L.)

If the paraboloid does not touch the top, the difference in heights is found by putting $x = 0$ and $x = R$ in $y = \omega^2x^2/(2g)$, R being the radius of the vessel. This difference is $\dfrac{\omega^2R^2}{(2g)}$ and varies as ω^2.

If the water touches the top, the volume V_0 of the paraboloid of revolution must remain constant. Let r be the radius of the section at the top; r is variable. Then, if y be the depth of the vertex, $\pi r^2y = 2V_0$. But the equation $y = \dfrac{\omega^2x^2}{(2g)}$ still applies, and the difference of levels is now $y = \dfrac{\omega^2r^2}{(2g)}$ in which y, ω, and r are variables. To eliminate r we have

$$y = \frac{\omega^2}{2g}r^2$$

$$= \frac{\omega^2}{2g} \times \frac{2V_0}{\pi y}$$

i.e.
$$y = \omega\sqrt{\frac{V_0}{\pi g}}$$

and the difference of levels varies as ω.

(40) Define the metacentre of a floating body, and show that, with the usual meaning of the symbols, its height above the centre of buoyancy is Ak^2/V.

A uniform solid right circular cylinder of length l, radius a, and specific gravity $\frac{3}{4}$, floats in water with its axis vertical. Show that it is stable for small displacements, if $l < \dfrac{3a}{2}$. (U.L.)

(41) Quote the mathematical inequality which gives the condition for stability of a floating body for rotations in a particular plane.

If a uniform cylinder floats stably with its axis horizontal, show that the rectangular water-line area must be longer in the direction of the axis of the cylinder than in the perpendicular direction. (U.L.)

For stability, $\overline{HM} > \overline{HG}$ where H is the centre of buoyancy, i.e. the centre of gravity of the displaced fluid, G is the centre of gravity of the floating body, and M the metacentre. Since $\overline{HM} = \dfrac{Ak^2}{V}$, where A is the area of the water-line section, k the radius of gyration of A about the axis of angular displacement, and V the volume displaced by the floating body, the inequality becomes $HG < \dfrac{Ak^2}{V}$.

Fig. 144 shows two views of the cylinder floating in the fluid. The water-line surface is ab in the elevation and AB in the end view. GX and GY are rectangular

axes through G, the centre of the circle in the end view, and x, y are the co-ordinates of K. For movements about an axis in the water-line surface parallel to the cylinder axis, the equilibrium is neutral, because the centre of gravity and of buoyancy are always in the same vertical line. The stability referred to in the example is that about an axis perpendicular to the axis of the cylinder.

Consider the small strip shown, below FK, of depth Δx. The equation to the circle ABK is $x^2 + y^2 = r^2$, r being the radius of the cylinder.

Let $\overline{CD} = l$, $\overline{AB} = 2b$, and $\overline{GL} = a$. For stability, $\overline{HG} < \dfrac{Ak^2}{V}$, or if $\bar{x} = \overline{HG}$, $V\bar{x} < Ak^2$. $A = 2bl$, $k^2 = \dfrac{1}{12}\,l^2$.

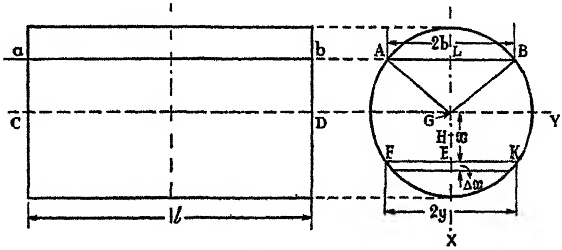


FIG. 144

To find $V\bar{x}$, we use the method of Art. 114. Thus

$$
\begin{aligned}
V\bar{x} &= l\int_{-a}^{r} 2yx\,dx \\
&= 2l\int_{-a}^{r} \sqrt{r^2 - x^2}\,x\,dx \\
&= -2l\left[\tfrac{1}{3}(r^2 - x^2)^{\frac{3}{2}}\right]_{-a}^{r} \\
&= \tfrac{2}{3}l\left(r^2 - a^2\right)^{\frac{3}{2}} \\
&= \tfrac{2}{3}lb^3
\end{aligned}
$$

Hence for stability,

$$\tfrac{2}{3}lb^3 < \tfrac{1}{6}bl^3$$

i.e.

$$l^2 > 4b^2$$

or

$$l > 2b$$

The length of the water-line area must be greater than its breadth.

(42) Starting from the equation $\dfrac{d^4y}{dx^4} = \dfrac{w}{EI}$, where y is the deflection of a beam at a point distant x from one end, due to a distributed load of intensity w, E is Young's modulus, and I is the second moment of the section about the neutral axis, find the central deflection of a uniform beam of length l clamped at each end and loaded with a uniformly distributed load. (U.L.)

(43) A flywheel, whose moment of inertia about its horizontal axis of rotation is I, is set in motion about this axis by means of a cord wrapped round the axle (of radius r), which carries a load P at its free end, there being no slipping of the cord on the axle. If there is a frictional torque $k\omega^2$, where ω is the angular speed of the wheel at any instant and k is constant, find the angular velocity acquired by the wheel when P has descended a vertical distance x. (U.L.)

(44) If a body, free to move about a fixed axis, is acted upon by a given torque G, show that the acceleration produced is G/I, where I is the moment of inertia of the body about the axis.

A wheel, free to turn about a fixed axis, is acted upon by a constant torque G, while the frictional resistance at the axis produces a retarding torque $k\omega^2$, where k is constant and ω is the angular speed at any instant. If the wheel starts from rest and acquires an angular speed ω_1 after N revolutions, show that the angular speed ω_2 after $2N$ revolutions is given by

$$\frac{\omega_2^2}{\omega_1^2} = 2 - \frac{k}{G}\,\omega_1^2 \qquad \text{(U.L.)}$$

(45) Prove that the equation of the curve assumed by a heavy uniform chain, with its ends attached to fixed points, may be expressed in the form $y = c \cosh \dfrac{x}{c}$, and show that, if c is so large that powers of $\dfrac{x}{c}$ above the square may be neglected, the curve is approximately a parabola.

A wire rope weighing 1 lb per ft is stretched between two points at the same level 40 yd apart. If the central sag is 1 ft, find approximately the tension at the supports to the nearest lb-wt. (U.L.)

[Assume that the wire hangs in the form of a parabola.]

(46) Two equal homogeneous solid spheres, each of radius r, are fixed together by a light rigid bar whose direction passes through the centres, so that their centres are at a distance $2l$ apart. The system is caused to oscillate as a compound pendulum about a point P in the bar distant x from the centre of the bar. Find the period of a small oscillation. Show that the period would be least if it were possible for the distance of P from the centre of the bar to be $\sqrt{\dfrac{5l^2 + 2r^2}{5}}$

Show that this point is inside one of the spheres. (U.L.)

(47) A ring of mass m can slide freely on a smooth horizontal rod. To the ring is attached a light cord of length $2a$, and a particle of mass m is attached to the other end of the cord. The system starts from rest with the particle in contact with the rod and the cord taut, and is then allowed to move freely under gravity. Show that when the ring has moved a distance x and the cord makes an angle θ with the horizontal,

$$\frac{dx}{dt} = a \sin\theta \frac{d\theta}{dt} \text{ and } \left(\frac{d\theta}{dt}\right)^2 = \frac{2g \sin\theta}{a(1 + \cos^2\theta)} \qquad \text{(U.L.)}$$

(48) Show that the work done in stretching a string from a length a to a length b is $(b - a)T$, where T is the arithmetic mean of the initial and final tensions.

An elastic string of natural length $2a$ has its ends attached to fixed points at a horizontal distance $2a$ apart. A weight w is attached to the mid-point of the string, and is found to rest in equilibrium at a depth $\dfrac{3}{4}a$ below the supports. Prove that the modulus of the string is $\dfrac{10}{3}w$, and show that the work done in

pulling w down through an *additional* small vertical distance x is $122wx^2/75a$, if powers of x/a above the second are neglected. (U.L.)

(49) The effective horse-power of a ship of mass 10 000 tons is 6 000, and its full speed is 20 m.p.h. Assuming that the resistance to motion varies as the square of the speed, and that the horse-power is constant, find the distance travelled from rest in attaining a speed of 16 m.p.h. (U.L.)

(50) A train of mass 300 tons travels along the level at a uniform speed of 80 ft per sec against resistances of 14 lb-wt per ton. It then climbs an incline of 1 in 160. Assuming that the horse-power and the resistances remain constant, show that, when the speed has dropped to v ft per sec, the retardation is $(2v - 80)/5v$ ft per sec². Find the time taken for the speed to fall from 80 to 60 ft per sec. (Take $g = 32$ ft per sec².) (U.L.)

$$\text{Resistance on level} = 300 \times 14 = 4\,200 \text{ lb}$$

$$\text{Rate of working} = 4\,200 \times 80 = 336\,000 \text{ ft-lb per sec}$$

$$\text{Tractive force on incline} = \frac{336\,000}{v}$$

$$\text{Force resisting motion} = 4\,200 + \frac{300 \times 2\,240}{160}$$

$$= 8\,400 \text{ lb}$$

$$\text{Unbalanced force} = \text{Mass} \times \text{retardation}$$

$$\frac{336\,000}{v} - 8\,400 = -\frac{300 \times 2\,240}{32}\frac{dv}{dt}, \text{ from which assuming } g = 32$$

$$\frac{dv}{dt} = -\frac{2}{5}\frac{v - 40}{v} \text{ which is the retardation}$$

Rearranging, $\left(1 + \dfrac{40}{v - 40}\right) dv = -\dfrac{2}{5}\,dt$

and integrating, $v + 40 \log_e(v - 40) = -\dfrac{2}{5}t + c$

Substituting $v = 80$,

$$80 + 40 \log_e 40 = -\frac{2}{5}t_1 + c$$

where t_1 is the time when $v = 80$.

Substituting $v = 60$,

$$60 + 40 \log_e 20 = -\frac{2}{5}t_2 + c$$

where t_2 is the time when $v = 60$.

By subtraction,

$$20 + 40 \log_e 2 = \frac{2}{5}(t_2 - t_1)$$

from which $\quad t_2 - t_1 = 50 + 100 \log_e 2$

or, time taken $\quad = t_2 - t_1 = 119$ sec

(51) A sphere of radius a and specific gravity 0.75 floats in water. If x is the distance of the centre below the surface, show that x is the real root of the equation

$$x^3 - 3a^2 x + a^3 = 0$$

and determine $\frac{x}{a}$ correct to three decimals. (U.L.)

(52) A train of weight W tons moves on the level under the action of a pull P tons-wt. against a resistance R tons-wt., and the speed at any instant is v ft. per sec. Show that the distance travelled while the speed varies from v_0 to v_1 is

$$\frac{W}{g}\int_{v_0}^{v_1}\frac{v\,dv}{P-R}$$

If $W = 300$ and $R = 0.9 + 0.007v^2$, show that the distance travelled in slowing down from 45 to 30 m.p.h. with power cut off is about 520 ft. (U.L.)

(53) A shaft of radius a is subject to a torque T. Assuming that when plastic flow occurs, the shear stress is proportional to the radius for radii less than R, reaching f at radius R, and thereafter remaining constant equal to f, for radii between R and a, show that

$$T = \pi f(4a^3 - R^3)/6 \qquad \text{(U.L.)}$$

(54) A heavy bob B is attached to a pivot P by a light rod 20 in. long. P is a horizontal arm rigidly attached to a vertical shaft, and is 6 in. from the axis. The shaft rotates steadily at 4 radians per sec. Find the equation which gives the angle θ which PB makes with the vertical, and (graphically or otherwise) determine the value of θ to the nearest degree. (U.L.)

ANSWERS TO EXAMPLES

EXAMPLES 1 Page 34

1. 4   2 0   3. 4

4 -4 5 $-$ 6. $\frac{1}{2\sqrt{a}}$

7. $\frac{(2n+1)\pi}{2}$, where n is any integer or zero   10 $\frac{a}{b} = \tan\theta$

11. Convergent if $|x| < 1$, divergent if $|x| = 1$ or > 1 12. Divergent

13. Convergent if $|x| < 1$ or $= 1$, divergent if $|x| > 1$ 14. Convergent

15. Convergent if $|x| < 1$, otherwise divergent

16. $1 - \frac{r^2}{2l^2}\sin^2\omega t - \frac{r^4}{8l^4}\sin^4\omega t - \frac{r^6}{16l^6}\sin^6\omega t - \frac{5r^8}{128l^8}\sin^8\omega t$

17. $\frac{n(n-1)(n-2) \quad (n-r+1)}{\underline{|r}} x^{n-r} a^r$ $25\ 344x^5$

21. $A = -\frac{1}{2}$ $B = -\frac{1}{8}$, $C = -\frac{1}{1\,1}$.

22. 1 03, 1 025, 1 017

24. $N + \frac{a}{5N^4} - \frac{2a^2}{25N^9} + \frac{6a^3}{125N^{14}}$

25. $e = 2\ 7183$, $e^{\frac{1}{2}} = 1\ 6487$, $e^{\frac{1}{4}} = 1\ 2840$

26. Sum $= (1 - \frac{1}{2})^{\frac{2}{3}} = 2^{\frac{2}{3}} = 1\ 587$

27. Sum $= (1 - 4x)$.   30. 0 2231

32. $1,\ 0\ 3090 \pm 0\ 9511i,\ -0\ 8090 \pm 0\ 5878i$

33. $3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$, $8\ 569\ [30^\circ\ 19] = 7\ 397 + 4\ 326i$, $8\ 569\ [210^\circ\ 19] = -(7\ 397 + 4\ 326i)$

34. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

35. $$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta \cdot \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\ \sin^4\theta -$$

$$\sin n\theta = n\cos^{n-1}\theta\ \sin\theta - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\ \sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\cos^{n-5}\theta\ \sin^5\theta - \ldots$$

36. Resultant $2\ 086\ [-7^\circ\ 42] = 2\ 067 - 0\ 2792i$

37. $1(\cos 53^\circ 8' + i\sin 53^\circ 8')$

38. $12 + 5i = 13\,[22^\circ\,37']$, $r = 1.445$, $\theta = 26^\circ\,3'$

40. $0.6 + 0.8i$, $-0.28 + 0.96i$, $0 - 0.32i$

41. -1, $0.8090 \pm 0.5878i$, $-0.3090 \pm 0.9511i$

42. $\pm(3 + 2i)$, $\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}$, $-1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, product $= -1$

43. $\cos^5\alpha = \frac{1}{16}[10\cos\alpha + 5\cos 3\alpha + \cos 5\alpha]$
$\sin^4\alpha = \frac{1}{8}[3 - 4\cos 2\alpha + \cos 4\alpha]$

45. $A = \frac{5 + 7i}{2}$, $B = \frac{5 - 7i}{2}$

46. $x = (a_1 + b_1)\cos nt - (a_2 - b_2)\sin nt$,
$y = (a_2 + b_2)\cos nt + (a_1 - b_1)\sin nt$
$x^2[(a_1 - b_1)^2 + (a_2 - b_2)^2] + y^2[(a_1 + b_1)^2 + (a_2 - b_2)^2] - 4xy(a_1b_2 + a_2b_1)$
$= (a_1^2 + a_2^2 - b_1^2 - b_2^2)^2$

47. $\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{240}$

48. (i) $0 - 0.6434i$ (ii) $\frac{\pi}{2} - i\log_e 2$

50. $x = -6.546$, $y = -7.621$

51. $A = \frac{RLp}{\sqrt{R^2(1 - LCp^2)^2 + L^2p^2}}$, $\theta = \tan^{-1}\frac{R}{Lp}(1 - LCp^2)$

52. 0.5211, 1.1276

54. $\left\{\sin\left(\alpha + \frac{n-1}{2}\delta\right)\sin\frac{n\delta}{2}\right\}\Big/\sin\frac{\delta}{2}$

55. $\frac{x\sin\theta - (-1)^n x^{n+1}\{x\sin n\theta + \sin(n+1)\theta\}}{x^2 + 2x\cos\theta + 1}$ $e^{x\cos\theta}\cos(x\sin\theta)$

56. $1 + \left(2\cos\frac{\theta}{2}\right)^n \sin\frac{n\theta}{2}$.

58. See Ex. 54. $l\Big/\left(2\sin^2\frac{\pi}{2n}\right)$, where l = length of a side of polygon

EXAMPLES II Page 73

1. (i) $-\frac{1}{2x}$ (ii) $\frac{1}{2}\sec^2\frac{x}{2}$. (iii) $\tan x\sec x$

2. $-\frac{12}{x^5} + \frac{5}{x^2} + \frac{2}{3\sqrt[3]{x^2}} - 7$

3. $\frac{8}{3\sqrt[3]{x}} + \frac{15}{4\sqrt[4]{x^7}}$

4. $4(x - 3)^3$

5. $-\frac{12}{(x - 1)^3}$.

6. $3\cos(3x - 4)$

7. $2\sec^2(2x + 1)$

8. $5\tan 5x\sec 5x$

9. $5e^{5x} + \frac{3}{e^x}$

10. $\frac{7}{3}e^{\frac{1}{3}x} - \frac{4}{5}e^{-\frac{1}{5}x}$.

11. $\dfrac{1}{x+a}$.

12. $3\cosh 3x - 8\sinh 2x$.

13. $\frac{1}{4}\operatorname{sech}^2 \frac{1}{2}x - \frac{1}{3}\operatorname{cosech}^2 \frac{1}{3}x$

14. $\dfrac{2}{\sqrt{2x-x^2}}$

15. $\dfrac{1}{1+(a+x)^2}$

16. $\dfrac{8}{1+64x^2}$.

17. $\pm\dfrac{1}{\sqrt{x^2-16}}$

18. $e^{2x}(2x+1)$

19. $\dfrac{x(2\log_e x - 1)}{(\log_e x)^2}$

20. $(2x\cos x + \sin x)/(2\sqrt{x})$

21. $\dfrac{1}{(1+\cos x)}$ or $\frac{1}{2}\sec^2\dfrac{x}{2}$

22. $2apx + aq + bp$.

23. $\dfrac{aq-bp}{(px+q)^2}$

24. $\dfrac{2x(3x+8)}{(3x+4)^2}$.

25. $\cot x - x\operatorname{cosec}^2 x$

26. $(1-7x)(2+x)^2(3-x)^3$

27. $10^x\log_e 10 - 1 - \log_e x$

28. $k\cos kx\cos lx - l\sin kx\sin lx$

29. $(k\cos kx\cos lx + l\sin kx\sin lx)/(\cos^2 lx)$

30. $x^{n-1}\left(\dfrac{px}{\sqrt{1-p^2x^2}} + n\sin^{-1}px\right)$.

31. $\dfrac{1}{\sqrt{1-x^2}}$.

32. $\dfrac{2px+q}{2\sqrt{px^2+qx+r}}$.

33. $\dfrac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$.

34. $\dfrac{2x}{3\sqrt[3]{(1+x^2)^2}}$.

35. $\dfrac{1-2x^2}{\sqrt{1-x^2}}$.

36. $\dfrac{x(4-3x)}{2(1-x)^{\frac{3}{2}}}$.

37. $\cosh 2x$

38. $3a^{3x-4}\log_e a$

39. $\dfrac{2}{1+x^2}$. [Note that $\operatorname{cosec}^{-1}\dfrac{1+x^2}{2x} = \tan^{-1}\dfrac{2x}{1-x^2} = 2\tan^{-1}x$].

40. $-3x\sqrt{p^2-x^2}$.

41. $-\dfrac{4x}{3(2x^2-5)}$.

42. $\sinh x\cosh x(3\sinh x - 2)$

43. $-\dfrac{2}{x\sqrt{x^2+4}}$.

44. $e^{-x}(\frac{1}{2}\sec^2\frac{1}{2}x - \tan\frac{1}{2}x)$

45. $2e^{4x}[\cos(2x+7) + 2\sin(2x+7)]$

46. $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

47. $-\dfrac{2}{x^2-1}$

48. $\dfrac{10x-3}{2(5x^2-3x+2)}$

49. $\cot x$

50. $\dfrac{2x}{x^2-1}$

51. $\sin^2 kx\cos lx\,(3k\cos kx\cos lx - 2l\sin kx\sin lx)$.

52. $\dfrac{(q-p)\sin 2x}{2\sqrt{p\cos^2 x + q\sin^2 x}}$ **53.** $\dfrac{\sin\alpha}{1+\cos\alpha\cos x}$

54. $-\dfrac{2\sinh x\sin x}{(\sinh x-\sin x)^2}$ **55.** $x^4 e^{3x}\sin 2x\left(\dfrac{4}{x}+3+2\cot 2x\right)$

56. $\dfrac{2}{a^2}\left[2x-\dfrac{2x^2+a^2}{\sqrt{x^2+a^2}}\right]$ **57.** $(x-2)(8x^2-11x-4)$

58. $\pm\dfrac{3}{\sqrt{9x^2-1}}$ **59.** $\sec 2x$ **60.** $\operatorname{sech} x$

61. $e^{5x}\left[\dfrac{1}{1+2x}+5\log_e\sqrt{1+2x}\right]$ **62.** $\dfrac{2}{x}+\dfrac{1}{(1+x^2)\tan^{-1}x}$

63. $\dfrac{x^2}{\sqrt{1-x^2}\sin^{-1}x}+2x\log_e(\sin^{-1}x)$ **64.** $\dfrac{x}{\sqrt{1-x^2}}+\sin^{-1}x-\dfrac{1}{x^2-1}$

65. $4x\operatorname{cosec}2x^2$ **66.** $2x+\dfrac{2x^2+1}{\sqrt{1+x^2}}$

67. (i) $5^n e^{3x}\sin[4x+n\tan^{-1}(\frac{4}{3})]$

(ii) $(-1)^n\dfrac{5\,\underline{|n}}{8}\left[\left(\dfrac{3}{3x-2}\right)^{n+1}-\left(\dfrac{1}{x+2}\right)^{n+1}\right]$

70. $\alpha=0,\ K=-\dfrac{a}{6p^2}$

72. Acceleration $\dfrac{d^2x}{dt^2}=-p^2x$, $x=\sqrt{A^2+B^2}\sin\left(pt+\tan^{-1}\dfrac{B}{A}\right)$, period $=\dfrac{2\pi}{p}$.

75. $\dfrac{27W}{16\sqrt{5}}$. **76.** $\dfrac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$

77. 0 when n is odd, $2^n\left\{\underline{\left|\frac{n}{2}\right.}\right\}^2$ when n is even

78. Thrust in rod $BF=\dfrac{5\sqrt{3}}{2}W$, and thrust in rod $CE=\dfrac{\sqrt{3}}{2}W$

79. $10W\tan\theta$ **80.** $d\sec^2\theta\,\Delta\theta$ ft.

81. $\frac{1}{2}ab\cos C\,\Delta C$ **82.** $\cosh x\,\Delta x$, 7·4437

83. 1·1, 8·1, 8 −1 **84.** 58·3 m.p.h.

EXAMPLES III. Page 105

1. $\frac{8}{9}x^{\frac{9}{8}}-3\log x+\frac{5}{7}x^7-3x$ **2.** $3\sqrt[3]{u}$.

3. $-\dfrac{10}{s^{0\cdot2}}-1\cdot47\,s^{3\cdot4}+2\sqrt{s}$ **4.** $\dfrac{(2x+7)^4}{8}$.

5. $-\dfrac{1}{3(3x-10)}$. **6.** $\frac{1}{4}\log_e(4t-1)$.

7. $\dfrac{10^x}{\log_e 10} = 10^x \log_{10} e = 0.4343 \times 10^x$ 8. $-\,{}^1 \cos(5t+2)$

9. 2

10. $\dfrac{\pi}{8}$

11. $\dfrac{1}{4}\log_e \dfrac{2+x}{2-x}$

12. 0.06931

13. $\sinh^{-1}\dfrac{x}{2}$ or $\log_e\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

14. $\dfrac{\pi}{3}$

15. $\cosh^{-1}\dfrac{x}{5}$

16. $\log_e(3t^2-5t+4)$

17. 0

18. 15.17

19. $\dfrac{3x}{2}-\dfrac{4}{7}\log_e(x+2)-\dfrac{5}{28}\log_e(2x-3)$

20. $\dfrac{1}{41}\log_e\dfrac{4y-7}{3y+5}$

21. -1.611

22. $\dfrac{1}{39}\log_e\dfrac{3p-4}{6p+5}$

23. $\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\dfrac{x+3}{\sqrt{5}}\right)$

24. $\dfrac{\sqrt{13}}{104}\log_e\left(\dfrac{2y+7-2\sqrt{13}}{2y+7+2\sqrt{13}}\right)$

25. $\dfrac{5}{4}\log_e(2u^2-u-9)-\dfrac{27\sqrt{73}}{292}\log_e\left(\dfrac{4u-1-\sqrt{73}}{4u-1+\sqrt{73}}\right)$

26. $\dfrac{17\sqrt{41}}{41}\tan^{-1}\left(\dfrac{3x+2}{\sqrt{41}}\right)-\dfrac{3}{2}\log_e(3x^2+4x+15)$

27. $\sqrt{x^2-3x+5}+\dfrac{3}{2}\sinh^{-1}\left(\dfrac{2x-3}{\sqrt{11}}\right)$

28. $\dfrac{11}{4}\cosh^{-1}\left(\dfrac{4v+5}{6\sqrt{2}}\right)-\dfrac{1}{2}\sqrt{16v^2+40v-47}$

29. $\dfrac{\sqrt{5}}{5}\sin^{-1}\left(\dfrac{5x+1}{4}\right)$

30. $31\sin^{-1}\left(\dfrac{2x-7}{\sqrt{65}}\right)-12\sqrt{4+7x-x^2}$

31. $\log_e\dfrac{y-9}{\sqrt{y^2-9y+1}}-\dfrac{9\sqrt{77}}{154}\log_e\dfrac{2y-9-\sqrt{77}}{2y-9+\sqrt{77}}$

32. $\dfrac{2}{x+1}+\log_e(x+1)+\dfrac{\sqrt{3}}{6}\log_e\dfrac{x-\sqrt{3}}{x+\sqrt{3}}$.

33. $\sinh^{-1}\left(\dfrac{v+3}{\sqrt{2}}\right)+\dfrac{(v+3)\sqrt{v^2+6v+11}}{2}$.

34. $\dfrac{(2x-1)\sqrt{2x^2-2x-1}}{4}-\dfrac{3\sqrt{2}}{8}\cosh^{-1}\dfrac{2x-1}{\sqrt{3}}$

35. $\dfrac{17\sqrt{6}}{36}\sin^{-1}\left(\dfrac{6x-2}{\sqrt{34}}\right)+\dfrac{(3x-1)\sqrt{5+4x-6x^2}}{6}$

36. 663 9 **37.** 0

38 0 **39.** $\frac{1}{3}\tan^{-1}(e^{3t})$

40 $\frac{1}{13}\log_e\tan^{-1}(x+\tan^{-1}\quad)$ **41** $-\ ^{1}\cot(t-\tan^{-1}\ ^{t})$

42 0 4303 **43** $\dfrac{\sqrt{5}\pi}{20}=0\ 3511$

44 $-\sqrt{1-x^2}$ **45** 5 757

46 $\dfrac{1}{7}\log_e\dfrac{y-2}{y+2}+\dfrac{\sqrt{3}}{7}\tan^{-1}\left(\dfrac{y}{\sqrt{3}}\right)$ **47** ^{1}

48 2 **49.** $\dfrac{\pi}{3}\quad\dfrac{\sqrt{3}}{8}=1\ 2637$

50. $\dfrac{\sqrt{2}}{12}=0\ 1178$ **51.** $\frac{1}{3}\sin^3\theta-\ ^{1}\sin^{5}\theta$

52. 0 **53** 0 **54.** $\frac{1}{2}\tan\ x-\log_e\sec x$

55. π **56** $\frac{1}{2}[y^{2}+y\sqrt{y^2-1}-\cosh^{-1}y]$

57. $\dfrac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2}+\dfrac{3a^4}{8}\log_e(x\quad\sqrt{x^2+a^2})$

58. $\dfrac{x}{a^2\sqrt{x^2+a^2}}$ **59.** $\dfrac{1}{\sqrt{a^2-x^2}}$

60. $\frac{3}{16}\pi a^2$. **61.** $-\sinh^{-1}\left[\dfrac{t+1}{\sqrt{3}(t-1)}\right]$

62. $2\left[\log_e\left(\dfrac{\sqrt{4-z}-2}{\sqrt{4-z}+2}\right)-\sqrt{4-z}\right]$

63 $\dfrac{\pi}{4}$ **64.** $\dfrac{x^6}{36}(6\log_e x-1)$ **65.** $x\tan^{-1}x-\log_e\sqrt{1+x^2}$

66. $\dfrac{e^{4x}}{10}(2\sin 2x-\cos 2x)$ **67.** $\dfrac{e^{-x}}{10}(2\sin 2x-\cos 2x-5)$

68. $\dfrac{e^{kt}}{k^2+p^2}[(ak+bp)\sin pt+(bk-ap)\cos pt]$

69. $\frac{4}{15}$ **70.** $\dfrac{5\pi}{32}$ **71.** $\frac{1}{12}$ **72.** $\frac{5}{1}$

73. 0 **74.** $\dfrac{\pi}{4}-\dfrac{2}{3}=0\ 1187$ **75.** 132

77. $\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\log_e(x+\sqrt{x^2+a^2})$, $\frac{1}{3}(x^2+a^2)$,

$\dfrac{k}{a^2}\cdot\dfrac{4\sqrt{5}-5\sqrt{2}}{10}=\dfrac{0\ 1874k}{a^2}$

78. $\dfrac{\pi}{4},\ \dfrac{\pi}{8}$. [Note that $x^4+4=(x^2+2x+2)(x^2-2x+2)$.]

79 $\frac{a}{y}, 2y \quad \frac{c}{y} \quad \frac{1}{a}\log\ (ax + y), \tfrac{1}{2}\left\{xy + \frac{c}{a}\log\ (ax + y)\right\}$

80 $\frac{27\pi}{8}$

81 $\pi\left[1 + \tfrac{1}{4}\sin^2\frac{\alpha}{2} + \quad \sin^4\frac{\alpha}{2} + \quad {}^{5}\ \sin\ \frac{\alpha}{2}\right]$, 0 14

82 $\frac{3\pi}{32} + \frac{1}{4} = 0\ 5445$

83. (i) $x\log_e(1 - x^2) + \log\left(\frac{1 + x}{1 - x}\right) - 2x$ (ii) $\frac{\pi^2}{4}$

84 (i) $-\frac{e^{-x}}{5}(\sin 2x + 2\cos 2x)$ (ii) $\frac{2\pi}{3}$

85. (i) $-\frac{\pi}{2}$ (ii) $3 - \frac{\pi}{4}$ (iii) $\pi + 2$

86. $\frac{x}{2}\sqrt{x^2 + 4} + 2\log_e(x + \sqrt{x^2 + 4})$

88. $\frac{1}{2}\log_e(x^2 + 2x - 4) - \frac{\sqrt{5}}{10}\log_e\left(\frac{x + 1 - \sqrt{5}}{x + 1 + \sqrt{5}}\right)$,

$\frac{x^3}{6} - \frac{x^2}{4}\sin 2x - \frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x$

89. $8\ 25 + \log_e 4 = 9\ 636$, $\frac{\sqrt{3}\pi}{18}$

90. $\frac{2n^{p+3}}{(p + 1)(p + 2)(p + 3)}$

91. (i) $-e^{-x}(x^2 + 2x + 2)$

(ii) $\frac{13}{15} - \frac{\pi}{4} - 0\ 0813$

(iii) $\frac{2}{5}x^{2\ 5} - {}^{6}\ x^{3} + \frac{2}{9}x^{4\ 5} - {}_{11}\ x^{5\ 5}$

(iv) $\frac{\sin x}{2 + \cos x}$

92 $\frac{\pi}{4} - \frac{1}{2}\log_e 2 = 0\ 4388$,

$\frac{1}{2}[\sqrt{2} + \log_e(\sqrt{2} + 1)] = 1\ 1477$

$\frac{\pi\sqrt{2}}{4} = 1\ 111$,

$\frac{\pi}{4} - \log_e 2 = 0\ 0923$

93. $\frac{1}{8}(3 - 19e^{-2}) = 0\ 0537$,

0 4,

$\frac{1}{8}[81\sinh^{-1}\frac{x + 2}{3} + (x + 2)(2x^2 + 8x + 53)\sqrt{x^2 + 4x + 13}]$

94. (i) $\frac{\pi}{4} - \frac{1}{2} = 0{\cdot}2854$

(ii) [illegible].

(iii) [illegible] $(3\pi$ [illegible] $2 \log_e 2) = 1{\cdot}0811$.

EXAMPLES IV. Page 135

2. (*a*) $x + \frac{x^3}{6} + \frac{3x^5}{40}$ (*b*) $x - \frac{x^3}{3} + \frac{x^5}{5}$.

5. $e^y\left(1 + h + \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 3} + \ldots\right)$.

6. $\tan x = x + \frac{1}{3}x^3 +$ [illegible] x^5; $\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + 2x^2 +$ [illegible] $x^3 +$ [illegible] x^4 $+$ [illegible] x^5. $\tan 3° = 0{\cdot}0524$; $\tan 44° = 0{\cdot}9657$.

7. $\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots$; $\cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \ldots$; percentage error $= 0{\cdot}0004$.

8. $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

9. $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots$, $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots$

13. $\log_e 5 = 1{\cdot}6094$.

15. $\log_{10} 73{\cdot}55 = 1{\cdot}86658$

16. $\sin 42° = 0{\cdot}669131$; $\cos 42° = 0{\cdot}743145$.

17. Error $= \frac{2}{3(2N+1)^3} = 0{\cdot}6471 \times 10^{-6}$ when $N = 50$

18. 1·608.

19. 4·275

20. $-2{\cdot}9$[illegible].

22. 2·120

23. 4·4936 radians $= 257° 28'$.

24. 2·013

25. 1·86.

26. 1·49.

27. $\theta = \alpha - \frac{x^2}{2}\tan\alpha$.

29. 3·9903122.

30. One root between 0 and 1, the other between 1 and 2; 0·305.

31. $\frac{1}{2}(1 + 2\log_e a)$; $\frac{1}{2}(2\log_e a - 1)$; $\frac{1}{2}\sec^2\alpha$.

32. See Art. 62.

33. $-\frac{1}{2}$; [illegible].

34. $\frac{1}{2}$; $\frac{1}{6}$; 1.

35. $\frac{n}{2}a^{n-2}$; [illegible].

37. $\log_e\left(\frac{a}{b}\right)$.

38. 0.

EXAMPLES V. Page 161

3. 1 (max.), 9 (min.). **4.** $x = 0, \frac{1}{8}$.

5. $\theta = \frac{\pi}{4}, p_{max} = \frac{3P}{bt^2}$ lb per in.^{2} **6.** $v = \sqrt{\frac{gT}{3w}}$.

7. (i) $b = d = \frac{\sqrt{2}D}{2}$. (ii) $b = \frac{\sqrt{3}D}{3}, d = \frac{\sqrt{6}D}{3}$. (iii) $b = \frac{D}{2}, d = \frac{\sqrt{3}D}{2}$.

8. Side of base $= \frac{\sqrt{3A}}{3}$, depth $= \frac{\sqrt{3A}}{6}$; side of base depth $= \frac{\sqrt{6A}}{6}$.

9. Radius of base = depth $= \sqrt{\frac{A}{3\pi}}$; radius of base $= \sqrt{\frac{A}{6\pi}}$, depth $= 2\sqrt{\frac{A}{6\pi}}$.

10. $x = \frac{1}{\sqrt{e}} = 0.606$.

11. Turning points where $x = 1$ and $-1{\cdot}5$; point of inflexion where $x = -0.25$; $x = 2{\cdot}024$.

13. Two-fifths of vessel full of water. **16.** $\alpha = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{1-\gamma}}$; 0·5270.

20. $R = 0{\cdot}0746$. **21.** 14·22 knots.

22. $x = \frac{2l}{9}$; min. period $= \frac{4\pi}{3}\sqrt{\frac{l}{g}}$.

23. Points of inflexion where $x = -2, -1, 1$; curve convex upwards between $x = -1$ and $x = 1$ and also between $x = -\infty$ and $x = -2$.

24. (i) Concave downwards at all points. (ii) Concave upwards at all points. (iii) When n is even, curve is concave upwards at all points; when n is odd, curve is concave upwards for positive values of x and concave downwards for negative values of x. (iv) (*a*) Curve is concave upwards at all points. (*b*) Curve is concave upwards for positive values of x, and concave downwards for negative values of x.

25. $\frac{W}{4BL}(2Ll + 2Bb - BL)$; $\frac{W}{4BL}(2Ll - 2Bb + BL)$;

$\frac{W}{4BL}(2Bb - 2Ll + BL)$; $\frac{W}{4BL}(3BL - 2Ll - 2Bb)$.

26. In AD, tension 14·9 lb; in DC, tension 11·2 lb; in CB, thrust 17·1 lb; in LD, thrust 18·7 lb; in AC, tension 21·3 lb.

27. (i) $9x^2 - 2y^2$; $-4xy - 12y^2$. (ii) $-\frac{y}{x\sqrt{x^2 - y^2}}$; $\frac{1}{\sqrt{x^2 - y^2}}$.

(iii) $2y \operatorname{cosec}(2xy)$; $2x \operatorname{cosec}(2xy)$.

29. (i) $x + 2y + 3z = 14$. (ii) $2z = 4x - 20y + 19$.

32. (i) $\frac{6x + 2y + 4}{2y - 2x + 5}$. (ii) $-\frac{x^3}{y^3}$. (iii) $-\frac{y(2x^2 - 1)}{x(2y^2 - 1)}$. (iv) $-\frac{y(x^2y + x + y)}{x(xy^2 + x + y)}$.

33. $abc(1 + k\theta)^3$.

35. (i) $dz = \dfrac{2}{(x-y)^2}[x\,dy - y\,dx]$

(ii) $dz = (6x^2y - 3y^3)dx + (2x^3 - 9xy^2 + 4y^3)dy$

(iii) $dz = \dfrac{1}{x^2}\tan\left(\dfrac{y}{x}\right)[y\,dx \quad x\,dy]$

36. 15 43 ft per sec

37. Volume decreasing at rate of $1\,980\pi$ in ^{3} per sec

45. Use the relations $dQ = \left(\dfrac{\partial Q}{\partial v}\right)_{v,T} dv + \left(\dfrac{\partial Q}{\partial T}\right)_{v,T} dt$ (1)

$$dQ = \left(\frac{\partial Q}{\partial p}\right)_{p,T} dp + \left(\frac{\partial Q}{\partial T}\right)_{p,T} dt \qquad (2)$$

and $$dp = -\frac{RT}{v^2}dv + \frac{R}{v}dt \quad . \quad . \quad . \quad (3)$$

Equate the right-hand sides of (1) and (2) and substitute in this the expression for dp from (3). This relation is identically true, hence equate the coefficients of dv and dt respectively on the two sides of the equation, $\left(\dfrac{\partial Q}{\partial p}\right)_{p,T} = -v$

46. 0 003 per cent nearly.

47. (i) $\delta\Delta = \dfrac{1}{8\Delta}[a(b^2 + c^2 - a^2)\delta a + b(c^2 + a^2 - b^2)\delta b + c(a^2 + b^2 - c^2)\delta c]$.

(ii) $\delta\Delta = \Delta\left(\dfrac{\delta a}{a} + \dfrac{\delta b}{b} + \delta C \cot C\right)$

(iii) $\delta\Delta = \Delta\left[\dfrac{2}{c}\delta c + \cot A \quad \delta A + \cot B \quad \delta B - \cot(A + B)\{\delta A + \delta B\}\right]$.

(iv) $\delta\Delta = -\dfrac{\Delta}{2}\left[\dfrac{\delta a}{s-a} + \dfrac{\delta b}{s-b} + \dfrac{\delta c}{s-c}\right]$, where $s = \dfrac{a+b+c}{2}$

48. 0 21 ft nearly   **49.** $-0\,0002$.   **50.** 5ι per cent.

52. Percentage error in $t = \frac{1}{2}\left[q + \dfrac{2ph}{r+h}\right]$

54. $\delta y = y[\delta x\,.\,p\cot(px + \alpha) - \delta t\,.\,q\tan(qt + \beta)]$

56. $\delta x = x\left[\dfrac{\delta r}{r} + \delta\theta\,.\,\cot\theta + \delta\lambda\,.\,\cot\lambda\right]$.

57. $\frac{1}{3}x = \frac{1}{4}y = z = 6^{\frac{1}{4}}$.

59. $x = 1{\cdot}723,\ y = 1\,990,\ z = 2\,223$

62. $x = 4,\ y = 2,\ z = 1$.   **63.** $x = y = 2$

64. $x = y = n\pi + (-1)^n\dfrac{\pi}{6}$, where n is zero or any positive or negative integer

65. $x = 2\sqrt{5}z,\ h = 2z$.   **67.** $a = 0\,250,\ b = 2\,833$.

EXAMPLES VI Page 190

1. $3my = 2(x + am^3)$

3. $(x = 4.303, y = 15.34)$, $(x = 0.6972, y = 6.244)$, max slope $= 2.887$

4. $y = t^2(3x - 2t)$, $t = \sqrt{\dfrac{r}{2}}$

5. Tangent, $2py = qt(3x - pt^2)$, normal, $3qty + 2px = t^2(2p^2 + 3q^2t^2)$

6. (i) Tangent, $\sqrt{3}x + y = 4$ Normal, $x = \sqrt{3}y$

Subtangent $= -\dfrac{\sqrt{3}}{3}$ Subnormal $= -\sqrt{3}$

(ii) Tangent, $x + 2y = 2c$ Normal, $2y = 4x - 3c$

Subtangent $= -c$ Subnormal $= -\dfrac{c}{4}$

(iii) Tangent, $y = x - 1$ Normal, $x + y + 1 = 0$

Subtangent $= -1$ Subnormal $= -1$

(iv) Tangent, $bx + \sqrt{3}ay = 2ab$ Normal, $2by = \sqrt{3}(2ax - a^2 + b^2)$

Subtangent $= -\dfrac{3a}{2}$ Subnormal $= -\dfrac{b^2}{2a}$

11. $90°$

12 Polar subtangent $= \pm a \tan 2\theta\sqrt{\sin 2\theta}$, polar subnormal

$= \pm a \cos 2\theta\sqrt{\operatorname{cosec} 2\theta}$

15. (i) $-\dfrac{(9x^2 + 4y^2)}{6xy}$; (ii) 0.566, (iii) 2.598, (iv) 0, (v) 2.828, (vi) a

16. $\tan^{-1}(\pm 2\sqrt{2}) = 70° \, 32'$ and $109° \, 28'$, $p = \dfrac{a}{4}$

17. $a^2 = b^2$, $\dfrac{1}{a^2}(a^4 + 4c^4)^{\frac{3}{2}}$, $\dfrac{1}{2c^2}(a^4 + 4c^4)^{\frac{3}{2}}$, $2c^2 = a^2$

18. $\dfrac{(a^2 + 12b^2)^{\frac{3}{2}}}{4ab}$.

22. (i) $\dfrac{r^3}{a^2}$, (ii) $\dfrac{l(1 + 2e\cos\theta + e^2)}{(1 + e\cos\theta)^3}$; (iii) $2a\sec^3\dfrac{\theta}{2}$, (iv) $\dfrac{(r^2 + a^2)}{r^2 + 2a^2}$

24. $\xi = a(\theta - \sin\theta)$, $\eta = a(3 + \cos\theta)$. Referred to parallel axes through the point $(\pi a, 2a)$, the parametric equations of the evolute become

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta).$$

The evolute of the given cycloid is therefore an equal cycloid

27. $\xi = \dfrac{(a^2 - b^2)\cos^3\phi}{a}$, $\eta = \dfrac{(b^2 - a^2)\sin^3\phi}{b}$

29. $a + b$

EXAMPLES VII Page 227

1. $5x - 3y - 21 = 0$, $3x - 11y - 31 = 0$; $9x + 7y - 13 = 0$

2. $(mx_1 + lx_2)/(l + m)$, $(my_1 + ly_2)/(l + m)$, $17 \cdot 49$, $13 \cdot 20$

3. $k = -10$

4. $x^2 + y^2 - 4x - 2y + 1 = 0$, $x = 2$, $y = 1$

5. $x = 6$, $y = 2$, $x = 0$, $y = -6$, $3x + 4y = 26$, $3x + 4y + 24 = 0$

6. $k = 2a \cos \dfrac{\alpha}{2}$

12. (i) $\dfrac{V^2 \sin^2 \alpha}{2g}$ (ii) $\dfrac{V^2 \sin 2\alpha}{g}$ (iii) $\dfrac{2V \sin \alpha}{g}$

13. $x = Vt \cos \theta_1$, $y = Vt \sin \theta_1 - \frac{1}{2}gt^2$, $x = Vt \cos \theta_2$, $y = Vt \sin \theta_2 - \frac{1}{2}gt^2$

16. The equation of the ellipse is $\dfrac{4x^2}{(R+r)^2} + \dfrac{4y^2}{(R+r)^2 - k^2} = 1$, its foci being the centres of the given circles and the axes of x and y being taken along and perpendicular to the line of centres R and r are the radii of the larger and smaller circles respectively and k is the distance between their centres The eccentricity of the ellipse $= \dfrac{k}{R+r}$

17. $101x^2 + 48xy + 81y^2 - 330x - 324y + 441 = 0$, $m = \dfrac{-8 \pm 4\sqrt{13}}{27}$ $= -0\,8305$ or $0\,2379$, $(1, 1\,1695)$, $(1, 2\,2379)$

22. $a^2p^2 - b^2q^2 = 1$

23. $e = \sqrt{6} - \sqrt{2} = 1\,035$

26 $\dfrac{x}{a} \sec \theta_1 - \dfrac{y}{b} \tan \theta_1 = 1$, $\dfrac{x}{a} \sec \theta_2 - \dfrac{y}{b} \tan \theta_2 = 1$

27. Tangent, $3t_1^2y + 4x = 24t_1$, normal, $4t_1y - 3t_1^3x = 16 - 9t_1^4$

30. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, $e = \dfrac{\sqrt{7}}{4} = 0\,6615$, point distant 3^1 in from either end

31. (i) $e = 0\,7141$, (ii) $4\,9$ in

32 The asymptotes are parallel to OX and OY respectively, and pass through the point $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ The equation of the hyperbola referred to its asymptotes is $xy = \dfrac{ab}{4}$

34.

x	$\frac{16}{5}$	$\frac{16}{5}$	$-\frac{16}{5}$	$-\frac{16}{5}$
y	$\frac{3\sqrt{7}}{5}$	$-\frac{3\sqrt{7}}{5}$	$\frac{3\sqrt{7}}{5}$	$-\frac{3\sqrt{7}}{5}$

, $y = \pm \dfrac{3x}{4}$

35. $136\,4$ lb

39. $0\,03$ ft

41. Total length $= l + \dfrac{8}{3}\dfrac{s^2}{l}$ approx, increase in sag $= \left(\dfrac{3l^2}{16s} + \dfrac{s}{2}\right) \alpha \Delta t$, decrease in tension $= H\left(\dfrac{3l^2}{16s^2} + \dfrac{1}{2}\right) \alpha\Delta t$

42 Greatest tension = 51.30 lb, least tension = 11.45 lb

43 Greatest tension = 56.32 lb, least tension = 11.50 lb

47 $x = c\sinh\left(\sqrt{\frac{g}{l}}\,t + \alpha\right)$ where $c = \frac{1}{g}\sqrt{u^2l - a^2g}$ and $\sinh\alpha = \frac{ag}{\sqrt{u^2l - a^2g}}$

EXAMPLES VIII Page 257

1. $y = x$ is axis of symmetry 4 $A = e^{l\,a}$

8

Original Relation	Altered Form	Substitutions	Modified Relation	Gradient	Intercept
$y = b + a\log x$	$y = a\log x + b$	$Y = y,\ X = \log x$	$Y = aX + b$	a	b
$y = \frac{cx}{a(x+1)}$	$\frac{1}{y} = \frac{a}{c}\frac{1}{x} + \frac{a}{c}$	$Y = \frac{1}{y},\ X = \frac{1}{x}$	$Y = \frac{a}{c}X + \frac{a}{c}$	$\frac{a}{c}$	$\frac{a}{c}$
$y = ax + b\sqrt{x+y}$	$\frac{y}{x} = a + \frac{b\sqrt{x+y}}{x}$	$Y = \frac{y}{x},\ X = \frac{\sqrt{x+y}}{x}$	$Y = a + bX$	b	a

The cases of $y = ax^n$ and $xy = a + bx$ are given in the Table of Art 102

9 $y = 0.99x + 3.16$ Percentage errors are (i) 0.17, (ii) 21, (iii) 6

10. See Art 103

11 $a = 0.302$, $b = 0.0004$

12. See Art 103

13 $a = 2.94$, $b = 0.0021$, $n = 1.99$

14. $a = 0.0014$, $b = 2.078$

15 $W = 1.974e^{0.2332\theta}$

16. $a = -0.031$, $b = 6.11$ Greatest error of 4.8 per cent

17. $a = 0.842$, $b = 4.17$, $n = 0.162$

18. $a = 0.296$, $b = 0.0004$

19. Best values are $m = 3.269$, $c = 0.849$, from a graph $m = 3.25$, $c = 0.85$

20. $a = 0.621$, $b = 2.133$, $c = 0.123$

21. $a = 0.115$, $b = 11.80$

23. (i) £530.66 (ii) £537.02 (iii) £540.30 (iv) £543.66

24. $\frac{100}{r}$ years

25. See Arts 103 to 105

26. See Art 103

27. (i) $a = 0.305$, $b = 0.125$ (ii) $a = 0.310$, $b = 0.125$

28 $y = 0.935x + 0.666$

29. $a = 6.727$, $b = 303.0$, $P = 0.0328W + 1.478$

30. $\mu = 0.266$

31. $a = 2.16$, $b = 1.06$, $c = 3.11$

32. $a = 1.307$, $b = 2.424$, $n = 0.0527$

33. $a = 0.229$, $b = 0.0912$, $c = 1.824$

EXAMPLES IX Page 314

1 64.

2. 117

3 $3\log_e 2 = 2.079$

4. $\frac{c}{1-n}\left(x_2^{1-n} - x_1^{1-n}\right)$

6. $c^2 \sinh \frac{\lambda_1}{c}$ **7.** $k = 0·004$, $l = 100$, $666\frac{2}{3}$

9. $c \log_e\left(\frac{v_2}{v_1}\right)$, $\frac{p_2v_2 - p_1v_1}{1-n}$ **10.** a – ', b = ', 135·1.

11. (i) 15 820 ft-lb (ii) 12 800 ft-lb **12.** $\frac{1\,496\frac{1}{4}}{EI}$.

13. $\frac{\tau q}{2r_2}(r_2^4 - r_1^4)$ **14.** $\frac{2}{3}\mu Pr$ **15.** $\frac{1}{2}\mu Pr$

16. (i) $\frac{2}{3}\mu P\left(\frac{R^2 + Rr + r^2}{R + r}\right)$ (ii) $\frac{1}{2}\mu P(R + r)$

17. (i) $\frac{2}{\pi}$ (ii) 0

20. (i) 7 (ii) $\frac{1 - e^{-2a}}{2a}$ (iii) 48·85 (iv) $\frac{a}{a-1}\log_e a - 1$

(v) $\frac{a^2}{2(a-1)}\log_e a - \frac{a+1}{4}$

21. 1, , 2. **23.** (i) $\frac{1}{2}$ (ii) . (iii) $2·5 + 2\cos\alpha$

24. (i) Period $= \frac{\pi}{p}$, $MV = \frac{\cos(\alpha - \beta)}{2}$ (ii) Period $= \frac{2\pi}{p}$, $MV = 0$.

(iii) Period $= \frac{\pi}{p}$, $MV = 14·5$

26. $\frac{1}{3}ab$. **28.** $10\frac{2}{3}$ **29.** $-1\frac{1}{3}$.

30. $\frac{1}{\,}\pi^3a^2$. **31.** See Art 77.

33. $\frac{a^2}{4\cot\alpha}\left(e^{2\theta_2\cot\alpha} - e^{2\theta_1\cot\alpha}\right) = \frac{r_2^2 - r_1^2}{4\cot\alpha}$

34. $\frac{\pi a^2}{2}$. Two circles **35.** $\frac{1}{ab}\tan^{-1}\left(\frac{b\tan\theta}{a}\right)$

36. The equation represents a circle of radius $\frac{a-b}{2}$ Area $= \frac{\pi}{4}(a-b)^2$. [a assumed $> b$.]

38. $3\pi a^2$.

39. (i) (*a*) $\frac{1}{\sqrt{2}}I$; (*b*) $\sqrt{\frac{I_1^2 + I_2^2}{2}}$. (ii) $\frac{1}{2}IE\cos(\alpha - \beta)$

40. $MV = \frac{1}{2}I_0^2\sqrt{R^2 + L^2p^2}\cos\alpha = \frac{1}{2}I_0^2R$

42. 234·5 in.^{3}, 670·1 in.^{3} **43.** $2\pi aL^2$.

45. 57·38 in.^{3} **46.** (i) $\frac{30}{\pi x^2}$ ft/min. (ii) 2·388 ft/min

47. (i) 2·297. (ii) 340·8. **48.** $\frac{56}{3}\pi a^2$.

50. (i) 140 100. (ii) 9·326 **51.** (i) 30·16. (ii) 25·13.

52. $V = \frac{\pi x}{300}(x^2 - 60x + 1\,200)$ in^3, $\frac{dV}{dt} = 0.003\pi(x^2 - 40x + 400)$ in^3/sec

53. (i) 23.18 ft (ii) 22.95 ft (iii) 22.98 ft

54. Length $= 2\int_0^b \sqrt{1 + \frac{4h^2x^2}{b^4}}\,dx$

55. Mean value $= \frac{a^2}{2} - \frac{4am}{\pi} + \frac{m^2\pi^2}{12}$, $m = \frac{24a}{\pi^3}$

56. Length of vertical portions = 5 ft. 57. $8a$

58. 7.254, 14.508

59. Percentage error negligible < 0.01.

62. $1.1752c$, (i) $1.1752c^2$, (ii) $1.4067\pi c^3$, (iii) $1{\cdot}2642\pi c^3$.

63. $\bar{x} = \frac{3c}{4}, \bar{y} = \frac{6ac^2}{5}$, $\bar{x} = \frac{5c}{6}, \bar{y} = 0$.

64. $\bar{x} = \frac{3l}{4}, \bar{y} = \frac{3wl^2}{20}$, deflection $= \frac{wl^4}{8EI}$

66. (i) On symmetrical radius at distance $\frac{4r}{3\pi}$ from centre (r = radius of circle).

(ii) On radius perpendicular to plane base and at distance $\frac{3r}{8}$ from centre (r = radius of hemisphere).

(iii) On axis at distance $\frac{h(3r_2^2 + 2r_2r_1 + r_1^2)}{4(r_2^2 + r_2r_1 + r_1^2)}$ from smaller end

67. On join of mid-points of parallel sides and at distance $\frac{2a + b}{3(a + b)}h$ from shorter side

68. $(h - 1)\,e^h + 1$, radius of section $= \sqrt{\frac{W}{\pi f}}\,e^{\frac{wy}{2f}}$; distance of c.g. from smaller end $= \dfrac{he^{\frac{wh}{f}}}{e^{\frac{wh}{f}} - 1} - \frac{f}{w}$.

69. $\bar{x} = \frac{3}{5}L$.

70. Area $= \frac{2}{3}a^{\frac{1}{2}}b$; $\bar{x} = \frac{3}{5}b$ $\bar{y} = \frac{3}{8}\sqrt{ab}$, vol. of ring $= \frac{1}{5}\pi a^{\frac{1}{2}}b$.

71. See Art. 106, Ex. 1; $\bar{x} = \frac{19a}{7}, \bar{y} = 0$.

72. On radius bisecting central angle and at distance $\dfrac{4r\sin\frac{\theta}{2}}{3\theta}$ from centre in (i), and $\dfrac{2r\sin\frac{\theta}{2}}{\theta}$ in (ii).

73. $\bar{x} = \frac{5a}{6}, \bar{y} = 0$, $\bar{x} = \frac{a(5\pi + 16)}{2(3\pi + 8)} = 0.9099a$; $\bar{y} = \frac{10a}{3\pi + 8} = 0.5741a$.

74. $\bar{x} = \pi a, \bar{y} = \dfrac{5a}{6}$.

75. $\bar{x} = \dfrac{\pi}{2}, \bar{y} = \dfrac{\pi}{8}$.

76. $\bar{x} = 0$, $\bar{y} = \dfrac{2a + c \sinh \dfrac{2a}{c}}{8 \sinh \dfrac{a}{c}}$.

77. If O is the centre of the circle and axes OX, OY are taken along and perpendicular to the radius through the less dense end of the arc, then

$$\bar{x} = \frac{2r}{a^2}(a \sin \alpha + \cos \alpha - 1), \bar{y} = \frac{2r}{a^2}(\sin \alpha - a \cos \alpha)$$

78. (i) On the symmetrical radius at distance $\dfrac{2r}{\pi}$ from centre of circle.

(ii) ,, ,, ,, ,, $\dfrac{4r}{3\pi}$,, ,,

(r = radius of circle.)

(iii) $\bar{y} = \dfrac{4b}{3\pi}, \bar{x} = 0$.

79. 4·402 in. from centre of circle.

80. (i) $I_{OX} = 357{\cdot}94$ (length)^4 units. (ii) $I_{OY} = 75{\cdot}6$ (length)^4 units.

81. $I_{OX} =$ [illegible]; $I_{OY} = 2(3\pi^2 - 4)$.

82. If A = area of circle, $I_0 = A\dfrac{a^2}{2}$; I about diameter $= A\dfrac{a^2}{4}$; I about tangent $= A\dfrac{5a^2}{4}$.

83. (i) $I = A\dfrac{H^2}{6}$ where A = area of $\Delta = \dfrac{BH}{2}$. (ii) $I = M\dfrac{H^2}{6}$, where H = height. (iii) $I = A\dfrac{H^2}{2}$ and $I = M\dfrac{H^2}{2}$.

84. (i) $I = M\tfrac{2}{5}r^2$. (ii) $I = M\dfrac{r^2}{2}$. (iii) $I = M\left(\dfrac{l^2}{12} + \dfrac{r^2}{4}\right)$.

(iv) $I = M\left(\dfrac{l^2}{3} + \dfrac{r^2}{4}\right)$.

85. (i) $I = M\tfrac{3}{10}r^2$. (ii) $I = M\tfrac{2}{3}r^2$.

86. $I = M\dfrac{a^2 + b^2}{12}$; $I = M\left[\dfrac{a^2}{12} + \dfrac{b^2}{3}\right]$. ($M$ = mass of solid.)

88. $56x^2 + 11y^2 = 12$; I about axis at 30° to $OX = 44{\cdot}75$ (inch)^4 units.

89. $I = M\left(\dfrac{h^2}{10} + \dfrac{3r^2}{20}\right)$.

90. Let base $BC = a$ and height of triangle $= h$. (i) $I = M\frac{h^2}{6}$ (ii) $I = M\frac{a^2}{24}$, I about $AB = M\frac{2a^2h^2}{3(a^2 + 4h^2)}$; I about axis through $A = M\left(\frac{a^2}{24} + \frac{h^2}{2}\right)$

91. $V = \frac{1}{3}\pi a^3 \operatorname{cosec} \alpha(1 + \sin \alpha)^2$.

92. $I_{OX} = \frac{\pi ab^3}{4}$; $I_{OY} = \frac{\pi a^3b}{4}$; $b^2x^2 + a^2y^2 = 4$.

93. $I = M\left[\frac{a^2b^2}{6(a^2 + b^2)} + \frac{l^2}{3}\right]$, where M = mass of rod $= ablm$.

94. K.E. $= \frac{7}{10} Mv^2$, if t_1 = time for solid sphere and t_2 = time for hollow sphere, then $\frac{t_1}{t_2} = \sqrt{\frac{98}{101}} = 0{\cdot}9703$.

95. $\frac{b}{\sqrt{12}}$ and $\frac{a}{\sqrt{12}}$; block in stable equilibrium with the 3 ft edges vertical.

98. Distance of metacentre from base of cylinder $= \frac{a^2}{4sh} + \frac{sh}{2}$; condition for stability is $a^2 > 2h^2s(1 - s)$.

99. Least radius = 4 in.

101. (*a*) On vertical diameter and at depths $\frac{17}{10}$ (diam.).

(*b*) At depth $\frac{17}{10}$ (diam.) and at distance 0·2122 (diam.) from vertical diameter.

105. 14·7 ft-lb.

EXAMPLES X. Page 341

1. Let P, Q be the fixed points, and let the fixed lines intersect at O, θ being the constant angle POQ. The space-centrode is a circle through P and Q such that $P\hat{I}Q = 180° - \theta$ = constant; the body-centrode is a circle, centre O and radius twice that of the space-centrode.

2. Let C be the centre of the fixed circle and P the position of the peg; let also Q be the point at which the slot meets the straight edge. The space-centrode is the circle on PC as diameter; the body-centrode is the circle whose radius is twice that of the space-centrode and whose centre is the foot of the perpendicular from C to PQ.

3. Let C_1 and C_2 be the centres of the fixed circles and let lines through C_1 and C_2 parallel respectively to the lines touching the circles meet at O. The space-centrode is the circle on C_1C_2 as diameter; the body-centrode is a circle, centre O and radius twice that of the space-centrode.

5. $(x^2 + y^2)(r^2 - l^2 + x^2)^2 = 4r^2x^4$.

8. The x-axis.

9. $8a$.

10. $\rho = \frac{4r(R + r)}{R + 2r} \sin\left(\frac{R\theta}{2r}\right)$.

12. $s = 4r(1 - \sin \psi)$.

15. A parabola, vertex at foot of perpendicular from the fixed point to the fixed straight line and latus-rectum equal to four times the length of this perpendicular. The fixed straight line is the tangent at the vertex of the parabola.

16. Equation of body-centrode is $a^2y^2 = x^4(x^2 - a^2)$, where a is the length of the perpendicular from the fixed point to the fixed straight line. The space-centrode is the parabola $ay = a^2 + x^2$, the x-axis being along the fixed line and the y-axis through the fixed point.

17. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$.

19. $n' = 4,\ \theta' = \theta;\ n' = -3,\ \theta' = \theta$.

21. $\omega = \sqrt{\dfrac{\cdot 3g(\sqrt{3} - 1)}{a}} = \dfrac{2{\cdot}66}{\sqrt{a}}$ radians per sec.

22. See end of Art. 126.

26. Similar to Ex. 3, except that the lines are not necessarily at right angles. The space-centrode is the circle through C_1, C_2 such that $C_1\hat{I}C_2 =$ supplement of angle between the rods. The body-centrode is as in Ex. 3.

28. The space-centrode is the parabola $dy = x^2 + d^2$.

31. $A = a^2,\ B = -\dfrac{4b(a - b)}{(a - 2b)^2}$.

33. The instantaneous centre immediately after the impulse is on the line through the centre of the square perpendicular to the direction of the impulse and at a distance $\dfrac{a}{3}$ from the centre. (See Art. 124, Ex. 2.)

34. Any point on the body-centrode generates a diameter of the space-centrode.

35. The two ends of that diameter of the body-centrode which is parallel to the given line will move along two perpendicular diameters of the space-centrode and will therefore envelop a four-cusped hypocycloid. The given line being parallel to and at a fixed distance from this diameter in the lamina, will envelop a parallel curve.

36. Relative to the ellipse, the origin O describes the director-circle; hence, the centre of the ellipse describes a circle, centre O and radius $\sqrt{a^2 + b^2}$, where a and b are the semi-major and semi-minor axes respectively of the ellipse.

37. (i) $PQ > OP$ and $\omega_1 = -\omega_2$. Produce OP to a point I such that $PI = OP$; then I is the instantaneous centre of the crank PQ, for if we imagine a point rigidly attached to PQ to be in the position I at the instant, this point will have a velocity $OP \,.\, \omega_1$ perpendicular to OI due to P's motion and a velocity $PI\omega_1$ in the opposite direction due to the rotation of PQ. The point will then be instantaneously at rest. The motion will be the same as that produced by the rolling of a circle inside a fixed circle of twice its radius. The path of Q will be a hypotrochoid—in this case an ellipse. (See Art. 128, Ex. 1.)

(ii) $PQ < OP$ and $\omega_1 = -\omega_2$. As in (i).

(iii) $PQ = OP$ and $\omega_1 = -\omega_2$. Same construction for I as in (i). Q now lies on the circumference of the rolling circle so that its path is a straight line, a diameter of the fixed circle.

(iv) $\omega_1 = \omega_2$. I is now the mid-point of OP, and the motion is the same as that produced by the rolling of a circle on the outside of another fixed circle of

equal radius. If $PQ = OP$, the path of Q is a cardioid; if $PQ \neq OP$, the path of Q is a limaçon.

38. $x = R(\cos\theta + \theta\sin\theta)$, $y = R(\sin\theta - \theta\cos\theta)$

43. See end of Art. 126.

EXAMPLES XI. Page 374

1. $\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^3 = 0$

2. $\dfrac{d^2y}{dx^2} = -y$

5. $y^2 = \dfrac{c}{x}$.

6. $x = ce^{\frac{1}{y}}$

7. $x^2 + y^2 = c$.

8. $y^2 - x^2 = c$.

9. $s = \frac{1}{3}at^3 + \frac{1}{2}bt^2 + ct + d$.

10. $s = \frac{1}{4}a^2(2\omega t + \sin 2\omega t) + c$.

11. $y = \frac{1}{2}x^2 - x + 2\log_e(x+1) + c$.

12. $y = \frac{1}{2}cx^2 + k$.

13. $y = ce^{\frac{x}{2}(x-2)}$.

14. $y = k + \dfrac{c}{\sqrt{x^2 - 1}}$.

15. $\sin^{-1}\dfrac{y}{a} - \sin^{-1}\dfrac{x}{a} = c$, which leads to $y\sqrt{a^2 - x^2} - x\sqrt{a^2 - y^2} = A$.

16. $y + 1 = c\sin x$.

17. $r^2(a - 2p) = c$.

18. $x^2 - 2xy - y^2 = c$.

19. $y^3 = x^3\log_e cx^3$.

20. $\dfrac{b}{\sqrt{ac}}\tan^{-1}\left(\dfrac{y\sqrt{c}}{x\sqrt{a}}\right) = \log_e k\sqrt{ax^2 + cy^2}$.

21. $y(1 - cx^2) = 2cx^3$.

22. $cx^2 = e^{y^2/x^2}$.

23. $x^2 + 2xy + 3y^2 = c$.

24. $x^3 + xy + 2y^3 = c$.

25. $x\sin y + y\sin x + xy = c$.

26. $\log_e(x^2 + y^2) = c$, which is equivalent to $x^2 + y^2 = k$.

27. $y = ce^{6x}$, and $c = 6$.

29. $q = ce^{-kt}$, and $c = q_0$.

30. $vr = c$; $h = k - \dfrac{c^2}{2gr^2}$.

31. (i) $y = ce^{-4x}$ and $c = 6$. (ii) $4y = 5 + ke^{-4x}$, and $k = 19$.

32. $by = a + ce^{-bx}$, and $c = -a$.

33. $2y = e^x + ce^{-x}$; $y = (x + c)e^{-x}$.

34. $y = \dfrac{c}{b^2 + a^2p^2}(b\sin pt - ap\cos pt) + ke^{-\frac{b}{a}t}$, and $k = \dfrac{acp}{b^2 + a^2p^2}$; $ay = (ct + k)e^{-\frac{b}{a}t}$.

35. $Ay = e^{-\frac{B}{A}x}\left\{\int e^{\frac{B}{A}x}f(x)dx + c\right\}$; 43·7 ft per sec nearly.

36. $6y = cx^3 - 3x + 2$.

37. $y = \frac{1}{\alpha + n} e^{\alpha t} + ce^{-nt}$.

38. $i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$.

39. $i = 0{\cdot}01956\,(15 \sin 300t + 100 \cos 300t - 100e^{-2000t})$; when t becomes large the last term tends to zero.

40. $y = ke^{cx}$, where k and c are constants.

41. $2x + y - 4 = c(2x - y + 8)^5$; $y - 3x - 1 = c(y + 2x + 4)^6$

43. $\frac{1}{y^{n-2}} = k(n - 2)x + c$

45. Terminal velocity $V = \sqrt{\frac{g}{k}}$; $u = V \tanh \frac{gt}{V}$; $x = \frac{V^2}{g} \log_e \cosh \frac{gt}{V}$.

46. $C = \omega_0^2 - \frac{2gh}{k^2 + h^2}$.

49. $v^2 = \frac{a}{k} + ce^{-\frac{2kx}{m}}$.

50. Density vanishes at height $\frac{n}{n-1} \cdot \frac{p_0}{\rho_0}$, where p_0 and ρ_0 are the pressure and density respectively at zero height; $\theta = \frac{\theta_0}{h_1}(h_1 - h)$ where θ and θ_0 are the absolute temperatures at heights h and zero respectively and h_1 is the height of the top of the atmosphere.

51. 110°C nearly.

54. The family of straight lines $y = mx$.

55. $x^2 + y^2 + 2fy = a^2$, f being the variable parameter; the family of coaxial circles $x^2 + y^2 + a^2 = 2kx$, k being arbitrary.

56. The family of curves given by $y^1 = cx$; the family of curves given by $y^1 = \frac{c}{x}$.

58. (i) $\omega^2 = -\frac{a}{1 + b^2}(b \sin \theta - \cos \theta) + ce^{-b\theta}$, where $c = \frac{a(\sqrt{3}b - 1)}{2(1 + b^2)} e^{b\pi/3}$.

(ii) $x = e^{-2t} + \frac{1}{3}t^2 -$ [illegible] $t +$ [illegible] $+ ce^{-3t}$.

60. 4·953 ft (g being taken as 32 ft per sec per sec).

61. $a - bv^3 = (a - bV^3)e^{-3b(x-c)}$; 0·0083 ft.

62. (i) $y = \frac{A - x}{\cos x}$. (ii) $(2y - x)(y + x)^2 = c$.

(iii) $9y^2 - 6xy + 4x^2 - 3y - 2x + c = 0$.

64. (i) $a = 3$, (ii) $3 \log_e (x + y) = x - y + 1$.

66. $\omega = \frac{G_0}{k} + \frac{GI}{k^2 + p^2I^2}\frac{k}{I}\left(\sin pt - p \cos pt\right) + Ce^{\frac{-kt}{I}}$.

EXAMPLES XII. Page 409

1. $x = x_0 + u_0 t + \frac{1}{2}gt^2$.

2. $x = \frac{29}{6} - 3t + \frac{3t^2}{2} + \frac{2t^3}{3}$.

3. $y = \frac{k}{12}(t^4 - 4tt_0^3 + 3t_0^4)$.

4. $y = y_0 + \frac{k}{12}(2ax^3 - x^4)$.

5. $y = 9 \sin x + 4 \cos x - \left(\frac{8}{\pi} + 1\right)x - 3$.

6. $y = 3(e^{\sqrt{6}x} + e^{-\sqrt{6}x})$.

7. $y^2 = 4(16 + x)$.

8. $x = 6(\sin 3t + \cos 3t) = 6\sqrt{2} \sin \left(3t + \frac{\pi}{4}\right)$.

9. $3x = 8 \cosh^{-1} \sqrt{\frac{y}{8}} + \sqrt{y^2 - 8y}$.

10. $t = 4\left[\cos^{-1}\sqrt{\frac{y}{8}} + \frac{\sqrt{8y - y^2}}{8}\right]$.

11. $y = 6(1 + \cos t)$.

12. $y = 5 + 3\sqrt{2} \sin \left(2x - \frac{\pi}{4}\right)$.

13. $y = Ae^{2\sqrt{2}x} + Be^{-2\sqrt{2}x} - 2{\cdot}5$.

14. (i) $M = \frac{5Wl}{24}$; points of inflexion $0{\cdot}55l$ from centre. (ii) $M = \frac{7Wl}{36}$; points of inflexion $0{\cdot}58l$ from centre.

15. (i) $y = Ae^{-2x} + Be^{-3x}$. (ii) $y = Re^{-x} \sin (4x + \varepsilon)$.

16. (i) $y = (Ax + B)e^{-3x}$. (ii) $y = Re^{-0{\cdot}6x} \sin (4{\cdot}964x + \varepsilon)$.

17. $x = Re^{-at} \sin (kt + \varepsilon)$; $x = Ae^{-(a - \sqrt{2a^2 + k^2})t} + Be^{-(a + \sqrt{2a^2 + k^2})t}$.

18. $x = Ae^{at} + Be^{-4at}$.

19. $x = A \cos (nt + \alpha)$.

20. $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$, where θ is the angle which the string makes with the vertical.

21. K.E. at distance $x = E + kM\left(\frac{1}{x} - \frac{1}{a}\right)$ $x^{3/2} = a^{3/2} - \frac{3\sqrt{2k}}{2}t$.

22. Stable equilibrium when $P\hat{A}O = \frac{\pi}{3}$ and period $= 2\pi\sqrt{\frac{4r}{3\mu}}$, where r is the radius; unstable equilibrium when P is at other end of diameter AO.

23. (i) $2\pi\sqrt{\frac{Wl}{Eg}}$. (ii) $2\pi\sqrt{\frac{\left(W + \frac{w}{3}\right)l}{Eg}}$.

25. $2\pi\frac{l}{a}\sqrt{\frac{W + 3w}{3Eg}}$.

28 Ratio of amplitudes $e^{-\frac{l\pi}{2l}}$, $x = Re^{-\frac{lt}{2}} \cos[\sqrt{n^2 - \frac{1}{4}a^2}\, t + \varepsilon]$

29. Period $= 2\pi\sqrt{\frac{Ml^2 + ma^2}{(Ml - ma)g}}$, which is less than $2\pi\sqrt{\frac{l}{g}}$, since $a < l$ For minimum period $a = \sqrt{\frac{Ml}{m}}\,(l - 2a)$

30 $2\pi\sqrt{\frac{2r^2 + 5a^2}{5ag}}$

31 Length on table varies from a to $\frac{V}{4Mg}(mV + \sqrt{m^2V^2 + 8aMmg})$, and length on table $= \frac{3a}{2}$ (max) when $V = \sqrt{\frac{1\ 8aMg}{m}}$

33 $x = (A + \frac{1}{4}t)\sin 6t \quad B\cos 6t,\ M = \frac{3}{4}g$

35. Highest point reached in time $\frac{V}{g\sin\alpha}$ from instant of projection

36. $y = Ae^{ax} + Be^{ax} + Ce^{ax}$

37. $s = 4e^{-t} + Be^{-8t} + \frac{1}{8}(18\sin 3t - 25\cos 3t)$

39. See Arts 139 and 142 $y = Ae^{3x} + Be^{-4x} - \frac{1}{6}e^{x}$

40. (i) $y = Ae^{x} + Be^{-7x}$ (ii) $y = (Ax + B)e^{-2x}$
(iii) $y = Ae^{-7\ 372x} + Be^{-1\ 628x}$

41. (i) $x = Ae^{5t} + Be^{6t} + \frac{1}{2}e^{7t}$ (ii) $x = A\sin(5t + \alpha) + 0\ 4e^{-5t}$
(iii) $y = Ae^{x} + Be^{4x} + \frac{1}{4}e^{-x} - 1$
(iv) $y = Ae^{x}\sin(2x + \alpha) - \frac{\sqrt{5}}{50}\sin(5x + \tan^{-1}2)$

42. $x = \frac{q}{2\sqrt{R^2 - 4CL}}\Big[(\sqrt{R^2 - 4CL} + R)e^{\frac{(-R + \sqrt{R^2 - 4CL})}{2L}t}$

$+ (\sqrt{R^2 - 4CL} - R)e^{\frac{(-R - \sqrt{R^2 - 4CL})}{2L}t}\Big]$,

$R^2 \nless 4CL$ if no periodic terms occur in value of x.

43. (i) $y = Ae^{2x} + Be^{-2x} - 2\ 5$ (ii) $Q = \frac{1}{10} + Ae^{-10000t}\cos(10\ 000t + \varepsilon)$

44. $x = A\sin t + (B - \frac{1}{2}at)\cos t$

45. $y = 4e^{-3t}\cos(4t + \varepsilon) + \frac{1}{13}(7\sin 2t - 4\cos 2t)$ When t is large, the first term on the right-hand side becomes negligible, $y = (\frac{1}{2}x^2 + Ax + B)e^{ax}$

46. $\alpha = 0,\ k = -\frac{a}{6p^2}$. **47.** $u = k(\theta\sin\theta - \cos\theta)$

48. $x = e^{-at}\left[\frac{1}{b}\left\{ah + \frac{pk(a^2 - b^2 + p^2)}{(a^2 + b^2 - p^2)^2 + 4a^2p^2}\right\}\sin bt + \left\{h + \frac{2apk}{(a^2 + b^2 - p^2)^2 + 4a^2p^2}\right\}\cos bt\right] + \frac{k(a^2 + b^2 - p^2)\sin pt - 2apk\cos pt}{(a^2 + b^2 - p^2)^2 + 4a^2p^2}$

49 (i) When $b^2 > 4ac$, $x = Ae^{\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)t} + Be^{\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)t}$

(ii) When $b^2 = 4ac$, $x = (At + B)e^{-\frac{bt}{2a}}$

(iii) When $b^2 < 4ac$, $x = Re^{-\frac{bt}{2a}}\cos\left(\frac{\sqrt{4ac - b^2}}{2a}t + \varepsilon\right)$

50. Period $= \dfrac{4\pi L}{\sqrt{\dfrac{4L}{c} - R^2}}$, frequency $= \dfrac{1}{2\pi\sqrt{LC}} = 65\,000$ nearly

51. $a\dfrac{d^2x}{dt^2} + 2b\dfrac{dx}{dt} + \dfrac{x}{k} = 0,\ a = kb^2$

52. (i) $T_1 = 20\pi\sqrt{\dfrac{2}{51g}}$ (ii) $T_2 = 4\pi\sqrt{\dfrac{14}{27g}}$, hence, $\dfrac{T_1}{T_2} = 1.38$ nearly.

53. $\dfrac{\pi}{24a}\sqrt{\dfrac{L}{g}(192l^2 + d^2)}$

54. Radius of gyration = 6.8 in

55. The point (x, y) describes a circle of radius a with angular acceleration $2p$, angular velocity of particle is n

EXAMPLES XIII Page 452

1. $\bar{x} = 11.82$ ounces, $\sigma = 0.440$ ounces, corrected value 0.436

6. $x = \pm\sigma$ **7.** 0.9104 **8.** 0.1914, 0.3413 **9.** $y = 15.8e^{-0.0784(x - 20.1)^2}$

13. 1.74 quarter-ounces **14.** 0.866, 0.816, 0.943 **15.** 2.85

16. (*a*) 42, 42, 32, 35, 41, (*b*) 4.91 and 6.32, (*c*) 2.77, (*d*) 1.46 against 1.41

17. (*b*) 4.91 and 6.32, (*c*) 2.72 (*d*) 1.54

18. 734 and 2, 741 and 4

19. (*a*) 0.667, (*b*) 0.670

21. 0.368 **22.** $r = 0.891$, yes

23. $Y = 0.0710X - 0.756$, $X = 14.09Y + 10.6$

24. 0.183 and 0.171, 0.48 against 0.63 in table, n is small and the distribution of r is not normal

25. No **26.** No

27 0 959, $Y = 4\ 723X + 4\ 514$, $X = 0\ 1992Y - 0\ 6154$

28 $-0\ 959$, $Y = -4\ 723X + 56\ 73$, $X = -0\ 1922Y + 11\ 62$

29. (*a*) 0 5, (*b*) 0 304, (*c*) − 0 414 (*d*) − 1

30. 0 968, − 0 952

31. 0 1309 and 0 1623, 0 1432 and 0 1580

EXAMPLES XIV Page 460

1. $\frac{44}{27}\mu W$, μ = coefficient of friction

3 0 748a, where a equals edge of cube.

6 $V \tanh \frac{ft}{V}, \frac{V^2}{f} \log_e \cosh \frac{ft}{V}$

10. The principal axes are inclined at the angle $\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2 - b^2)}$, which becomes 45° in the case of the square

13. 12 94 ft per sec

15. 2 51 ft, 24 1 ft-tons

16. Tension $= \frac{W\omega^2}{4ag}(a^2 - x^2)$

18. 10 1 radians per sec

19. 67 7°

20. Acceleration $= \dfrac{Tg}{r\left[M + 4m\left(1 + \frac{k^2}{r^2}\right)\right]}$

Friction forces on one wheel,

$$\frac{T}{4r}\left[1 \pm \frac{M + 4m}{M + 4m\left(1 + \frac{k^2}{r^2}\right)}\right]$$

taking the plus sign for a near wheel and the minus sign for a front wheel

25. $\frac{\pi + 4}{\pi} M\omega^2 a^2$.

28. $\omega^2 r = f\theta$

30. $\frac{1}{7} m\sqrt{32\ 5u^2 - 4\sqrt{2}u\omega a + \omega^2 a^2}$, at angle $\tan^{-1}\left[\frac{\sqrt{2}}{7}\left(2 - \frac{\omega a}{u}\right)\right]$

31. $T = \frac{d}{V} + \frac{V}{2}\left(\frac{1}{f_1} + \frac{1}{f_2}\right)$, $V = \sqrt{\frac{2f_1 f_2 d}{f_1 + f_2}}$

33. 29 1 min.

34. 8 min 49 sec.

35. 1 510 lb 33 7° to horizontal

36 2 02 ft

42. $\dfrac{wl^4}{384EI}$

43 $\omega = \sqrt{Pr(1 - e^{-cx})}$, where $c = \dfrac{2kg}{r(I + Pr^2)}$

45. 1 801 lb

49. 1 272 ft

51 1 532

54. 36°

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The numerals refer to the pages